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by

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Aarno Hohti

1. Introduction

The notion of *n-cardinality* is due to van Douwen and Przymusinski [7]. The n-cardinality of a subset of X^n, where X is any set, is the minimum cardinality of a set of hyperplanes of codimension 1, parallel to the coordinate axes, needed to cover the subset. In other words, the n-cardinality $|A|_n$ of $A \subseteq X^n$ is the minimum cardinality of a subset $Y \subseteq X$ such that

$$A \subseteq Y \times X^{n-1} \cup X \times Y \times X^{n-2} \cup \cdots \cup X^{n-1} \times Y,$$

or, equivalently,

$$A \subseteq \cup \{\pi_i^{-1}[Y]: \ 1 \leq i \leq n\}.$$

The basic result proved by van Douwen and Przymusinski tells us that if an analytic subset of X^n, where X is a Polish space, has uncountable n-cardinality, then the n-cardinality of the subset equals 2^ω.

By using this concept, we proved in [3] that there are supercomplete spaces [4] X (topologically subspaces of the reals) such that all the finite powers X^n are supercomplete but X^ω is not. This shows that the action of the Ginsberg-Isbell *locally fine coreflection* λ [1] is not determined by *finite subpowers*, even on separable metrizable spaces. Extending results of [5], [6], Hušek and Pelant recently proved that the locally fine coreflection of any product of fine, Čech-complete paracompact spaces is fine. The question whether--in the class of separable metrizable spaces--only Čech-complete (i.e., Polish) spaces have this property will be answered in the negative. By Gleason's Factorization Theorem (see [5], p. 130), it will be enough to consider only countable powers, and this leads us to the study of ω-cardinality.

To prove our result on locally fine coreflections, we need a relativized version of ω-cardinality. The main result concerning this notion states that given a Polish space X, a subset $S \subset X$, a subset $\Lambda \subset \omega$ and an analytic subset A of the product space X^ω, if the Λ-cardinality of A relative to S is uncountable, then this cardinality equals 2^ω. It is used in the inductive proof of Theorem 3.2, which is an extension of the Bernstein construction of a non-analytic subset of $[0,1]$. The results on relative ω-cardinality might be of independent interest.

2. ω-cardinality

Let X be a set, and let $A \subseteq X^\omega$. The ω-cardinality of A, written $|A|_\omega$, is defined as the minimum cardinality of a subset $Y \subseteq X$ such that

$$A \subseteq \pi_0^{-1}[Y] \cup \pi_1^{-1}[Y] \cup \ldots,$$

where the π_i are the standard projections. That is, A can be "killed" by the hyperplanes $\pi_i^{-1}(y)$, $y \in Y$. In the same vein, we can consider the Λ-cardinality of A with respect to any subset $\Lambda \subseteq \omega$: $|A|_\Lambda$ is the minimum cardinality of a subset $Y \subseteq X$ such that

$$A \subseteq \cup \{\pi_i^{-1}[Y] \colon i \in \Lambda\}.$$

In this paper, we have to consider Λ-cardinality *relative to* given subsets $S \subseteq X$. Let $A \subseteq X^\omega$, let $\Lambda \subseteq \omega$ and let $S \subseteq X$. We define that the Λ-cardinality of A relative to S, written $|A,S|_\Lambda$, is the minimum cardinality of a subset $Y \subseteq S$ (if such a set exists) such that

$$A \subseteq \cup \{\pi_i^{-1}[Y] : i \in \Lambda\}.$$

In case there is no such $Y \subseteq S$, we define $|A,S|_\Lambda = |X|$.

In this section we prove the analogue of the result of van Douwen and Przymusinski for relative Λ-cardinality, where Λ is a finite subset of ω.

Theorem 2.1. *Let X be a Polish space, let $A \subseteq X^\omega$ be analytic, let $S \subseteq X$ and let $\Lambda \in [\omega]^{<\omega}$. Then $|A,S|_\Lambda > \omega$ implies $|A,S|_\Lambda = 2^\omega$.*

Proof. The result is proved by induction on $|\Lambda|$. Obviously it is valid for $|\Lambda| = 1$. Suppose that we have proved it for $1 \leq |\Lambda| \leq n$, and let $|\Lambda| = n + 1$. By the definition of relative Λ-cardinality, we can assume that $A \subseteq \cup \{\pi_i^{-1} [S] : i \in \Lambda\}$. We consider two cases.

