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CLOSED PREIMAGES OF CERTAIN ISOCOMPACTNESS PROPERTIES

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In this paper we provide necessary and sufficient conditions that certain isocompactness properties are preserved in the closed preimage. A space is *isocompact* if every closed countably compact subset is compact. We consider the properties pure, closed complete, realcompact, Borel complete and neat in addition to isocompact itself. Also, we describe a new property, δ -neat, which is related to the others.

It has previously been shown that the perfect pre-images of closed complete, pure, neat and isocompact spaces are closed complete, pure, neat and isocompact, respectively. Since perfect mappings are closed mappings such that inverse images of points are compact, it is natural to ask by how much one can weaken the compactness condition. Later it will be shown that, at least in some cases, compactness can be reduced to the property under study. For example, the inverse image of a pure space under a closed mapping such that the inverse images of points are pure is itself pure. Analogous results hold for closed complete, δ -neat and isocompact. Related results are shown for realcompact and Borel complete spaces.

1. Definitions and Preliminary Results

Throughout the paper all spaces are assumed to be Hausdorff, all mappings are continuous unless otherwise specified. A *fiber* of a mapping is defined to be the inverse image of a singleton.

We begin with a theorem which sets a pattern for the remainder of the paper. For each property described, we prove a result analogous to Theorem 1.1.

Theorem 1.1. The inverse image X of an isocompact space Y under a closed mapping f is isocompact if and only if every fiber of f is isocompact.

Proof. First assume X is isocompact. Every fiber of f is closed and isocompactness is hereditary in closed subsets. Clearly then each fiber of f is isocompact.

Assume now that each fiber of f is isocompact. Let $A \subseteq X$ be closed and countably compact. Then $f^{-1}(A)$ is a closed countably compact subset of Y , and is therefore compact. Consider $\tilde{f} = f|_A$. $\tilde{f}: A \rightarrow f^{-1}(A)$ is closed and continuous. Let $y \in f^{-1}(A)$. Now $\tilde{f}^{-1}(\{y\}) \subseteq A$ is closed, implying that it is countably compact. It is also a closed subset of $f^{-1}(\{y\})$; by assumption $f^{-1}(\{y\})$ is isocompact, hence $\tilde{f}^{-1}(\{y\})$ is compact. Now we have that \tilde{f} is perfect. It is well known that the perfect preimage of a compact space is compact. Therefore A is compact. X is isocompact.

Definition [1]. A family $\mathcal{E} = \{E_n : n \in \omega\}$ of non-empty subsets of a space X is called an *interlacing* on X if $\bigcup \mathcal{E} = X$ and for each $n \in \omega$, $U \in \mathcal{E}_n$, U is open in $U \cap E_n$. An interlacing \mathcal{E} is δ -suspended from a family \mathcal{H} of subsets of a space X if for arbitrary $n \in \omega$, $x \in U \cap E_n$, $\exists F \subseteq \mathcal{H}$, $|F| \leq \omega$ such that $\text{st}(x, E_n) \cap (\bigcap F) = \emptyset$.

Definition [1]. A space X is called *pure* if for each free closed ultrafilter F on X there is an interlacing which is δ -suspended from F .

A collection of sets A has the *countable intersection property (cip)*, if every countable subset of A has non-empty intersection. The following proposition is easy to prove.

Proposition 1.2. A space X is pure if and only if each free closed ultrafilter with cip has an interlacing which is δ -suspended from it.

Definition [3]. A space X is called *closed complete* (also *α -realcompact* or *α -realcompact*) if every closed ultrafilter with cip has nonempty intersection.

Definition [5]. A Tychonoff space X is *realcompact* if every ultrafilter of zero sets (*z -ultrafilter*) with cip has nonempty intersection.

Definition [3]. A space X is called *Borel complete* if every Borel ultrafilter with cip has nonempty intersection.

Theorem 1.3. Borel complete \Rightarrow closed complete \Rightarrow pure \Rightarrow isocompact.

