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0. Introduction

A topological space X is said to be *homogeneous* if for every two points p and q in X there exists a homeomorphism $\phi: X \to X$ such that $\phi(p) = q$. A Cartesian product of homogeneous spaces is homogeneous. However, if at least one of the Cartesian factors is homeomorphic to the Menger curve M, then the Cartesian product does not have some of the stronger homogeneity-type properties, see [3], [6] and [7]. Even more interesting are continua which are not Cartesian products but whose every point has a neighborhood homeomorphic to a Cartesian product with one or more factors homeomorphic to M, see [5].

In this paper, twisted products are obtained by making certain identifications on $M \times M$, $M \times I$, or $M \times S^1$. The construction yields continua whose every point has a homogeneous neighborhood but the space might not be homogeneous, see [4]. It is shown here that twisted products of two Menger curves are not (with one obvious exception) homeomorphic to the Cartesian product $M \times M$, but many twisted products of M and I are homeomorphic to $M \times S^1$.

1. Preliminaries

Let M denote the Menger curve, a subset of the cube $\{(x,y,z) \in E^3: x, y, z \in [0,1]\}$ as described by R. D.

Anderson in [1], page 321. For every $c \in [0,1]$, let $M_c = \{(x,y,z) \in M\colon z = c\}$.

[1] and [2] contain several strong theorems concerning the Menger curve M. Mainly, it has been proved that every 1-dimensional continuum, with no local cut points, and no nonempty open subsets embeddable in the plane, is homeomorphic to M, and that M is homogeneous. Moreover, if U is an open and connected subset of M, and p and q are points in U, then there exists a homeomorphism $\phi\colon M \to M$ such that $\phi(p) = q$ and $\phi(v) = v$ for $v \in U$. A space X is *strongly k-homogeneous* if for any two ordered sequences $P = \{p_1,\ldots,p_k\}$ and $Q = \{q_1,\ldots,q_k\}$ of distinct points there exists a homeomorphism of X carrying P onto Q. One of the results in the above papers is that M is strongly k-homogeneous for any integer $k \geq 1$.

Let A and B be two disjoint and closed subsets of a compact space X. Suppose that A and B are homeomorphic and let $H\colon A \to B$ be a homeomorphism. Let $\sim$ be the equivalence relation defined on X so that $p \sim q$ iff $p = q$, or $p = H(q)$, or $q = H(p)$. The space of equivalence classes will be denoted by X/H.

A homeomorphism $g\colon X \to X$, where X is a topological space, is *periodic* if there is an integer $k > 1$ such that for every $x \in X$, we have $g^k(x) = x$, and for every $x \in X$ and every integer ℓ, $1 \leq \ell < k$, $g^\ell(x) \neq x$.

Let S^1 be the unit circle in E^2; $S^1 = \{(r,\theta) \in E^2\colon r = 1 \text{ and } \theta \in [0,2\pi)\}$, where (r,θ) denote the polar coordinates. Assume that $S^1 = \{\theta\colon \theta \in [0,2\pi)\}$, and that if θ_1 and θ_2 are in S^1, then the usual operations $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ modulo 2π can be performed. The unit interval in E^1 will be denoted by $[0,1]$ or I.

2. The Twisted Products $(M \times M)/H$

Let A and B be subsets of the Cartesian product $M \times M$ defined as follows: $A = M \times M_1$ and $B = M \times M_0$. Let $h: M \to M$ be a homeomorphism. Define $H: A \to B$, by

$$H((x_1,y_1,z_1), (x_2,y_2,1)) = (h(x_1,y_1,z_1), (x_2,y_2,0)).$$

The point p in the resulting continuum $(M \times M)/H$, corresponding to the point (m,n) in $M - A$, will be denoted by $(\overline{m},\overline{n})$.

