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ON THE MAPPING CLASS GROUP OF THE CLOSED ORIENTABLE SURFACE OF GENUS TWO

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The surface mapping class groups are involved in different domains of mathematics, especially in 3-manifold topology. In order to study the mapping class groups, choosing a convenient family of generators appears to be very helpful. The first finite set of generators was given by Lickorish [8] as a family of Dehn twists along some simple closed curves in the surface. Later, Humphries [7] found the smallest generating set by Dehn twists, which consists of $2g + 1$ elements for the closed surface of genus g.

The Lickorish's Dehn twist generators have been widely studied, since they are so nice in topology. But they seem hard to use directly in algebraic discussions, which are important in various studies in topology. So here, we will give first a family of generators whose algebraic description is straightforward, and whose topological interpretation is still very clear and strongly related to Lickorish generators. Moreover the number of generators is minimal.

In this paper, we will discuss only the case when the genus is two, i.e., the first nontrivial case. In the first section, we define some elementary homeotopy classes, show their topological and algebraic properties, relate them to Dehn twists, and prove that the mapping class group M_2 of the closed orientable surface F_2 of genus two is generated by two elements from those elementary classes.

In the second section we give another proof that those two classes generate the group M_2, by giving an algorithm to write an arbitrary mapping class in those specific classes in a unique (not canonical) way, which certainly also solves the word problem for the group M_2.

In the third section we give a simple presentation of the mapping class group M_2 of two generators and six relators, by using the presentation given by Birman [1]. More precisely, we have

Theorem 3.2. *The mapping class group* M_2 *of the closed orientable surface of genus two admits a presentation with two generators* L *and* N, *and six relations:*

$$N^6 = 1,$$
$$(LN)^5 = 1,$$
$$(L\bar{N})^{10} = 1,$$
$$L \leftrightarrow (L\bar{N})^5,$$
$$L \leftrightarrow N^3LN^3,$$
$$L \leftrightarrow N^2LN^4.$$

Moreover, $L = D_0$ *and* $N = D_0D_1D_2D_3D_4$, *where* $D_0, D_1, \ldots, D_5$ *are Dehn twists along the simple closed curves* $\gamma_0, \gamma_1, \ldots, \gamma_5$ *pictured in Figure* 3.1, *and any five of them form a Humphries' system of Lickorish generators.*

Notationally, we do not distinguish between a homeomorphism and its homeotopy class in the paper.

1. The Elementary Mapping Classes on the Surface F_2

Let $F_2 = S^1 \times S^1 \# S^1 \times S^1$ be the closed orientable surface of genus two, which is embedded standardly in S^3 and bounds a handlebody H_2. Let $\mathcal{O}$ be a chosen point, called the *basepoint* of F_2, and let $\mathcal{B} = \{a_1, b_1, a_2, b_2\}$ be a family of simple loops based at $\mathcal{O}$, called the *basecurves* of F_2, such that a_1 and a_2 are meridian, and b_1 and b_2 are longitudes of the handlebody H_2, as shown in Figure 1.1. Clearly the basecurves $\mathcal{B}$ generate the fundamental group $\pi_1(F_2, \mathcal{O})$ of the surface F_2. It is well-known that the isotopy class of a self-homeomorphism of a closed surface is uniquely determined by the homotopy classes of the image of the basecurves in the fundamental group of the surface relative to the fixed basepoint. Therefore, it is convenient to denote a homeotopy class f by

$$f = (\mathcal{B})f = [[(a_1)f], [(b_1)f], [(a_2)f], [(b_2)f]].$$

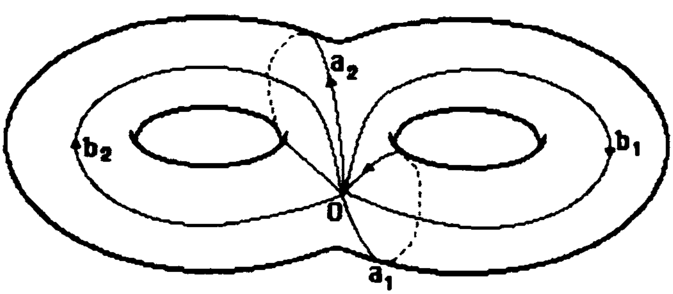


Figure 1.1 Basecurves $\mathcal{B}$ on F_2

Now we will define some *elementary operations* in this way.