Proof. The first implication appears in [6], Theorem 1.1. The second is clear, as in a closed complete space there are no free closed ultrafilters with cip. The final implication is Theorem 5 of [1].

The following is shown in [3].

Theorem 1.4. Realcompact spaces are closed complete.

2. Pure Spaces

The following is stated by Arhangel'skii without proof:

Theorem 2.1. [1, Proposition 7]. The inverse image of a pure space under a perfect mapping is a pure space.

With the following two constructions we will give necessary and sufficient conditions that the inverse image of a pure space under a closed mapping is a pure space.

Lemma 2.2. If $f: X \rightarrow Y$, H is a collection of subsets of X , $F = \{f^{-1}(H) : H \in H\}$, and E is an interlacing δ -suspended from F , then there is an interlacing G which is δ -suspended from H .

Proof. For each $n \in \omega$, let $G_n = \{f^{-1}(E) : E \in E_n\}$. Then $G = \{G_n : n \in \omega\}$ is an interlacing which is

δ -suspended from H . $\bigcup G$ covers X since $\bigcup E$ covers Y , and for each $n \in \omega$ and $G \in G_n$, there is some $E \in E_n$ with $G = f^+ (E)$. E is open in $\bigcup E_n$, so G is open in $f^+ (\bigcup E_n) = \bigcup G_n$. We have shown that G is an interlacing. Now, let $n \in \omega$, and $x \in \bigcup G_n$. Then $f(x) \in \bigcup E_n$, so there is some countable $F' \subseteq F$ such that $\text{st}(f(x), E_n) \cap (\cap F') = \emptyset$. This implies $f^+ (\text{st}(f(x), E_n)) \cap f^+ (\cap F') = \emptyset$. However, $f^+ (\text{st}(f(x), E_n)) = f^+ (\bigcup \{E: f(x) \in E \in E_n\}) = \bigcup \{f^+ (E): f(x) \in E \in E_n\} = \bigcup \{G: x \in G \in G_n\} = \text{st}(x, G_n)$. Also, since each $F \in F'$ is $f^+ (H)$ for some $H \in H$, there is an $H' \subseteq H$ which is countable and for all $F \in F'$, $F = f^+ (H)$ for some $h \in H'$. Thus $\cap H' \subseteq \cap \{f^+ (f^+ (H)): H \in H'\} \subseteq \cap \{f^+ (F): F \in F'\} = f^+ (\cap F')$. Combining this with the previous two equations we get that $\text{st}(x, G_n) \cap (\cap H') = \emptyset$. Therefore G is δ -suspended from H .

Lemma 2.3. If H is a free closed ultrafilter on X with *cip*, $D \in H$, and D with the subspace topology is pure, then there is an interlacing G which is δ -suspended from H .

Proof. Let $H_D = \{H \cap D: H \in H\}$. H_D is a free closed ultrafilter on D with *cip*. Since D is pure, there is an interlacing $G_D = \{G_n: n \in \omega \setminus \{0\}\}$ on D which is δ -suspended from H_D . Define $G_0 = \{X \setminus D\}$. Then $G = \{G_n: n \in \omega\}$ is an interlacing which is δ -suspended from H . It is clear that G is an interlacing, since $\bigcup G_D$ covers D and $X \setminus D$ is open in $X \setminus D = \bigcup \{X \setminus D\}$. For $x \in \bigcup G_0$,

$\emptyset = (X \setminus D) \cap D = \text{st}(x, G_0) \cap (\cap \{D\})$, and $\{D\}$ is a countable subset of H . For $n > 0$, $x \in \cup G_n$, there is a countable $H' \subseteq H_D \subseteq H$ such that $\text{st}(x, G_n) \cap (\cap H') = \emptyset$. The collection G is therefore δ -suspended from H .

Theorem 2.4. *The inverse image X of a pure space Y under a closed mapping f is pure if and only if every fiber of f is pure.*

Proof. Assume that X is pure. Since f is continuous, each fiber of f is closed in X . Closed subspaces of pure spaces are pure [1, Proposition 6].