Theorem 1. *If* h *is periodic, then* $(M \times M)/H$ *is homogeneous.*

Proof. Let $c \in [0,1)$ be a number. If $c \neq 0$, then the set M_c separates $M - M_1$ into two components V_1, containing M_0, and V_2. For $c = 0$, let $V_1 = \emptyset$ and $V_2 = M - (M_0 \cup M_1)$. Denote by $\tilde{M}$ the continuum obtained from M by identifying each point $(x,y,1)$ in M with the point $(x,y,0)$. For $c \in [0,1)$, let $\tilde{M}_c$ be the subset of $\tilde{M}$ corresponding to M_c. The point in $\tilde{M}$, corresponding to the point $n \in M - M_1$, will be denoted by $\tilde{n}$. Define an embedding $\psi_c: M \times (\tilde{M} - \tilde{M}_c) \to (M \times M)/H$ by

$$\psi_c(m,\tilde{n}) = \begin{cases} (\overline{m},\overline{n}) \text{ if } n \in V_1, \\ (\overline{h^{-1}(m)},\overline{n}) \text{ if } n \in V_2. \end{cases}$$

Denote the image $\psi_c(M \times (\tilde{M} - \tilde{M}_c))$ by U_c.

Suppose that $p = (\bar{m}_p, \bar{n}_p)$ and $q = (\bar{m}_q, \bar{n}_q)$ are two arbitrary points in $(M \times M)/H$.

There exists a number $c \in [0,1)$ such that both points p and q are in the set U_c; equivalently, the points $\tilde{n}_p$ and $\tilde{n}_q$ are in $\tilde{M} - \tilde{M}_c$. By [1] and [2], $\tilde{M}$ is homeomorphic to M, $\tilde{M} - \tilde{M}_c$ is connected, and there exists a homeomorphism $g: \tilde{M} \to \tilde{M}$ such that $g(\tilde{n}_p) = \tilde{n}_q$ and $g(\tilde{n}) = \tilde{n}$ for $\tilde{n} \in \tilde{M}_c$ (i.e. $n \in M_c$). Let $\mu_1: M \times \tilde{M} \to M \times \tilde{M}$ be such that $\mu_1(m,\tilde{n}) = (m, g(\tilde{n}))$. Let $h_1: (M \times M)/H \to (M \times M)/H$ be defined by

$$h_1(v) = \begin{cases} v & \text{if } v \notin U_c \\ \psi_c \circ \mu_1 \circ \psi_c^{-1}(v) & \text{if } v \in U_c. \end{cases}$$

Hence $h_1(p) = (\bar{s}, \bar{n}_q)$, where $s = m_p$, $s = h(m_p)$, or $s = h^{-1}(m_p)$.

Suppose that k is the period of h. There exists a finite cover $\mathcal{W}$, consisting of connected open sets such that if $W \in \mathcal{W}$, then the sets $W, h(W), \ldots, h^{k-1}(W)$ are pairwise disjoint. Hence, for each $W \in \mathcal{W}$, the set $\{(\bar{m},\bar{n}) \in (M \times M)/H\colon m \in \bigcup_{i=1}^{k} f^k(W)\}$ is homeomorphic to the Cartesian product of W and the Menger curve.

To prove that for any p and q in $(M \times M)/H$ there is a homeomorphism taking p onto q, it remains to show that the point $(\bar{s}, \bar{n}_q)$ can be taken onto q by a homeomorphism. In order to do that, it is enough to show that for any $W \in \mathcal{W}$, and any two points s and t in W, there is a homeomorphism $h_2: (M \times M)/H \to (M \times M)/H$ such that

$h_2(\bar{s},\bar{n}_q) = (\bar{t},\bar{n}_q)$. Let μ_2: $M \to M$ be a homeomorphism such that $\mu_2(s) = t$ and $\mu_2(m) = m$ for $m \notin W$. Define

$$h_2(\bar{m},\bar{n}) = \begin{cases} (\bar{m},\bar{n}) \text{ if } m \notin \bigcup_{i=1}^{k} f^i(W) \\ \\ (\overline{h^i \circ \mu_2 \circ h^{-i}(m)}, \bar{n}) \text{ if } m \in f^i(W). \end{cases}$$

Lemma 1. *Let* $X = X_1 \times X_2$, *where* X_i *is homeomorphic to* M *for* i = 1, 2. *Let* $U_i \subset X_i$ *be a connected open set for* i = 1, 2. *If* ϕ: $U_1 \times U_2 \to X$ *is an open embedding, then* $\phi = \phi_1 \times \phi_2$, *where either* 1) ϕ_1: $U_1 \to X_1$ *and* ϕ_2: $U_2 \to X_2$, *or* 2) ϕ_1: $U_1 \to X_2$ *and* ϕ_2: $U_2 \to X_1$.

This lemma appears in [5] as Lemma 1.