0) The *identity* $I: F_2 \to F_2$ is given by

$$I = [a_1, b_1, a_2, b_2].$$

1) An orientation-reversing mapping, called *reversion* $R: F_2 \to F_2$, is given by

$$R = [b_1, a_1, b_2, a_2].$$

2) The interchanging handles mapping, called *transport* $T: F_2 \to F_2$, is given by

$$T = [a_2, b_2, a_1, b_1].$$

Proposition 1.1

(a) $R^2 = I$,

(b) $T^2 = I$,

(c) $TR = RT$.

It can be easily proved by a direct verification.

3) Homeotopy classes called *linear cuttings*, are obtained in the following way: on the presentation polygon of the surface F_2 relative to the basis $\mathcal{B}$, we cut some triangle formed by two successive edges x and y and glue it back along one of the edges, e.g. along x, and get a new polygon. If we have a homeomorphism from the old polygon to the new one that maps the basepoint and the basecurves other than x and y invariantly, and defines a self-homeomorphism of F_2, then, it is unique up to isotopy, and we denote it by $L(x,y;x)$ (Figure 1.2).

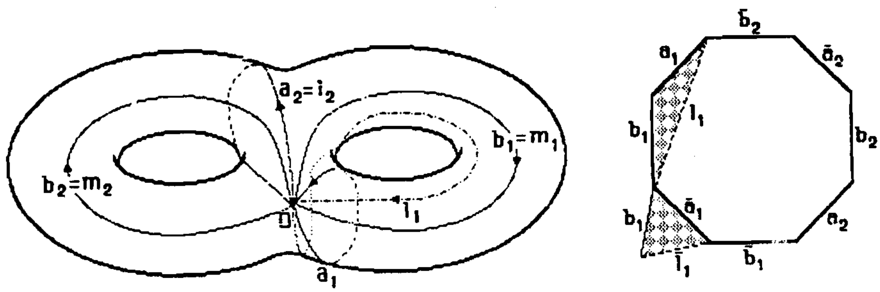


Figure 1.2 Linear cutting L

More explicitly we have

$$L(a_i,b_i;a_i),\ L(a_i,b_i;b_i),\ L(b_i,\bar{a}_i;b_i),$$
$$L(b_i,\bar{a}_i;\bar{a}_i),\ L(\bar{a}_i,\bar{b}_i;\bar{a}_i),\ L(\bar{a}_i,\bar{b}_i;\bar{b}_i),$$

where $i = 1,2$.

Among these linear cuttings, we will denote

$$L = L(a_1,b_1;a_1) \text{ and } M = L(b_1,\bar{a}_1;b_1),$$

and call them *longitude cutting* and *meridian cutting* respectively, since they are the Dehn twists along the longitude and meridian circles of the first handle of H_2. Their expressions in $\pi_1(F_2,0)$ are given by

$$L = [a_1b_1,b_1,a_2,b_2], \text{ and } M = [a_1,b_1\bar{a}_1,a_2,b_2].$$

Proposition 1.2

(a) $L(x_2,y_2;y_2) = T \cdot L(x_1,y_1;y_1) \cdot T,$

for any $(x,y) \in \{(a,b),(b,a),(b,\bar{a}),(\bar{a},b),(\bar{a},\bar{b}),(\bar{b},\bar{a})\}$.

(b) $L(b_1,\bar{a}_1;\bar{a}_1) = RMR = \bar{L},$

$L(\bar{a}_1,\bar{b}_1;\bar{b}_1) = RLR = \bar{M},$

$L(\bar{a}_1,\bar{b}_1;\bar{a}_1) = R \cdot L(a_1,b_1;b_1) \cdot R$

$= (L(a_1,b_1;b_1))^{-1},$

(c) $L(a_1,b_1;b_1) = R\bar{L}RL$.

These relations may be verified directly. As an immediate consequence, we have

Corollary 1.3 *Every linear cutting is a composition of the homeotopy classes* L,T *and* R.