Conversely, assume that every fiber of f is pure. Let H be a free closed ultrafilter on X with cip. Let $F = \{f^{-1}(H) : H \in H\}$. F is a closed ultrafilter on Y with cip. Assume F is free. Then there is an interlacing E which is δ -suspended from F . By Lemma 2.2, there is an interlacing G which is δ -suspended from H . If F is fixed, then $\cap F = \{y\}$ for some $y \in Y$. Then $D = f^{-1}(\{y\})$ is closed, pure, and a member of H since $y \in f^{-1}(H)$ for every $H \in H$. By Lemma 2.3, H has an interlacing which is δ -suspended from it. We conclude that X is pure.

3. Closed complete, Borel complete and realcompact spaces

The result analogous to Theorem 2.4 for closed complete spaces is more easily proved. This partially generalizes Theorem 1.5 in [3].

Theorem 3.1. The inverse image X of a closed complete space Y under a closed mapping f is closed complete if and only if every fiber of f is closed complete.

Proof. Assume X is closed complete. Closed subspaces of closed complete spaces are closed complete.

Now assume that every fiber of f is closed complete. Let \mathcal{H} be a closed ultrafilter on X with cip. Then $\mathcal{F} = \{f^{-1}(H) : H \in \mathcal{H}\}$ is a closed ultrafilter on Y with cip. Since Y is closed complete, \mathcal{F} is fixed; therefore, $\cap \mathcal{F} = \{y\}$ for some $y \in Y$. Then $D = f^{-1}(\{y\}) \in \mathcal{H}$ as in Theorem 2.4. $\mathcal{H}_D = \{H \cap D : H \in \mathcal{H}\}$ is a closed ultrafilter on D with cip. Thus \mathcal{H}_D has a nonempty intersection since D is closed complete. However, $\cap \mathcal{H}_D = \cap \mathcal{H}$ because $D \in \mathcal{H}$. \mathcal{H} has a nonempty intersection, so X is closed complete.

Borel completeness is a somewhat different property from both pure and closed complete. It is known that compact spaces need not be Borel complete. The space 2^{ω_1} with the product topology is an easy example [6, Corollary 2.10]. Thus $f: 2^{\omega_1} \rightarrow \{x\}$ is a perfect mapping onto a Borel complete space where the preimage is not Borel complete. However, the following result does hold.

Theorem 3.2. [6, Theorem 2.6]. If $f: X \rightarrow Y$ is one-to-one and Borel measurable, then X is Borel complete if Y is.

This result can be generalized using a proof similar to that of Theorem 3.1. Note that we no longer require the mapping to be continuous or closed.

Theorem 3.3. The inverse image X of a Borel complete space Y under a Borel measurable map f is Borel complete if and only if every fiber of f is Borel complete.

Proof. If X is Borel complete, then the fibers of f are Borel complete since this property is hereditary.

Assume the fibers of f are Borel complete. Let H be a Borel ultrafilter with cip on X , let $F = \{F \subseteq Y: F \text{ is a Borel set and } \exists H \in H \text{ with } H \subseteq f^{-1}(F)\}$. We show that F is a Borel ultrafilter on Y . Let $B \subseteq Y$ be a Borel set such that $B \notin F$. Then $f^{-1}(B)$ is a Borel set in X and $f^{-1}(B) \notin H$. This implies $X \setminus f^{-1}(B) \in H$. Since $f^{-1}(Y \setminus B) \subseteq X \setminus f^{-1}(B)$ and $f^{-1}(Y \setminus B)$ is a Borel set in X , $f^{-1}(Y \setminus B) \in H$. Thus $Y \setminus B \in F$, proving that F is a Borel ultrafilter on Y . Clearly F has cip . Y is Borel complete, so $\cap F = \{y\}$ for some $y \in Y$. $D = f^{-1}(\{y\})$ is Borel complete and $D \in H$. The collection $H_D = \{H \cap D: H \in H\}$ is a Borel ultrafilter on D with cip . Hence, H_D has nonempty intersection, and $\cap H = \cap H_D$ because $D \in H$. X is Borel complete.