Case 1: $|A|_\Lambda \leq \omega$. There is a countable set $D \subset X$ with $A \subseteq \cup\{\pi_i^{-1}[D]: i \in \Lambda\}$. Since $|A,S|_\Lambda > \omega$, there exist $i \in \Lambda$ and $x \in D \sim S$ with

$$|A_{i,x},S|_\Lambda > \omega,$$

where $A_{i,x} = \pi_i^{-1}(x) \cap A$. But

$$|A_{i,x},S|_\Lambda = |A_{i,x},S|_{\Lambda\sim\{i\}},$$

and the inductive hypothesis implies that $|A_{i,x},S|_\Lambda = 2^\omega$. But then $|A,S|_\Lambda = 2^\omega$, because $x \in D \sim S$.

Case 2. $|A|_\Lambda > \omega$. Let $\pi_\Lambda: X^\omega \to X^\Lambda$ be the natural projection. Then (as is easily seen) $|A|_\Lambda = |\pi_\Lambda[A]|_{n+1}$, and therefore $|A|_\Lambda = 2^\omega$. Clearly $|A,S|_\Lambda \geq |A|_\Lambda$, and the claim is proved.

Now we move to prove the analogue of 2.1 for relative ω-cardinality. Let $S \subseteq X$, let $E \subseteq X^\omega$ and let $i \in \omega$. Define

$$A(i,E,S) = \{x \in S: \pi_i^{-1}(x) \cap E \neq \emptyset\}.$$

The set of (i,S)-limit points of E, written $D_{i,S}(E)$ is defined as the set of all $p \in X^\omega$ such that $|A(i,U\cap E,S)| \geq \omega$ for all neighborhoods U of p in X^ω. We define the successive (i,S)-derivatives in the same way as the Cantor-Bendixon derivatives are defined by transfinite induction:

$$D_{i,S}^{(0)}(E) = E;$$

$$D_{i,S}^{(\alpha+1)}(E) = D_{i,S}(D_{i,S}^{(\alpha)}(E)), \quad \text{and}$$

$$D_{i,S}^{(\beta)}(E) = \cap\{D_{i,S}^{(\alpha)}(E): \alpha < \beta\}$$

if β is a limit ordinal. There is α such that $D_{i,S}^{(\alpha+1)}(E) = D_{i,S}^{(\alpha)}(E)$; the set $D_{i,S}^{(\alpha)}(E)$ is called the *perfect* (i,S)*-kernel* of E and denoted by $K_{i,S}(E)$. Notice that $K_{i,S}(E)$ is a closed set and $A(i,E \sim K_{i,S}(E),S)$ is countable, since E is separable. Therefore, if E is a closed subset of X^ω with $|A(i,E,S)| > \omega$, then $K_{i,S}(E)$ is a nonempty closed subset of E. In case $K_{i,S}(E)$ equals the closure of E, the set E is called (i,S)-perfect.

The following lemma is needed in the proof of the main result. Notice that it follows from 2.3 that the hypothesis of 2.2 is never satisfied; thus, 2.2 is of technical character only.

Lemma 2.2. *Let* $S \subseteq X$ *and let* $F \subseteq X^{\omega}$ *be a closed subset such that* $\omega < |F,S|_{\omega} < 2^{\omega}$. *Then for each* $k \in \omega$ *there is a* $j \geq k$ *such that* $|K_{j,S}(F),S|_{\omega} > \omega$.

Proof. Suppose that for each integer $j \geq k$ we have $|K_{j,S}(F),S|_{\omega} \leq \omega$. Define $M = \{j : j \geq k, |A(j,F,S)| > \omega\}$. For each $j \in M$, define $F_j = K_{j,S}(F)$. Then F_j is a closed non-empty subset of F, and hence there exists an increasing sequence (F_j^i) of closed subsets of F such that $F_j^i \subset F \sim F_j$ and

$$F = F_j \cup (\cup_{i\in\omega} F_j^i).$$

As $F_j \cap F_j^i = \emptyset$, we have $|A(j,F_j^i,S)| \leq \omega$ for each $i \in \omega$. It follows that the set

$$D_1 = \cup_{j\in M} \cup\{\pi_j[F_j^i] \cap S: i \in \omega\}$$

is countable. On the other hand, for $j \in \omega \sim M$, $j \geq k$, we have $|A(j,F,S)| \leq \omega$ and thus the set