4) A homeotopy class called *normal cutting*,

$$N = N(a,\bar{b};\bar{b}) = N(a_1,\bar{b}_2;\bar{b}_2/a_2,\bar{b}_1;\bar{b}_1),$$

is defined algebraically by

$$N = [\bar{a}_2b_1,\bar{a}_1,\bar{a}_1b_2,\bar{a}_2],$$

and topologically by cutting two triangles on the presentation polygon, one between the edges a_1 and $\bar{b}_2$ and the other between a_2 and $\bar{b}_1$, and sewing them along curves b_1 and b_2 respectively, (Figure 1.3).

By a similar discussion as for linear cuttings, we may have another normal cutting $N(a,\bar{b};a)$. But this is nothing new and is just the inverse of N by the next proposition.

Proposition 1.4

(a) $N^3 = T$, $N^6 = I$,

(b) $TN = NT$,

(c) $(RN)^2 = I$,

(d) $N(a,\bar{b};a) = RNR = \bar{N}$,

(e) $M = R\bar{L}R = \bar{N}LN$.

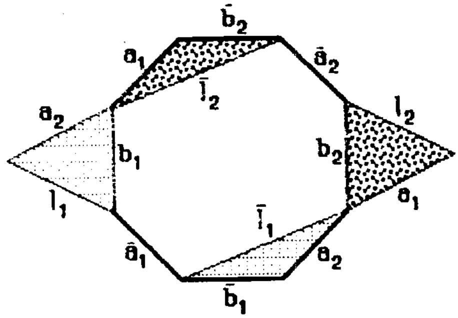


Figure 1.3 Normal cutting N

The proof is again obvious. For example, writing

$$N = [\bar{a}_2 b_1, \bar{a}_1, \bar{a}_1 b_2 \bar{a}_2],$$

we have

$$\begin{aligned} N^2 &= [(\bar{b}_2 a_1)\bar{a}_1, \bar{b}_1 a_2, (\bar{b}_1 a_2)\bar{a}_2, \bar{b}_2 a_1] \\ &= [\bar{b}_2, \bar{b}_1 a_2, \bar{b}_1, \bar{b}_2 a_1], \end{aligned}$$

and

$$\begin{aligned} N^3 = N^2 \cdot N &= [a_2, a_1(\bar{a}_1 b_2), a_1, a_2(\bar{a}_2 b_1)] \\ &= [a_2, b_2, a_1, b_1] = T. \end{aligned}$$

Therefore, $N^3 = T$, and easily $N^6 = T^2 = I$.

Remark. From the formulas 1.4.(d) and (e), we have that the reversion R commutes with the subgroup generated by the classes L and N in the homeotopy class group $\tilde{M}_2$.

5) The last type of homeotopy classes are called *parallel cuttings*, denoted by P(x), $x \in \{a,b,\bar{a},\bar{b}\}$, defined algebraically by

$$P(a) = [\bar{b}_1, b_1a_1\bar{b}_1, \bar{b}_2, b_2a_2\bar{b}_2],$$
$$P(\bar{a}) = [\bar{b}_1, \bar{b}_2a_1b_1, \bar{b}_2, \bar{b}_1a_2b_2],$$
$$P(b) = [a_1b_1\bar{a}_2, \bar{a}_1, a_2b_2\bar{a}_1, \bar{a}_2],$$
$$P(\bar{b}) = [\bar{a}_1b_1a_1, \bar{a}_1, \bar{a}_2b_2a_2, \bar{a}_2],$$

and obtained by cutting the presentation polygon in three quadrilaterals (Figure 1.4), such that the center one contains the edges x_1 and x_2, and gluing them along the curves given by x_1 and x_2.