Strong preimage theorems are available in the literature for realcompactness. The following is given by Isiwata in [7].

Theorem 3.4. [7, Theorem 5.3]. Let $f: X \rightarrow Y$ be a \mathbb{Z} -mapping with C^ -embedded realcompact fibers. If Y is realcompact, then so is X .*

A Z -mapping is one which maps zero sets into closed sets. It is easily seen that Z -mappings are generalizations of closed mappings. Note, however, that the fibers are C^* -embedded in addition to being realcompact. An immediate corollary to this theorem is the following.

Corollary 3.5. [7, Theorem 5.4]. *If $f: X \rightarrow Y$ is a Z -mapping, X is normal, and fibers of f are realcompact, then if Y is realcompact, so is X .*

If we consider the relationships between a -realcompact and realcompact spaces, a similar result can be proved.

Theorem 3.6. [3, Corollary 1.10]. *A Tychonoff a -realcompact cb space is realcompact.*

Theorem 3.7. *If $f: X \rightarrow Y$ is a closed mapping with a -realcompact fibers from a Tychonoff cb -space X onto an a -realcompact space Y , then X is realcompact.*

Proof. By Theorem 3.1, X is a -realcompact (recall that closed complete and a -realcompact are the same). Then Theorem 3.6 gives the result.

Application of the techniques of this section to the realcompactness property are partially successful. The nature of zero sets and Z -filters allow the mappings considered to be only continuous, but an embedding condition is required on the fibers. A subset F of a space X is said to be Z -embedded if for every zero set $Z \subseteq F$ in the subspace topology, there is a zero set Y in X such that $Z = F \cap Y$.

Lemma 3.8. [3, before Theorem 1.7]. *If \mathcal{Z} is a \mathcal{Z} -ultrafilter which contains a prime \mathcal{Z} -filter with cip, then \mathcal{Z} has cip.*

Proof. Let \mathcal{Z}' be such a prime filter. Suppose $\{Z_i: i \in \omega\} \subseteq \mathcal{Z}$ such that $\bigcap \{Z_i: i \in \omega\} = \emptyset$. Choose $f_i: X \rightarrow I$ such that each $Z_i = f_i^+(\{0\})$, and let

$$f = \sum \frac{f_i}{2^i}. \quad \text{For each } n \in \omega, \text{ define } J_n = \{x: f(x) \geq 2^{-n}\}$$

and $K_n = \{x: f(x) \leq 2^{-n}\}$. Then $J_n \cup K_n = X \in \mathcal{Z}'$ for every $n \in \omega$; J_n and K_n are also zero sets. For

$$x \in \bigcap_{i=0}^{n+1} Z_i \text{ if } f(x) \leq 2^{-n}, \text{ so } \bigcap_{i=0}^{n+1} Z_i \subseteq K_n \text{ and } \bigcap_{i=0}^{n+1} Z_i \cap J_n = \emptyset.$$

This shows that $J_n \notin \mathcal{Z}'$ and therefore $K_n \in \mathcal{Z}'$ since \mathcal{Z}' is prime. But $\bigcap_{n \in \omega} K_n = \bigcap_{n \in \omega} Z_n = \emptyset$, contradicting the

fact that \mathcal{Z}' has cip. Thus \mathcal{Z} has cip.

Theorem 3.9. *Let $f: X \rightarrow Y$ be a continuous mapping with \mathcal{Z} -embedded realcompact fibers. If X is Tychonoff and Y is realcompact, then X is realcompact.*

