TOPOLOGY PROCEEDINGS

Volume 13, 1988

Pages 181–188

http://topology.auburn.edu/tp/

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by

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Topology Proceedings

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ISSN:	0146-4124

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It is shown that span zero is preserved under finite to one open mappings. 54C10, 54F20

Introduction

The notion of span of a compact metric space and its natural generalization semispan were defined by A. Lelek in 1964 and 1977, respectively [6, 7]. It follows from the definitions that semispan is greater than or equal to span and both functions are monotone with respect to closed subsets. It can be shown directly that a nonunicoherent continuum or a triod have span greater than zero. All chainable continua have semispan zero and those continua without the fixed point property have span greater than zero. J. F. Davis has shown that span zero and semispan zero are equivalent [2]. In [11] I. Rosenholtz has shown that an open image of a chainable continuum is also chainable. A natural question is whether open mappings preserve span zero.

Notations and Definitions

A metric continuum is a compact connected metric space. We let d represent the distance function and Π_1 and Π_2 the natural projection mappings of the Cartesian Product $X \times X$ onto X. The semispan of a compact metric space, denoted by $\sigma_0(X)$, is the least upper bound of all real numbers ε such that there is a subcontinuum Z of $X \times X$ with the properties that $\Pi_2(Z) \subset \Pi_1(Z)$ and $d(x,y) > \varepsilon$ for all $(x,y) \in Z$. The definition of span of X, denoted by $\sigma(X)$, is the same except that we require $\Pi_2(Z) = \Pi_1(Z)$. A mapping $f(X) = Y$ is weakly confluent provided that for each subcontinuum $K \subset Y$, there is a component H of $f^{-1}(K)$ such that $f(H) = K$. A continuum Y is in class W provided that for each mapping f from a continuum onto Y, f is weakly confluent. A continuum is indecomposable provided it cannot be the union of two proper subcontinua. For $x \in X$, define K_x as follows; $K_x = \cap\{K$, K is a subcontinuum of X and x is in the interior of K$\}$. The properties of K_x for certain continua are developed in [1]. A subcontinuum B is **terminal** in the space X if whenever H and K are two subcontinua with $B \cap H \neq \emptyset$ and $B \cap K \neq \emptyset$, then $B \cup H \subset B \cup K$ or $B \cup H \supset B \cup K$.

Preliminary Results

A mapping is finite to one if each point inverse is a finite set.

Lemma 1. *Let* $f(X) = Y$ *be a finite to one open mapping, where* X *and* Y *are locally compact separable metric spaces. The set* D *of all points* x *of* X *such that* f *is a local homeomorphism at each point of* $f^{-1}f(x)$ *is an open dense subset of* X.

Proof. Let $D_n = \{x | x \in X$ and $f^{-1}f(x)$ has less than or equal to n points$\}$. Each D_n is a closed set by the openness of f. Since $X = \bigcup_{n=1}^{\infty} D_n$ and X is a Baire Space some D_n contains interior points. Let n_0 be the least integer for which D_{n_0} has interior points and let x be an interior point. Each point of $f^{-1}f(x)$ is interior to D_{n_0} since $f^{-1}f(\text{int}.M_{n_0})$ is an open set. Let $f^{-1}f(x) = \{x_1, x_2, \ldots, x_{n_0}\}$ and take pairwise disjoint open sets $U(x_i)$, $i = 1,2,\ldots,n_0$ about each point of $f^{-1}f(x)$ and contained in $\text{int}.M_{n_0}$. Now $f \mid U(x_i)$, $i = 1,2,\ldots,n_0$ is an open and 1-1 mapping of $U(x_i)$ onto $f(U(x_i))$ and is therefore a homeomorphism. To show D is dense in X apply the argument above to an arbitrary non-empty open set U. That is $U = \bigcup_{n=1}^{\infty} (U \cap D_n)$ and U as a subspace is a Baire space so there is a least integer n_0 such that $U \cap D_{n_0}$ has interior points. See [12], VII, 3.5 for a similar result.

Lemma 2. *If* $f(X) = Y$ *is a finite to one open mapping, where* X *and* Y *are locally compact separable metric spaces, and* B *is a closed subset of* X *such that int.*$f(B)$ *is not empty, then int.*B *is not empty.*

Proof. By the preceding result there is a point $y_0 \in \text{int}.f(B)$ such that f is a local homeomorphism at each point of $f^{-1}(y_0)$. Suppose $f^{-1}(y_0) = \{x_1,\ldots,x_{n_0}\}$ and choose pairwise dispoint open sets $U(x_i), i = 1,\ldots,n_0$ so that $W = f(U(x_i) = f(U(x_j)) \subset \text{int. } f(B)$ for all $i,j = 1,\ldots,n_0$ and each $f|U(x_i)$ is a homeomorphism of $U(x_i)$ onto $f(U(x_i))$. We have $W = \bigcup_{i=1}^{n_0} f(B \cap U(x_i))$ and each $f(B \cap U(x_i))$ is an F_σ set and W as a subspace is a Baire space so some $f(B \cap U(x_i))$ has interior points. If V is an open set in $f(B \cap U(x_i))$ then the set U in $B \cap U(x_i)$ which maps onto V is open in X.