Actually, for the convenience of our future discussion, we will call *parallel cutting* the homeotopy class

$$P = LML = MLM = [a_1b_1\bar{a}_1, \bar{a}_1, a_2, b_2].$$

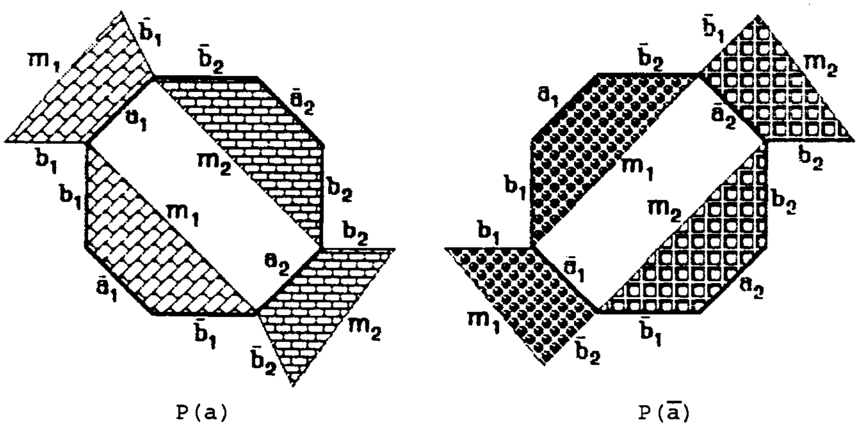


Figure 1.4 Parallel cuttings

Proposition 1.5

(a) $P(a) = \bar{P}T\bar{P}T$,

(b) $P(\bar{a}) = \bar{N}LTLT$,

(c) $P(b) = R \cdot P(\bar{a}) \cdot R = N\bar{M}T\bar{M}T$,

(d) $P(\bar{b}) = R \cdot P(a) \cdot R = PTPT$,

(e) $RPR = \bar{P}$, since $RLR = \bar{M}$ and $RMR = \bar{L}$.

Proof. They can be proved by a direct algebraic verification. However, there is topological interpretation hidden inside. Here we show (a) as an example to illustrate the topological aspect.

As shown in Figure 1.5, first we do the operation

$$O_1 = L(\bar{a}_1,\bar{b}_1;\bar{a}_1)L(\bar{a}_2,\bar{b}_2;\bar{a}_2) = \bar{L}RLR \cdot T\bar{L}RLRT,$$

and then do the second one

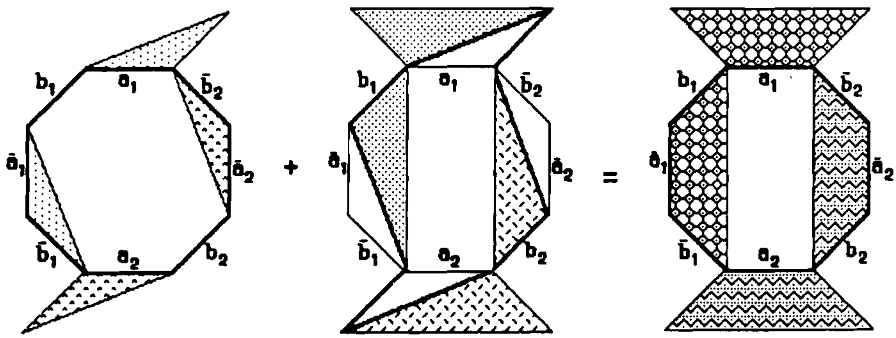
$$O_2 = L(\bar{a}_1,\bar{b}_1;\bar{b}_1)L(\bar{a}_2,\bar{b}_2;\bar{b}_2) = RLR \cdot TRLRT.$$

Then, as shown in the figure, we have

$$\begin{aligned} P(a) = O_2 \cdot O_1 &= L(\bar{a}_1,\bar{b}_1;\bar{a}_1) \cdot L(\bar{a}_2,\bar{b}_2;\bar{a}_2) \cdot L(\bar{a}_1,\bar{b}_1;\bar{b}_1) \cdot \\ &\quad L(\bar{a}_2,\bar{b}_2;\bar{b}_2) \\ &= L(\bar{a}_1,\bar{b}_1;\bar{a}_1) \cdot L(\bar{a}_1,\bar{b}_1;\bar{b}_1) \cdot L(\bar{a}_2,\bar{b}_2;\bar{a}_2) \cdot \\ &\quad L(\bar{a}_2,\bar{b}_2;\bar{b}_2) \\ &= RLR \cdot \bar{L}RLR \cdot TRLRT \cdot T\bar{L}RLRT = (RLR\bar{L}RLRT)^2 \\ &= (RLMLRT)^2 = RPRTRPRT = \bar{P}T\bar{P}T, \end{aligned}$$

since $RPR = \bar{P}$ and $RTR = T$ by the formulas (1.1.c), (1.5.e).