Proof. Let \mathcal{Z} be a \mathcal{Z} -ultrafilter on X with cip. Then $\mathcal{U} = \{Z \subseteq Y: f^+(Z) \in \mathcal{Z} \text{ and } Z \text{ is a zero set in } Y\}$ is a prime \mathcal{Z} -filter on Y with cip [9, Problem 12F] and is contained in a unique \mathcal{Z} -ultrafilter \mathcal{U} on Y [9, Problem 12E] which, by Lemma 3.8, has cip. Since Y is realcompact, \mathcal{U} is fixed. Let $D = f^+(\{y\})$ for some $y \in \bigcap \mathcal{U}$ and $\mathcal{Z}_D = \{D \cap Z: Z \in \mathcal{Z}\}$. For all $Z \in \mathcal{Z}$, $D \cap Z \neq \emptyset$, so $\emptyset \notin \mathcal{Z}_D$. We verify that \mathcal{Z}_D is a \mathcal{Z} -filter. Let $A \in \mathcal{Z}_D$ and

let B be a zero set in D such that $A \subseteq B$. There exists $Z_A \in \mathcal{Z}$ and, since D is \mathcal{Z} -embedded in X , a zero set Z_B in X such that $A = D \cap Z_A$ and $B = D \cap Z_B$. Now because \mathcal{Z} is a \mathcal{Z} -filter and $Z_A \in \mathcal{Z}$, $Z_A \cup Z_B$ is also in \mathcal{Z} . Thus $D \cap Z_A \subseteq D \cap Z_B$ gives $B = D \cap Z_B = (D \cap Z_A) \cup (D \cap Z_B) = D \cap (Z_B \cup Z_A) \in \mathcal{Z}_D$. We have shown that \mathcal{Z}_D is a \mathcal{Z} -filter on D . Now let A be a zero set in D such that $A \cap (D \cap Z) \neq \emptyset$ for every $Z \in \mathcal{Z}$. There is a zero set Z_A on X such that $A = Z_A \cap D$, so for $Z \in \mathcal{Z}$, the intersection $Z \cap Z_A \supseteq Z \cap D \cap Z_A = Z \cap D \cap A$ is nonempty. Since \mathcal{Z} is an ultrafilter, we have $Z_A \in \mathcal{Z}$; hence $A \in \mathcal{Z}_D$, proving that \mathcal{Z}_D is a \mathcal{Z} -ultrafilter. Clearly \mathcal{Z}_D has cip. Since D is *realcompact*, $\emptyset \neq \bigcap \mathcal{Z}_D \subseteq \bigcap \mathcal{Z}$. We conclude that X is *realcompact*.

An additional question may be posed in this area. In [4], Frolik defines a property which is weaker than *realcompact* and stronger than *a-realcompact*. A space is *almost realcompact* if for every open filter \mathcal{U} such that the intersection of the closures of every countable subcollection of \mathcal{U} is nonempty, $\bigcap \{\bar{U} : U \in \mathcal{U}\} \neq \emptyset$. Does a similar theorem hold for almost *realcompact* spaces; i.e., is the closed preimage of an almost *realcompact* space under a mapping with almost *realcompact* fibers almost *realcompact*?

4. Neat Spaces

Definition [8]. For an ultrafilter H , $\lambda(H) = \min\{|F| : F \subseteq H \text{ and } \cap F = \emptyset\}$.

Definition [8]. A space X is called *neat* if for every free closed ultrafilter H with cip on X there is a system $\langle X_\gamma, V_\gamma, f_\gamma \rangle \gamma \in \Gamma$ such that

- (1) $\Gamma < \lambda(H)$ and $\bigcup_{\gamma \in \Gamma} X_\gamma = X$,
- (2) for each $\gamma \in \Gamma$, V_γ is an open collection in X and $X_\gamma \subseteq \bigcup V_\gamma$,
- (3) each $f_\gamma: X_\gamma \rightarrow V_\gamma$ is such that if $A \in [X]^{<\omega}$ and $f_\gamma|_A$ is injective, then $\overline{A}^{\bigcup V_\gamma} \subseteq \bigcup_{x \in A} f_\gamma(x)$,
- (4) for each $\gamma \in \Gamma$ and $x \in X_\gamma$, $\exists H \in H$ such that $f_\gamma(x) \cap X_n \cap H = \emptyset$.

Such a system is called a *neat system* for H .