$$D_2 = \cup\{\pi_j[F] \cap S: j \in \omega \sim M, j \geq k\}$$

is countable, too. Finally, as by our assumption $j \geq k$ implies $|F_j,S|_{\omega} \leq \omega$, there is a countable set $D_3 \subset S$ such that

$$\cup_{j\in M} F_j \subset \cup_{i\in\omega} \pi_i^{-1}[D_3].$$

Define $D = D_1 \cup D_2 \cup D_3$ and let

$$F' = F \sim \cup_{i\in\omega} \pi_i^{-1}[D].$$

Notice that $j \geq k$ implies $\pi_j[F'] \cap S = \emptyset$. Thus, $|F',S|_{\omega} = |F',S|_{\Lambda}$, where Λ is the set $\{0,\cdots,k\}$. Clearly

$$|F,S|_{\omega} \leq |F',S|_{\omega} + |F \sim F',S|_{\omega} \leq |F',S|_{\Lambda} + \omega,$$

which implies (by the assumption of 2.2) that $|F',S|_{\Lambda} > \omega$. Since F' is analytic (being a G_δ-set), Theorem 2.1 now gives

$$2^\omega = |F',S|_{\Lambda} = |F',S|_\omega \leq |F,S|_\omega,$$

contradicting the hypothesis of 2.2. Hence, there is a $j \geq k$ with $|F_j,S|_\omega > \omega$.

Now we are ready to state the main result of this section.

Theorem 2.3. *Let* X *be a Polish space, let* $S \subseteq X$ *and let* $F \subseteq X^\omega$ *be closed. Then* $|F,S|_\omega > \omega$ *implies* $|F,S|_\omega = 2^\omega$.

Proof. Assume that $|F,S|_\omega > \omega$. We shall prove the claim by the method of contradiction. Thus, assume that $|F,S|_\omega < 2^\omega$. We shall construct a map $\varphi\colon 2^\omega \to F$ with the property that if $s,s' \in 2^\omega$, $s \neq s'$, then $\varphi(s),\varphi(s')$ do not both belong to any hyperplane $\pi_i^{-1}(x)$, where $x \in S$. To start with, let $\mathcal{B}$ be a countable base for open subsets of X^ω. Put

$$F' = F \sim \cup\{B \in \mathcal{B} : |F \cap \overline{B},S|_\omega \leq \omega\}.$$

Then F' is a closed subspace of F such that given any open subset U of X^ω, either $F' \cap U = \emptyset$ or $|F' \cap U,S|_\omega > \omega$. By Lemma 2.2 there is the least $i \geq 0$ such that $|K_{i,S}(F'),S|_\omega > \omega$. Let

$$F'' = K_{i,S}(F') \sim \cup\{B \in \mathcal{B} : |K_{i,S}(F') \cap \overline{B},S|_\omega \leq \omega\}.$$

Then F" is a closed subspace of $K_{i,S}(F')$ such that given any open subset U of X^ω, either $F'' \cap U = \emptyset$ or $|F'' \cap U,S|_\omega > \omega$. For each $r < i$ we have $K_{r,S}(F'') \neq F''$. (If $K_{r,S}(F'') = F''$, then $|K_{r,S}(F'),S|_\omega \geq |K_{r,S}(F''),S|_\omega = |F'',S|_\omega > \omega$, which would yield a contradiction with the definition of i given above.) Then for each such an r, there is an open set U_r with $U_r \cap F" \neq \emptyset$ and $A(r,\overline{U}_r,S) = \emptyset$. Indeed, suppose that $r < i$ and that $|A(r,\overline{U_r} \cap F",S)| > 1$ for each open set U_r for which $U_r \cap F" \neq \emptyset$. Then F" would have no (r,S)-isolated points and hence would be (r,S)-perfect. Thus, we would have $K_{r,S}(F") = F"$, which contradicts the result just obtained above. Thus, there is an open set U'_r such that $U'_r \cap F" \neq \emptyset$ and $|A(r,\overline{U'_r} \cap F",S)| \leq 1$. In case $A(r,\overline{U'_r} \cap F",S) = \emptyset$, we are done, otherwise let $A(r,\overline{U'_r} \cap F",S) = \{p\}$. Since $|U'_r \cap F",S|_\omega > \omega$, we have