All above homeotopy classes are called *elementary operations*, and among them clearly only the reversion R is orientation-reversing, and all others are orientation-preserving and generated by only the operations L and N.



O_1 O_2 P(a)

Figure 1.5 $P(a) = O_1 \cdot O_2$

In the next part in this section, we relate these elementary operations to Lickorish generators of Dehn twists.

Considering the surface given in Figure 1.6, the Lickorish generators are exactly the five Dehn twists along the simple curves a_1, a_2, b_1, b_2, and c, we denote them by A_1, A_2, B_1, B_2, and C respectively. Remembering our elementary operations, and we have

$$A_1 = M,$$
$$A_2 = TMT,$$
$$B_1 = L,$$
$$B_2 = TLT$$

For C, write the curve c in the basecurves of $\mathcal{B}$, thus

$$c = a_2\bar{b}_2\bar{a}_2b_1$$

Therefore,

$$C = [a_1c, \bar{c}b_1c, \bar{c}a_2, b_2]$$
$$= [a_1a_2\bar{b}_2\bar{a}_2b_1, \bar{b}_1a_2b_2\bar{a}_2b_1a_2\bar{b}_2\bar{a}_2b_1, \bar{b}_1a_2b_2, b_2],$$

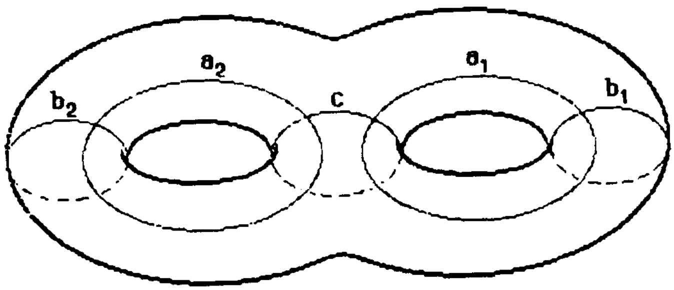


Figure 1.6 Twist curves of Lickorish generators

and then it is not difficult to show

$$C = \bar{P}NL\bar{N}P = \bar{N}\bar{L}N\bar{L}NL\bar{N}L\bar{N}LN.$$

By Lickorish's result we have that,

Theorem 1.6 *The surface mapping class group* M_2 *is generated by two elements* L *and* N, *and the homeotopy class group* $\tilde{M}_2$ *is generated by three elements* R, L *and* N.

Finally, it is reasonable to write our generators in Lickorish's.

Proposition 1.7

(1) $L = A_1$,

(2) $N = B_1A_1(A_1B_1C\bar{B}_1\bar{A}_1)B_2A_2$.

Proof. Indeed, since $(LN)^5 = I$ and $N^6 = I$, we have

$$N = \bar{N}(LN)^5N^2$$
$$= \bar{N}LN \cdot L \cdot NL\bar{N} \cdot N^2L\bar{N}^2 \cdot N^3L\bar{N}^3$$
$$= B_1A_1(A_1B_1C\bar{B}_1\bar{A}_1)B_2A_2.$$

Remark. Later in Section 3 we will give a different correspondence between the Lickorish generators and ours, which is simpler and nicer, and which is conjugate to that given in Proposition 1.7.

2. The Algorithm for Writing Homeotopy Classes in the Generators

In this section we give an algorithm to write an aribitrary homeotopy class in the generators L, N and R, which gives a direct proof of Theorem 1.6.