The following two results given by Sakai in his paper which introduces the concept of neat spaces suggested the line of investigation leading to the group of pre-image results given in this paper:

Theorem 4.1. [8, Theorem 3.2]. Let f be a closed map from X onto a neat space Y . If each fiber of f is Lindelöf, then X is neat.

Theorem 4.2. [8, Theorem 3.7]. Let f be a closed map from X onto a closed complete space Y . If each fiber of f is neat, then X is neat.

By employing techniques used in proving Theorems 4.2 and 2.4, closed complete can be replaced in the statement of 4.2 by pure; using a similar method of proof, one may also substitute property θL described by S. W. Davis in [2]. Either substitution makes a slight strengthening of the theorem. The nature of the definition of $\lambda(H)$ has so far prevented us from proving a result for neat spaces which is similar to Theorems 2.4 and 3.1.

However, a slight modification to the definition of neat produces a technically stronger property which preserves all the theorems stated for neat save one. In addition, one may prove a theorem in the manner of 2.4 and 3.1 for this new isocompactness property. The author has unfortunately been unable to provide an example of a space which distinguishes the following from neat.

Definition. A space X is called δ -neat if for every free closed ultrafilter H with cip on X there is a system $\langle X_n, V_n, f_n \rangle$ $n \in \omega$ such that

- (1) $\bigcup_{n \in \omega} X_n = X$,
- (2) for each $n \in \omega$ V_n is an open collection in X and $X_n \subseteq \bigcup V_n$,
- (3) each $f_n: X_n \rightarrow V_n$ is such that if $A \in [X]^{<\omega}$ and $f_n|_A$ is injective, then $\overline{A}^{\bigcup V_n} \subseteq \bigcup_{x \in A} f_n(x)$,
- (4) for each $n \in \omega$ and $x \in X_n$, $\exists H \in \mathcal{H}$ such that $f_n(x) \cap X_n \cap H = \emptyset$.

Such a system is called a δ -neat system for H .

Proposition 4.3. The following spaces are δ -neat:

- (a) *neighborhood F spaces*
- (b) *spaces satisfying property θL*
- (c) *$\delta\theta$ -penetrable spaces*
- (d) *pure spaces*

The proof of this proposition is identical to that of [8, Proposition 2.3]. While it is known that spaces which are weakly $[\omega_1, \infty)^{\mathbb{R}}$ -refinable are neat [8, Proposition 2.3], it is still unknown whether weakly $[\omega_1, \infty)^{\mathbb{R}}$ -refinable spaces are δ -neat. Sakai provides a CH example which shows that neat spaces are different from all of those mentioned in Proposition 4.3 as well as weakly $[\omega_1, \infty)^{\mathbb{R}}$ -refinable spaces. This example [8, Example 3.8] serves also to distinguish δ -neat from these spaces.

Proposition 4.4. δ -neat spaces are neat

Proof. Since all closed ultrafilters H must have $\text{cip}, \omega < \lambda(H)$. Otherwise the definition of δ -neat corresponds exactly to that of neat.

Theorem 4.5. [8, Theorem 2.6]. Neat spaces are isocompact.

We now show the principal theorem which is true for δ -neat spaces but is not yet known for neat spaces. The following preliminary lemma does hold when formulated for neat systems.

Lemma 4.6. Every free closed ultrafilter H with cip on a space X which contains a δ -neat subspace of X has a δ -neat system for H .

Proof. Let $H \in \mathcal{H}$ be the δ -neat subspace, and define $F = \{H' \in \mathcal{H} : H' \subseteq H\}$. Then F is a free closed ultrafilter with cip on H . Let $\langle F_n, V_n, g_n \rangle$ $n \in \omega \setminus \{0\}$ be a δ -neat system for F . For each $n \in \omega \setminus \{0\}$, let $W_n = \{V \cup (X \setminus H) : V \in V_n\}$, $X_n = F_n$ and define $f_n: X_n \rightarrow W_n$ by $f_n(x) = g_n(x) \cup (X \setminus H)$. Also let $X_0 = X \setminus H$, $W_0 = \{X \setminus H\}$, and $f_0: X_0 \rightarrow W_0$. Then $\langle X_n, W_n, f_n \rangle$ $n \in \omega$ is a δ -neat system for H . We need only verify conditions (2) - (4) for $n = 0$.