$$(U'_r \cap F") \sim \pi_r^{-1}(p) \neq \emptyset.$$

Choose $q \in U'_r \cap F"$ with $\pi_r(q) \neq p$. Since $\pi_r^{-1}(p)$ is closed, we can find an open neighbourhood V of q such that $\overline{V} \cap \pi_r^{-1}(p) = \emptyset$. Now take $U_r = V \cap U'_r$. By redefining F" as $F" \cap \overline{U_r}$, we still have $|F",S|_\omega > \omega$. By repeating this procedure for all $r < i$, we get a set F" such that $A(r,F",S) = \emptyset$ for all $r < i$. Let ρ be some fixed compatible complete metric for X^ω. Choose two points $p_0,p_1 \in F"$ and open sets $U_0,U_1 \subset X^\omega$ satisfying the following conditions:

1) $p_j \in U_j$, $j = 0,1$;
2) $\overline{\pi_i[U_0]} \cap \overline{\pi_i[U_1]} = \emptyset$;
3) $\mathrm{diam}_\rho(U_j) < 1/2$, $j = 0,1$.

Define

$$\begin{cases} F_0 = F" \cap \overline{U}_0, \\ F_1 = F" \cap \overline{U}_1. \end{cases}$$

Then the sets F_j satisfy the conditions $\omega < |F_j,S|_\omega < 2^\omega$. Define $\Lambda_j = \{i\}$, where $j = 0,1$.

For the inductive step, let $n \in \omega$ and suppose that we have defined for all $s \in 2^n$ the points p_s and the sets F_s, Λ_s and that they satisfy the following properties:

1) if $s, s' \in 2^n$, $s \neq s'$, and $j \in \Lambda_s \cap \Lambda_{s'}$, then $\overline{\pi_j(F_s)} \cap \overline{\pi_j(F_{s'})} = \emptyset$;
2) if $j \in \{0,\cdots,n\} \sim \Lambda_s$, then $A(j,F_s,S) = \emptyset$;
3) $|F_s,S|_\omega > \omega$ for all $s \in 2^n$;
4) $\mathrm{diam}_\rho(F_s) < 2^{-(n+1)}$ for all $s \in 2^n$.

Let $\{s_0,\cdots,s_{2^n-1}\}$ be an enumeration of 2^n. (We consider 2^n, $n \in \omega$, as the set of all sequences $(t_0,\cdots,t_n)$ with terms in $\{0,1\}$. In the sequel the symbol $\sigma \mid m$ denotes the restriction of $\sigma \in 2^n$ to the set $\{0,\cdots,m\}$. For $i \in \{0,1\}$, the symbol $\sigma \wedge i$ denotes the concatenated sequence $\sigma(0)\cdots\sigma(n)i$. Similarly, 2^ω denotes the set of all sequences $(t_i)_{i\in\omega}$ with terms in $\{0,1\}$, and for each $\sigma \in 2^\omega$, $\sigma|n$ denotes the corresponding element of 2^n.) Let $t = s_0$ and let $k = \max(\Lambda_t)$. By 2.2 there is the least $i > k$ with $|K_{i,S}(F_t),S|_\omega > \omega$. Let

$$F_t' = K_{k,S}(F_t) \sim \cup\{B \in \mathcal{B} : |K_{k,S}(F_t) \cap \overline{B},S|_\omega \leq \omega\}.$$

As before, we can reduce F_t' so that if we have $K_{r,S}(F_t') \neq F_t'$ for some $r < i$ (i.e., F_t' is not (r,S)-perfect), then $A(r,F_t',S) = \emptyset$. Define

$$\Lambda_t' = \{r\colon 0 \leq r \leq i, A(r,F_t',S) \neq \emptyset\}.$$

Notice that we can use the inductive hypothesis (for F_t' instead of F) to find points $q,q' \in F_t'$ such that $\pi_j(q) \neq \pi_j(q')$ for all $j \in \Lambda_t' \subset \Lambda_t$. Choose neighborhoods V and V' of q,q', respectively, such that $\pi_j[V] \cap \pi_j[V'] = \emptyset$ for all $j \in \Lambda_t'$. Since F_t' is (i,S)-perfect, we can choose distinct $x, x' \in S$ with $P = \pi_i^{-1}(x) \cap V \cap F_t' \neq \emptyset$, $P' = \pi_i^{-1}(x') \cap V' \cap F_t' \neq \emptyset$. Choose any points $p \in P$, $p' \in P'$; then $\pi_j(p) \neq \pi_j(p')$ for all $j \in \Lambda_t' \cup \{i\}$. Put $p_{t\wedge 0} = p$, $p_{t\wedge 1} = p'$. Define $\Lambda_{t\wedge 0} = \Lambda_{t\wedge 1} = \Lambda_t' \cup \{i\}$.