Let $p = (\bar{m}_p,\bar{n}_p)$ be a point in $(M \times M)/H$. Assume the following notation:

$$M_p = \{(\bar{m},\bar{n}) \in (M \times M)/H\colon m = m_p\},$$
$$N_p = \{(\bar{m},\bar{n}) \in (M \times M)/H\colon n = h^i(n_p),\ i = 1,\dots,k\},$$
$$O_p = M_p \cap N_p.$$

Lemma 2. *If* ϕ: $(M \times M)/H \to (M \times M)/H$ *is a homeomorphism, then either* 1) $\phi(M_p) = M_{\phi(p)}$ *and* $\phi(N_p) = N_{\phi(p)}$ *for all* $p \in (M \times M)/H$, *or* 2) $\phi(M_p) = N_{\phi(p)}$ *and* $\phi(N_p) = M_{\phi(p)}$ *for all* $p \in (M \times M)/H$.

The proof of this lemma is based on Lemma 1, and it is almost identical to the proof of Lemma 5 in [5].

Lemma 3. *If* ϕ: $(M \times M)/H \to (M \times M)/H$ *is a homeomorphism, then* $\phi(O_p) = O_p$ *or* $\phi(O_p) \cap O_p = \emptyset$.

This is an immediate consequence of Lemma 2.

Theorem 2. *If* h *is periodic, then* $(M \times M)/H$ *is not homeomorphic to* $M \times M$.

Proof. By Lemma 3, it is enough to show that for every finite set $A = \{p_1, \ldots, p_k\}$ in $M \times M$, where $k \geq 2$, there is a homeomorphism $\phi: M \times M \to M \times M$ such that $\phi(A) \cap A \neq \emptyset$ and $\phi(A) \neq A$. Suppose that $p_i = (m_i, n_i)$, where m_i and n_i are points in M.

Without loss of generality, we may assume that $m_1 \neq m_2$. Let s be a point in M such that $s \notin \{m_1, \ldots, m_k\}$. Let $\eta: M \to M$ be a homeomorphism taking m_1 onto m_1, and m_2 onto s. Set $\phi(m,n) = (\eta(m), n)$. Clearly $\phi(p_1) = p_1$ and $\phi(p_2) \notin A$.

Remark 1. Using Lemma 1, one can show that if $h: M \to M$ is a homeomorphism having a fixed or periodic point p, and having a point q with an infinite orbit, then $(M \times M)/H$ is not homogeneous.

Question 1. Does the homogeneity of $(M \times M)/H$ imply that h is periodic or h is the identity?

Question 2. Does there exist a homeomorphism $h: M \to M$ such that the orbit $\{p, h(p), h^2(p) \ldots\}$ is dense for every $p \in M$, and $(M \times M)/H$ is homogeneous?

Question 3. Does there exist a homeomorphism $M \to M$ such that the orbit $\{p, h(p), h^2(p) \ldots\}$ is dense for every $p \in M$?

3. The Twisted Products $(M \times I)/H$ and $(M \times S^1)/F$

Let $A = M_1 \times S^1$ and $B = M_0 \times S^1$ be subsets of the Cartesian product $M \times S^1$. Let $f: S^1 \to S^1$ be a homeomorphsim. Define $F: A \to B$ by $F((x,y,1),s) = ((x,y,0), f(s))$. The point p in the resulting continuum $(M \times S^1)/F$ corresponding to the point (m,s), where $m \in M - M_1$, will be denoted by $(\bar{\bar{m}},\bar{\bar{s}})$.

Theorem 3. *If* f *is orientation preserving, then* $(M \times S^1)/F$ *is homeomorphic to* $M \times S^1$.

Proof. We will exhibit a homeomorphism $\phi: \tilde{M} \times S^1 \to (M \times S^1)/F$, where $\tilde{M}$ is obtained (see Section 2) from M by identifying each point $(x,y,1)$ with the point $(x,y,0)$.