$$(1) \bigcup_{n \in \omega} X_n = (X \setminus H) \cup \left(\bigcup_{n \in \omega \setminus \{0\}} H_n \right) = (X \setminus H) \cup H = X,$$

$$(2) X \setminus H \text{ is open in } X \text{ and } X \setminus H \subseteq X \setminus H,$$

$$(3) \text{ If } A \subseteq X \setminus H, |A| \leq \omega, \text{ and } f_0|_A \text{ is injective, then}$$

$$|A| \leq 1, \text{ so } \bar{A} = A. \text{ Thus } \bar{A}^{X \setminus H} \subseteq X \setminus H = \bigcup_{x \in A} f_0(x),$$

$$(4) \text{ Since } H \in \mathcal{H}, f_0(x) \cap (X \setminus H) \cap H = \emptyset.$$

Lemma 4.7. Let $f: X \rightarrow Y$ be a closed mapping, H be a free closed ultrafilter on X , and $F = \{f^+(H) : H \in \mathcal{H}\}$. If F has a δ -neat system $\langle Y_n, U_n, g_n \rangle$ $n \in \omega$, then H has a δ -neat system.

Proof. Define, for each $n \in \omega$, $X_n = f^+(Y_n)$, $V_n = \{f^+(U) : U \in U_n\}$, and $f_n: X_n \rightarrow V_n$ by $f_n(x) = f^+(g_n(f(x)))$. Then $\langle X_n, V_n, f_n \rangle$ $n \in \omega$ is a δ -neat system for H .

(1) and (2) are clear since f is a continuous surjection.

(3) Let $n \in \omega$, $A \subseteq X_n$, $|A| \leq \omega$. Assume $f_n|_A$ is an injection. Then $g_n|_{f^+(A)}$ must also be an injection.

Let $a \in \overline{A}^{\cup \cup n} = \overline{A} \cap (\cup \cup n)$. This gives $f(a) \in f^+(\overline{A}) \cap (\cup \cup n) =$

$\overline{f^+(A)} \cap (\cup \cup n)$. Since $f^+(A) \subseteq Y_n$, $|f^+(A)| \leq \omega$, and

$g_n|_{f^+(A)}$ is an injection, we have $\overline{f^+(A)}^{\cup \cup n} \subseteq \bigcup_{y \in f^+(A)} g_n(y) =$

$\bigcup_{x \in A} g_n(x)$. Thus $f^+ \left[\overline{f^+(A)}^{\cup \cup n} \right] \subseteq f^+ \left(\bigcup_{x \in A} g_n(f(x)) \right) =$

$\bigcup_{x \in A} f_n(x)$. Working in the other direction

$$\overline{A}^{\cup \cup n} \subseteq \overline{f^+(f^+(A))}^{\cup \cup n} \subseteq f^+ \left[\overline{f^+(A)}^{\cup \cup n} \right] \subseteq \bigcup_{x \in A} f_n(x).$$

(4) For each $n \in \omega$ and $x \in X_n$ there is an $F \in \mathcal{F}$ such that $g_n(f(x)) \cap Y_n \cap F = \emptyset$. Thus, mapping back into X , we get $f_n^+(x) \cap X_n \cap f^+(F) = f^+(g_n(f(x))) \cap f^+(Y_n) \cap f^+(F) = \emptyset$. However, $f^+(F) \in \mathcal{H}$, completing the proof.

Copying the proof of Theorem 2.4, using Lemmas 4.6 and 4.7 to replace Lemmas 2.2 and 2.3 where appropriate, we obtain the following result.

Theorem 4.8. The inverse image X of a δ -neat space Y under a closed mapping f is δ -neat if and only if every fiber of f is δ -neat.

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