For the subinductive hypothesis, let $0 \leq m < 2^n - 1$ and suppose that the points $p_{s_j\wedge 0}$, $p_{s_j\wedge 1}$ have been defined for all $j \leq m$. Let $t = s_{m+1}$, and let Λ_t', F_t' and i be defined as before. By repeating the procedure used above for finding p, p', if necessary, sufficiently many times, we can find points $p_{t\wedge 0}$, $p_{t\wedge 1} \in F_t'$ such that

$$\{\pi_j(p_{t\wedge 0}), \pi_j(p_{t\wedge 1})\} \cap \{\pi_j(p_{s\wedge 0}), \pi_j(p_{s\wedge 1})\} = \emptyset$$

whenever $s \in \{s_0, \cdots, s_m\}$ and $j \in (\Lambda_t' \cup \{i\}) \cap \Lambda_s$. (Use the inductive condition 1) above and in choosing x and x' in the preceding paragraph, notice that any finite set can be avoided.) This finishes the subinductive step.

Thus, we have defined the points p_s for all $s \in 2^{n+1}$. Choose neighbourhoods U_s such that

1) $p_s \in U_s$;
2) if $s, s' \in 2^{n+1}$, $s \neq s'$ and $j \in \Lambda_s \cap \Lambda_{s'}$, then $\overline{\pi_j[U_s]} \cap \overline{\pi_j[U_{s'}]} = \emptyset$;
3) $\operatorname{diam}_\rho(U_s) < 2^{-(n+2)}$.

For each $s \in 2^{n+1}$, let

$$F_s = F'_{s|n} \cap \overline{U}_s.$$

We get a map

$$F : 2^{<\omega} \to 2^{X^\omega}$$

defined by $F(s) = F_s$. Notice that for all $s \in 2^\omega$, we have

$$\cdots \supseteq F(s|n) \supseteq F(s|(n+1)) \supseteq \cdots$$

and $\operatorname{diam}_\rho F(s|n) \to 0$, whence we can define a map $\varphi: 2^\omega \to X^\omega$ by setting

$$\varphi(s) = \cap \{F(s|n): n \in \omega\}.$$

Moreover, as F is closed, $\varphi[2^\omega]$ lies in F. Now let $s, s' \in 2^\omega$, $s \neq s'$. Choose the least $n \in \omega$ with $s(n) \neq s'(n)$. Then by the construction of the sets F_t, $k \geq n$ implies

1) if $j \in \Lambda_{s|k} \cap \Lambda_{s'|k}$, then $\overline{\pi_j[F_{s|k}]} \cap \overline{\pi_j[F_{s'|k}]} = \emptyset$ and thus $\pi_j(\varphi(s)) \neq \pi_j(\varphi(s'))$;

2) if $j \in \{0, \cdots, k\} \sim (\Lambda_{s|k} \cap \Lambda_{s'|k})$, then either $F_{s|k} \cap \pi_j^{-1}[S] = \emptyset$ or $F_{s'|k} \cap \pi_j^{-1}[S] = \emptyset$ and therefore either $\varphi(s) \not\in \pi_j^{-1}[S]$ or $\varphi(s') \not\in \pi_j^{-1}[S]$.

It follows from 1) and 2) that $\varphi(s), \varphi(s')$ do not both belong to any hyperplane $\pi_j^{-1}(x)$, where $x \in S$. Therefore, the number of such hyperplanes needed to cover F is 2^ω. This condition shows that $|F,S|_\omega = 2^\omega$, as required.

The following result is a more general version of 2.3, proved in the same way as 2.3.

Theorem 2.4. *Let* X *be a Polish space, let* $S \subseteq X$, *let* $\Lambda \subseteq \omega$ *and let* $F \subseteq X^\omega$ *be closed. Then* $|F,S|_\Lambda > \omega$ *implies* $|F,S|_\Lambda = 2^\omega$.