Given a homeotopy class f, since exactly one of f and Rf is orientation-preserving, we may suppose that f is a mapping class. In fact, the mapping class group M_2 is a normal subgroup of index 2 of the homeotopy group $\tilde{M}_2$. Moreover,

$$RL = \bar{M}R,\ RM = \bar{L}R \text{ and } RN = \bar{N}R.$$

As in the last section, let

$$f = [l_1,m_1,l_2,m_2]$$

be written in the basis

$$\mathcal{B} = \{a_1,b_1,a_2,b_2\}.$$

i.e. $l_1 = (a_1)f$, $m_1 = (b_1)f$, $l_2 = (a_2)f$ and $m_2 = (b_2)f$ And suppose all curves intersect transversally. We will denote by $\#(\gamma_1 \cap \gamma_2)$ the geometric number of the intersection points other than the basepoint O of the curves γ_1 and γ_2, and

$$\#(\gamma_1 \cap B) = \#(\gamma_1 \cap a_1) + \#(\gamma_1 \cap b_1) + \#(\gamma_1 \cap a_2) + \#(\gamma_1 \cap b_2).$$

Given f an orientation-preserving self-homeomorphism of the surface F_2, we will denote the lexicographically ordered multi-index

$$\#(f) = (\#(l_1 \cap B), \#(m_1 \cap B), \#(l_2 \cap B), \#(m_2 \cap B)).$$

Our algorithm is based on the reduction of this multi-index.

Now we start our algorithm.

Step 1. *Given* f, *if* $m = \#(l_1 \cap B) \neq 0$, *there is a self-homeomorphism* h *which is a composition of elementary operations, such that* $\#((l_1)h^{-1} \cap B) < m$.

Suppose that $l_1 \cap B = \{O, P_1, \ldots, P_m\}$, and denote by s_i, $0 \leq i \leq n$, the arc between the points P_i and P_{i+1} of the curve l_1, where $P_0 = P_{m+1} = O$ by convention. Regarding s_i as an arc in the presentation polygon with ends in the boundary of the polygon, we will say s_i of the type $[x: y]$, and write $s_i \in [x: y]$, if one of its two end points is in the edge x and the other is in y, where $x, y \in \{a_1, b_1, \bar{a}_1, \bar{b}_1, a_2, b_2, \bar{a}_2, \bar{b}_2\}$. (Figure 2.1). We will denote by $\#(l_1 \cap [x: y])$ the number of arcs of the type $[x: y]$.

Now we consider cases.

Case 0. *There is an arc* s_i *of the type* $[x: x]$, *for some edge* x.

An isotopy of f decreases the number m. (Figure 2.2).

Case I. s_0 *or* s_m *is of one of the types* $[a_k: b_k]$, $[b_k: \bar{a}_k]$ *and* $[\bar{a}_k: \bar{b}_k]$, $k = 1$ *or* 2.

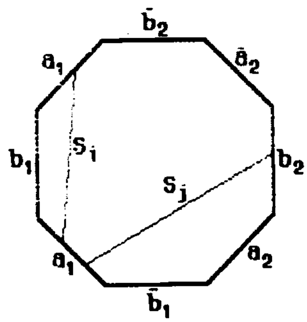


Figure 2.1 $s_i \in [a_1: \bar{a}_1]$ and $s_j \in [\bar{a}_1: b_2]$

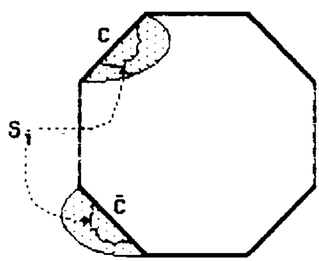


Figure 2.2 Case 0

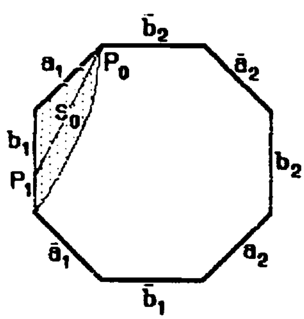


Figure 2.3 Case I

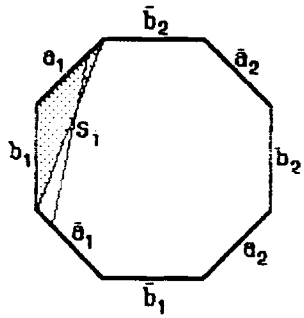


Figure 2.4 Case II

The homeomorphism h will be chosen to be a suitable linear cutting. For example, suppose $s_0 \in [a_1: b_1]$ and $\#(l_1 \cap b_1) > 0$. Apply $h = L(a_1,b_1;b_1)$, (Figure 2.3), and obviously after the cutting and sewing it is easy to see $\#((l_1)h^{-1} \cap B) = m - \#(l_1 \cap [a_1: b_1]) < m$.