Lemma 3. *Let* X *be a hereditarily unicoherent atroidic metric continuum. For* $x \in X$, *if* K_x *has non-empty interior then* K_x *is indecomposable or the union of two indecomposable continua with non-empty interiors.*

Proof. Suppose $K_x = L \cup M$, where L and M are proper subcontinua. We can assume $L = \overline{L - M}$ and $M = \overline{M - L}$. By the definition of K_x it follows that $x \notin L - M$ and $x \notin M - L$ so $x \in L \cap M$ and furthermore $x \in \text{int.}(L \cup M)$ otherwise the space X contains a triod. Suppose $L = L_1 \cup L_2$, where L_1 and L_2 are proper subcontinua. If $x \notin L_1$, then $x \in \text{int.}(L_2 \cup M)$ which contradicts the definition of K_x so that $x \in L_1 \cap L_2 \cap M_i$ this implies $L_1 \cup L_2 \cup M$ is a triod and this is not possible hence L and M are indecomposable with non-empty interiors.

From [3] we have: Theorem 2. If $\sigma_0(X) = \varepsilon > 0$, there exists an indecomposable continuum $I \subset X$ with

$\sigma_0(I) = \varepsilon$ and every proper subcontinuum of I has semispan less than ε. The space X in this result must be an atroidic hereditarily unicoherent metric continuum.

Main Result

The following theorem is the main result of this article.

Theorem. Let X *and* Y *represent metric continua. If* $\sigma_0(X) = 0$ *and* $f(X) = Y$ *is a finite to one open mapping, then* $\sigma_0(Y) = 0$.

Proof. $\sigma_0(X) = 0$ implies X is atroidic and hereditarily unicoherent. By [8] $\sigma_0(X) = 0$ implies X is tree-like and by [5] atroidic and tree-like implies X is in class W. By [9] Y is hereditarily unicoherent and by [10] Y is tree-like. The mapping f takes atroidic continua onto atroidic continua, so by [5], Y is in class W. By Theorem 2, if $\sigma_0(Y) = \varepsilon > 0$, then there exists an indecomposable subcontinuum I with $\sigma_0(I) = \varepsilon$ and every proper subcontinuum of I has semispan less than ε. Let H be a component of $f^{-1}(I)$, then, as is well known, $f(H) = I$ and f/H is a finite to one open mapping of H onto I. For $x \in H$, $K_x = \bigcap_\alpha K_x^\alpha$, $K_x \subset H$, $\alpha \in \Gamma$ and $x \in \text{int } K_x^\alpha$ relative to H. The continuum $f(K_x^\alpha) \subset I$ has interior points so $f(K_x^\alpha) = I$ since I is indecomposable. Thus, in this case, we can argue that $f(K_x) = I$ and since f is finite to one and open, by Lemma 2 int $K_x \neq \emptyset$. There are only two possible cases; (i) K_x is indecomposable, or (ii) $K_x = K_y \cup K_z$, where K_y and K_z have non-empty interior, are indecomposable, int $K_y \cap$ int $K_z = \emptyset$ and $x \in K_y \cap K_z$. This means that, since f is finite to one and open on H, we can consider H as a finite linear chain of indecomposable continua, i.e., $H = K_{x_1} \cup K_{x_2} \cup, \ldots, \cup K_{x_n}$, where {int K_{x_i}}, $i = 1, \ldots, n$ is a pairwise disjoint collection and only successive K_{x_i}'s intersect. The continuum H is irreducible from a point $p \in K_{x_1}$ to a point $q \in K_{x_n}$. There is a continuum $Z \subset I \times I$ such that for all $(x,y) \in Z$, $d(x,y) \geq \varepsilon$ and $\Pi_1(Z) = I = \Pi_2(Z)$. The continuum Z can be chosen so that it is indecomposable and Π_1/Z and Π_2/Z are irreducible mappings [4]. Let C be the composant of K_{x_1} which is accessible from K_{x_2} and let L be the composant of I which contains f(C). The composant L may be expressed as, $L = \bigcup_{i=1}^{\infty} D_1$, $D_i \subset D_{i+1}$, $i = 1,2,\ldots$ and each D_i is a continuum. For each i, $Z - \Pi_1^{-1}(D_i)$ is open in Z and connected and dense since $\Pi_1^{-1}(D_i)$ cannot meet all composants of Z. Thus $[\bigcap_{i=1}^{\infty} Z - \Pi_1^{-1}(D_i)] \cap [\bigcap_{i=1}^{\infty} Z - \Pi_2^{-1}(D_1)] \neq \emptyset$ by a Baire theorem and hence there exists an $(x,y) \in Z$ with $x,y \in I - L$. Since $(f \times f)(K_{x_1} \times K_{x_1}) = I \times I$ there is a point $(a,b) \in K_{x_1} \times K_{x_1}$ with $(f \times f)(a,b) = (x,y)$ and $a,b \in K_{x_1} - C$. Let B be the component of $(f \times f)^{-1}(Z)$ which contains (a,b). We have $f(\Pi_1(B)) = I$ so $\Pi_1(B)$ has interior points and thus cannot be a proper subcontinuum of K_{x_1}. The point $a \in \Pi_1(B)$ and $a \notin C$ so $\Pi_1(B) \supset K_{x_1}$. Similarly $\Pi_2(B) \supset K_{x_1}$. Since K_{x_1} is a terminal subcontinuum in H it follows that $\Pi_1(B) \supset \Pi_2(B)$ or $\Pi_1(B) \subset \Pi_2(B)$. Since B does not meet the diagonal of H, we have $\sigma_0(H) > 0$ and consequently $\sigma_0(X) > 0$, which is a contradiction.

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