Proof. If Λ is finite, then 2.4 can be proved by induction following the proof of 2.2; on the other hand, if Λ is infinite, then the proof of 2.3 applies, provided that only projections π_j with $j \in \Lambda$ are considered.

The concept of Λ-cardinality, relative to a subset of a Polish space X, can be generalized in a natural way to products in which not all the factors are the same space. Let $(X_i)_{i\in\omega}$ be a countable family of Polish spaces, and let $\Lambda \subseteq \omega$. For each $i \in \Lambda$, suppose that $X_i = X$ and let S be a subset of X. Then the Λ-cardinality of a subset $A \subseteq \Pi_{i\in\omega} X_i$ relative to S, written as usual $|A,S|_\Lambda$, is the least cardinality of a subset $Y \subseteq S$ such that

$$A \subseteq \cup \{\pi_i^{-1}[Y]: i \in \Lambda\}.$$

The following result is proved in the same way as 2.4.

Theorem 2.5. *Let* $(X_i)_{i\in\omega}$ *be a countable family of Polish spaces, let* $\Lambda \subseteq \omega$, *let* $X_i = X$ *for all* $i \in \Lambda$ *and let* $S \subseteq X$. *Then* $|F,S|_\Lambda > \omega$ *implies* $|F,S|_\Lambda = 2^\omega$ *for every closed subset* $F \subseteq \Pi_{i\in\omega} X_i$.

As a corollary, we obtain an easy proof of the extension of 2.4 to analytic subsets.

Corollary 2.6. *Let* X *be a Polish space, let* $S \subseteq X$, *let* $\Lambda \subseteq \omega$ *and let* $A \subseteq X^\omega$ *be analytic. Then* $|A,S|_\Lambda > \omega$ *implies* $|A,S|_\Lambda = 2^\omega$.

Proof. Since A is analytic, there is a closed subset $F \subseteq \omega^\omega \times X^\omega$ such that $A = \pi_0[F]$, where $\pi_0: \omega^\omega \times X^\omega \to X^\omega$ is the standard projection. Put $X_0 = \omega^\omega$ and for $i \in \omega \sim \{0\}$, put $X_i = X$. Define $\Lambda' = \{i + 1: i \in \Lambda\}$. Since $A = \pi_0[F]$, it is clear that

$$|F,S|_{\Lambda'} \text{ (in } \Pi_{i\in\omega} X_i) = |A,S|_\Lambda \text{ (in } X^\omega)$$

and the claim then follows from 2.5.

3. Locally Fine Coreflections

The locally fine coreflection of a uniform space uX, written λuX, is the coarsest uniformity on X, finer than u, with the property that every locally uniformly uniform cover is uniform. For this concept, see e.g. [1], [2]. Much of the importance of this notion derives from its connection with supercompleteness [4]. The proof of the following result is similar to that of Corollary 3.5 in [2].

Lemma 3.1. *Let* Y *be a dense subspace of a Polish space* X. *If for each closed* $K \subseteq X^{\omega} \sim Y^{\omega}$ *there is a* G_{δ}*-set* $G \subset X$ *with* $Y \subseteq G$ *and* $K \subseteq X^{\omega} \sim G^{\omega}$, *then* $\lambda((\mathcal{J}Y)^{\omega}) = \mathcal{J}(Y^{\omega})$. Thus, to prove that $\lambda((\mathcal{J}Y)^{\omega}) = \mathcal{J}(Y^{\omega})$, it is sufficient to show that given a closed subset $K \subseteq X^{\omega} \sim Y^{\omega}$, there is a countable $D \subset X \sim Y$ with

$$K \subseteq \cup \{\pi_i^{-1}[D] : i \in \omega\}.$$

Now we give the promised application of 2.3.

Theorem 3.2. *There is a non-analytic subset* $Y \subseteq [0,1]$ *such that* $\lambda((\mathcal{J}Y)^{\omega})) = \mathcal{J}(Y^{\omega})$.

Proof. Let $(F_{\alpha}: \alpha < 2^{\omega})$ be an enumeration of all closed subsets of $[0,1]^{\omega}$ and let $(A_{\alpha}: \alpha < 2^{\omega})$ be an enumeration of all analytic subsets of $[0,1]$ of cardinality 2^{ω}. We shall construct two sets $Y, Z \subseteq [0,1]$ by induction on α. To begin with, let $p \in F_0$ and put $Y_0 = \{\pi_i(p): i \in \omega\}$. Choose $q \in A_0 \sim Y_0$ and let $Z_0 = \{q\}$.