Let $\pi: M \to I$ be a continuous map such that $\pi^{-1}(0) = M_0$ and $\pi^{-1}(1) = M_1$. Define a homeomorphism $\psi: M \times S^1 \to M \times S^1$ by $\psi(m,s) = (m,[s + \pi(m)(f^{-1}(s) - s)] \bmod 2\pi)$. Next, let $\alpha: M \times S^1 \to \tilde{M} \times S^1$ and $\beta: M \times S^1 \to (M \times S^1)/F$ be continuous maps satisfying $\alpha(m,s) = (\tilde{m},s)$ and $\beta(m,s) = (\bar{\bar{m}},\bar{\bar{s}})$ for $m \in M - M_1$. Clearly, there is a homeomorphism $\phi: \tilde{M} \times S^1 \to (M \times S^1)/F$ such that the diagram

$$\begin{array}{ccc} M \times S^1 & \xrightarrow{\psi} & M \times S^1 \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ \tilde{M} \times S^1 & \xrightarrow{\phi} & (M \times S^1)/F \end{array}$$

commutes.

Let $h: M \to M$ be a homeomorphism. Define $H: M \times \{1\} \to M \times \{0\}$ by $H(m,1) = (h(m),0)$. The point p in the resulting continuum $(M \times I)/H$ corresponding to the point $(m,s) \in M \times [0,1)$ will be denoted by $(\bar{m},\bar{s})$.

Theorem 4. *For every integer* $k \geq 2$, *there exists a periodic homeomorphism* $h: M \to M$, *with period* k, *such that* $(M \times I)/H$ *is homeomorphic to* $M \times S^1$.

Proof. Denote by (r,θ,z) the cylindrical coordinates of a point in E^3.

Let F_0 be a set in E^3, homeomorphic to M, such that

(i) if $(r,\theta,z) \in F_0$, then $0 \leq \theta \leq \frac{2\pi}{k}$ and $r > 0$,

(ii) there is a homeomorphism $\mu: M \to F_0$ such that $\mu(M_0) = \{(r,\theta,z) \in F_0: \theta = 0\}$ and $\mu(M_1) = \{(r,\theta,z) \in F_0: \theta = \frac{2\pi}{k}\}$,

(iii) $\mu(x,y,0) = (r,0,z)$ iff $\mu(x,y,1) = (r,\frac{2\pi}{k},z)$.

Let $h(r,\theta,z) = (r,\theta + \frac{2\pi}{k},k)$. Set:

$F_i = h^i(F_0)$ (clearly $F_0 = F_k$),

$M' = \bigcup_{i=1}^{k} F_i$ (by Anderson's results M' is homeomorphic to M),

$G_i = \{(r,\theta,z) \in M': \frac{2\pi i}{k} < \theta < \frac{2\pi(i+1)}{k}\}$,

$A_i = \{(r,\theta,z) \in M': \theta = \frac{2\pi i}{k}\}$.

Consider h to be a homeomorphism defined on M', and assume similar notation for points in $(M' \times I)/H$ as for points in $(M \times M)/H$. Notice that the set $N \subset (M' \times I)/H$ defined by $N = \{(\bar{m},\bar{s}) \in (M' \times I)/H: m \in \bigcup_{i=1}^{k} G_i\}$ is homeomorphic to $G_0 \times S^1$. In fact, if $(\bar{m},\bar{s}) = (\overline{(r,\theta,z)}, \bar{s})$ is a point in N and $m \in G_i$, $1 \leq i \leq k$, then $\gamma(\overline{m},\overline{s}) = ((r, \theta - \frac{2\pi i}{k}, z), \frac{2\pi(s+i)}{k} \bmod 2\pi)$ defines a homeomorphism $\gamma: N \to G_0 \times S^1$.

Let $\Gamma: F_0 \times S^1 \to (M' \times I)/H$ be an extension of γ^{-1}. Notice that if $(r,\frac{2\pi}{k},z) \in \mu(M_1)$, then $\Gamma((r,\frac{2\pi}{k},z),s) = \Gamma((r,\theta,z),(s + \frac{2\pi}{k}) \bmod 2\pi)$. Therefore, N is homeomorphic to $(M' \times S^1)/F$, where f is a rotation. By Theorem 3, N is homeomorphic to $M \times S^1$.

Lemma 4. *Let* $U \subset M$ *and* $V \subset S^1$ *be connected open sets. If* $\phi: U \times V \to M \times S^1$ *is an embedding, then for every* $m \in U$ *there exists an* $n \in M$ *such that* $\phi(\{m\} \times V) \subset \{n\} \times S^1$.

The proof of this lemma is almost identical to the proof of Theorem 1 in [6] and it is omitted.

Let $p = (\overline{\overline{m}}_p,\overline{\overline{s}}_p)$ be a point in $(M \times S^1)/F$. Denote by S^1_p the set $\{(\overline{\overline{m}},\overline{\overline{s}}) \in (M \times S^1)/F\colon m = m_p\}$.