Case II. *There is some* s_i *of the type* $[a_k: \bar{a}_k]$ *or* $[b_k: \bar{b}_k]$, $k = 1$ *or* 2.

For example, $s_i \in [a_1: \bar{a}_1]$, (Figure 2.4). Evidently,

$$\#(l_1 \cap b_1) = \#(l_1 \cap [a_1: b_1]) + \#(l_1 \cap [\bar{a}_1: b_1]).$$

We have always $\#(l_1 \cap [a_1: b_1]) \neq \#(l_1 \cap [\bar{a}_1: b_1])$, since the endpoints of any arc of l_1 are disjoint. If $\#(l_1 \cap [a_1: b_1]) > \#(l_1 \cap [\bar{a}_1: b_1])$, let h be the linear cutting $L(a_1,b_1;a_1)$.

Clearly

$$\begin{aligned}\#((l_1)h^{-1} \cap B) &= m - \#(l_1 \cap a_1) + \#(l_1 \cap h(a_1)) \\ &= m - \#(l_1 \cap a_1) + [\#(l_1 \cap a_1) \\ &\quad + \#(l_1 \cap b_1) - 2\#(l_1 \cap [a_1: b_1])\end{aligned}$$

$$= m + \#(l_1 \cap [\bar{a}_1 : b_1] - \#(l_1 \cap [a_1 : b_1]) < m.$$

The other situations are exactly similar.

Case III. *Cases* 0, I *or* II *do not occur, and there is some* $s_i \in [a_k : \bar{b}_k]$, $k = 1$ *or* 2.

Thus,

$$\#(l_1 \cap b_1) = \#(l_1 \cap [a_1 : b_1]) + \#(l_1 \cap [b_1 : \bar{a}_1]),$$

and

$$\#(l_1 \cap a_1) = \#(l_1 \cap [b_1 : \bar{a}_1]) + \#(l_1 \cap [\bar{a}_1 : \bar{b}_1]).$$

III-i) When $\#(l_1 \cap [a_1 : b_1]) > \#(l_1 \cap [b_1 : \bar{a}_1])$.

We choose $h = L(a_1, b_1; b_1)$ and the number m is reduced. And analogously, if $\#(l_1 \cap [a_1 : b_1]) < \#(l_1 \cap [b_1 : \bar{a}_1])$ or $\#(l_1 \cap [b_1 : \bar{a}_1]) \neq \#(l_1 \cap [\bar{a}_1 : \bar{b}_1])$, we also may choose suitable linear cuttings (Figure 2.5).

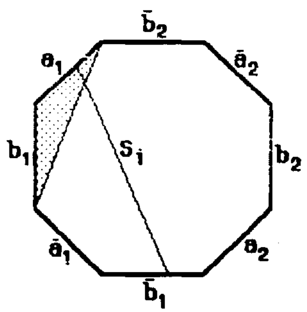


Figure 2.5 Case III-(i)

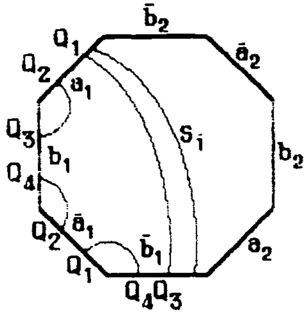


Figure 2.6 Case III-(ii)

III-ii) When $\#(l_1 \cap [a_1:b_1]) = \#(l_1 \cap [b_1: \bar{a}_1]) = \#(l_1 \cap [\bar{a}_1: \bar{b}_1]) = \lambda$.

Then $\#(l_1 \cap a_1) = \#(l_1 \cap B_1) = 2\lambda$. Denote by Q_1, Q_2, Q_3, and Q_4 the p-th and (p + 1)-th points of l_1 on the basecurves a_1 and b_1 as shown in Figure 2.6. This produces a closed curve which is a proper subset of the simple curve l_1, showing the impossibility of this case, since it does not contain the basepoint.