Suppose that $\alpha < 2^{\omega}$ and that the sets Y_{β}, Z_{β} have been defined for all $\beta < \alpha$ with the following properties:

1) $Y_{\beta}^{\omega} \cap F_{\beta} \neq \emptyset$ whenever $|F_{\beta}, [0,1] \sim Y_{\beta}|_{\omega} > \omega$;

2) $Y_{\beta} \cap Z_{\beta} = \emptyset$;

3) $\beta' < \beta < \alpha$ implies $Y_{\beta'} \subset Y_{\beta}$, $Z_{\beta'} \subset Z_{\beta}$;

4) $A_{\beta} \cap Z_{\beta} \neq \emptyset$;

5) $|Y_{\beta}|, |Z_{\beta}| < 2^{\omega}$.

If α is a limit ordinal, let $Y_{\alpha}^{*} = \cup \{Y_{\beta}: \beta < \alpha\}$ and $Z_{\alpha}^{*} = \cup \{Z_{\beta}: \beta < \alpha\}$. Otherwise, let $Y_{\alpha}^{*} = Y_{\beta}$, $Z_{\alpha}^{*} = Z_{\beta}$, where $\alpha = \beta + 1$. Now consider the set F_{α}. If $|F_{\alpha}, [0,1] \sim Y_{\alpha}^{*}|_{\omega} > \omega$, then by 2.3 $|F_{\alpha}, [0,1] \sim Y_{\alpha}^{*}|_{\omega} = 2^{\omega}$. In this case there is a point

$$p \in F_{\alpha} \sim \cup \{\pi_i^{-1}[Z_{\alpha}^{*}]: i \in \omega\},$$

because $Z_{\alpha}^{*} \subseteq [0,1] \sim Y_{\alpha}^{*}$ and $|Z_{\alpha}^{*}| < 2^{\omega}$. Put

$$Y_{\alpha} = \cup \{\pi_i(p): i \in \omega\} \cup Y_{\alpha}^{*};$$

clearly $Y_{\alpha}^{\omega} \cap F_{\alpha} \neq \emptyset$. On the other hand, if $|F_{\alpha}, [0,1] \sim Y_{\alpha}^{*}|_{\omega} \leq \omega$, then there is a countable $D \subset [0,1] \sim Y_{\alpha}^{*}$ such that $F_{\alpha} \subset \cup \{\pi_i^{-1}[D]: i \in \omega\}$. In this case choose $p \in A_{\alpha} \sim Y_{\alpha}^{*}$ (remember that $|A_{\alpha}| = 2^{\omega}$) and define $Y_{\alpha} = Y_{\alpha}^{*}$, $Z_{\alpha} = Z_{\alpha}^{*} \cup D \cup \{p\}$. This completes the inductive step.

Put $Y = \cup \{Y_{\alpha}: \alpha < 2^{\omega}\}$, $Z = \cup \{Z_{\alpha}: \alpha < 2^{\omega}\}$. Note that Y is not analytic, since its complement meets every analytic set of uncountable cardinality. To show that Y has the desired property, let $K \subset [0,1]^{\omega} \sim Y^{\omega}$ be closed. As $Y^{\omega} \cap K = \emptyset$, and $K = F_{\alpha}$ for some $\alpha < 2^{\omega}$, we have $|K, [0,1] \sim Y_{\alpha}^{*}|_{\omega} \leq \omega$ and thus by the construction of Z there is a countable set $D \subset Z$ with $K \subset \cup \{\pi_i^{-1}[D]: i \in \omega\}$. Therefore, 3.1 applies to show that $\lambda(\mathcal{J}(Y)^{\omega}) = \mathcal{J}(Y^{\omega})$.

Corollary 3.3. *There is a non-analytic subset* $Y \subset [0,1]$ *such that* $\lambda(\mathcal{J}(Y)^\kappa) = \mathcal{J}(Y^\kappa)$ *for every cardinal number* κ.

Proof. By 3.2 there is a non-analytic $Y \subset [0,1]$ such that $\lambda(\mathcal{J}(Y)^\omega) = \mathcal{J}(Y^\omega)$. By Gleason's factorization theorem, as given in [5], p. 130, $\lambda(\mathcal{J}(Y)^\kappa) = \mathcal{J}(Y^\kappa)$ for all κ.

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