Lemma 5. *If* $\phi: (M \times S^1)/F$ *is a homeomorphism, then for every* $p \in (M \times S^1)/F$, $\phi(S^1_p) = S^1_{\phi(p)}$.

Proof. $(M \times S^1)/F = Z_1 \cup Z_2$, where
$Z_1 = \{(\overline{\overline{m}},\overline{\overline{s}}) \in (M \times S^1)/F\colon m \in \cup \{M_c\colon c \in [\frac{1}{6},\frac{5}{6}]\}\}$ and
$Z_2 = \{(\overline{\overline{m}},\overline{\overline{s}}) \in (M \times S^1)/F\colon m \in \cup \{M_c\colon c \in [0,\frac{1}{3}] \cup [\frac{2}{3},1)\}\}$.
Notice that each of the sets $\cup \{M_c\colon c \in [\frac{1}{6},\frac{5}{6}]\}$ and $\cup \{M_c\colon c \in [0,\frac{1}{3}] \cup [\frac{2}{3},1)\}$ is homeomorphic to M. There is a finite cover $\{W_1,\ldots,W_\ell\}$ of $(M \times S^1)/F$ such that

(i) for $1 \leq i \leq \ell$, the set W_i is in form $U_i \times V_i$, where U_i is a connected open set in M and V_i is a connected open set in S^1,

(ii) $\phi(W_i) \subset Z_1$ or $\phi(W_i) \subset Z_2$ for $i = 1,\ldots,\ell$.

We may assume that $W_i = U_i \times V_i$. By Lemma 4, for every $u \in U_i$, the set $\phi(\{u\} \times V_i)$ is contained in S^1_q for some $q \in (M \times S^1)/F$. Since each S^1_p is connected, we have if $p \in \{u\} \times V_i$, then $\phi(S^1_p) \subset S^1_{\phi(p)}$.

Theorem 4. *If* f *is orientation reversing, then* $(M \times S^1)/F$ *is a homogeneous continuum which is not homeomorphic to* $M \times S^1$.

Proof. Let Z_1 and Z_2 be the sets defined in the proof of Lemma 5. It is easy to see that for i = 1, 2, any point in the interior of Z_i can be taken by a homeomorphism (defined on $(M \times S^1)/F$) onto any other point in the interior of Z_i; the homeomorphism may be the identity outside Z_i. Hence $(M \times S^1)/F$ is homogeneous.

If $X \subset M$, and $\psi: M \times S^1 \to M \times S^1$ is a homeomorphism, then there exists a $Y \subset M$ such that $\psi(X \times S^1) = Y \times S^1$, see Lemma 4 or Theorem 1 in [6]. Hence for any $X \subset M$ and any homeomorphism ψ, the set $\psi(X) \cap X$ is a union of pairwise disjoint simple closed curves. However, it is easy to show that if a nonempty closed set $P \subset M \times S^1$ is not in form $X \times S^1$, then there exists a homeomorphism $\psi: M \times S^1 \to M \times S^1$ such that $P \cap \psi(P)$ contains an isolated point. The only 2-dimensional manifolds in $M \times S^1$, which are in form $X \times S^1$, are homeomorphic to $S^1 \times S^1$.

Let $L \subset M$ be an arc with the end points $p = (x_0, y_0, 1)$ and $q = (x_0, y_0, 0)$. Assume that $L \cap M_1 = p$ and $L \cap M_0 = q$. The set $Q = \{(\bar{\bar{m}}, \bar{\bar{s}}) \in (M \times S^1)/F: m \in L - \{p\}\}$ is homeomorphic to the Klein bottle. Notice that for any homeomorphism $\psi: (M \times S^1)/F \to (M \times S^1)/F$ the set $\psi(Q) \cap Q$ is a union of pairwise disjoint simple closed curves. This proves that $(M \times S^1)/F$ and $M \times S^1$ are not homeomorphic.

Question 4. Is it true that $(M \times I)/H$ is homeomorphic to $M \times S^1$ for every periodic homeomorphism h?

Question 5. Does the homogeneity of $(M \times I)/H$ imply that h is periodic or h is the identity?

Question 6. Does there exist a homeomorphism $h: M \to M$ such that the orbit of every point is dense and $(M \times I)/H$ is homogeneous?

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