Case IV. *There is some* s_i *of the type* $[a_1: a_2]$, $[b_1: b_2]$, $[\bar{a}_1: \bar{a}_2]$ *or* $[\bar{b}_1: \bar{b}_2]$. We let h = P(a), P(b), $P(\bar{a})$ or $P(\bar{b})$, respectively, and obviously

$$\#((l_1)h^{-1} \cap B) = m - \#(l_1 \cap [x_1: x_2]) < m,$$

where x = a, b, $\bar{a}$, or $\bar{b}$. (Figure 2.7).

The remaining cases will be discussed in another way. Consider the starting point P_0 of s_0 in the presentation polygon (Figure 2.8), since the standard operations T and R leave the intersection numbers unchanged, it is sufficient to consider the cases P_0 = A, B, and C.

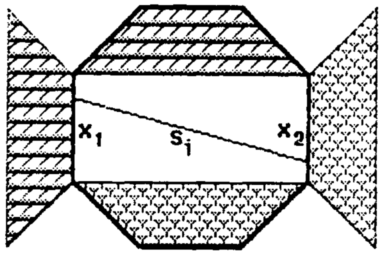


Figure 2.7 Case IV

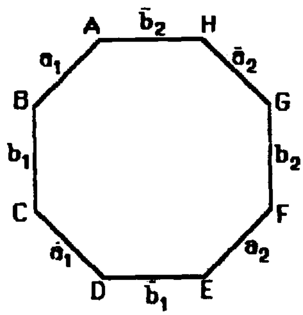


Figure 2.8 Presentation polygon

Case V. $P_0 = A$.

V-i) When $P_1 \in a_1$, $\bar{b}_2$, b_1 or $\bar{a}_2$, s_0 is of the types in Case I.

V-ii) When $P_1 \in \bar{a}_1$, b_2, $\bar{b}_1$, or a_2, s_0 is of the types in Cases II or III.

Case VI. $P_0 = B$.

VI-i) When $P_1 \in a_1$, b_1, $\bar{a}_1$, or $\bar{b}_1$, s_0 is of the types from Cases O-III.

VI-ii) When $P_1 \in a_2$ or b_2, s_0 is of the types in Case IV.

VI-iii) When $P_1 \in \bar{a}_2$,

(a) If $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$, there is no arc crossing the parallel band between b_1 and b_2. Do $P(b)$, creating a situation as in Case I, without changing the intersection number. (Figure 2.9).

(b) If $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$, (Figure 2.10). Then, of course $\#(l_1 \cap [a_2: b_1]) = 0$, i.e. there is no arc crossing the band between $\bar{a}_1$ and $\bar{a}_2$, similarly we have Case I after doing $P(\bar{a})$.

VI-iv) When $P_1 \in \bar{b}_2$,

(a) If $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) = 0$. We do $P(b)$ as in Case VI-(iii-a), and the situation becomes Case V, (Figure 2.11).

(b) If $\#(l_1 \cap [\bar{a}_2: \bar{b}_1]) \neq 0$ and $\#(l_1 \cap [\bar{a}_2: b_1]) = 0$. Analogously, do $P(\bar{b})$, obtaining Case I, (Figure 2.12).

(c) If not (a), not (b), and $\#(l_1 \cap [a_2: b_2]) >$ $\#(l_1 \cap [a_2: \bar{b}_1])$, let $h = L(a_2,b_2;b_2)$, (Figure 2.13), and obviously

$$\begin{aligned}\#((l_1)h^{-1} \cap B) &= m - \#(l_1 \cap b_2) + \#(l_1 \cap h(b_2)) \\ &= m - \#(l_1 \cap [a_2: b_2]) + \\ &\quad \#(l_1 \cap [a_2: \bar{b}_1]) < m.\end{aligned}$$

(d) If not (a), not (b), and $\#(l_1 \cap [a_2: b_2]) \leq$ $\#(l_1 \cap [a_2: \bar{b}_1])$. Then we have always $\#(l_1 \cap [a_1: b_1]) = 0$ $< 1 \leq \#(l_1 \cap [a_1: \bar{b}_2])$, let $h = N(a,\bar{b};\bar{b})$ be a normal cutting, (Figure 2.14). Clearly

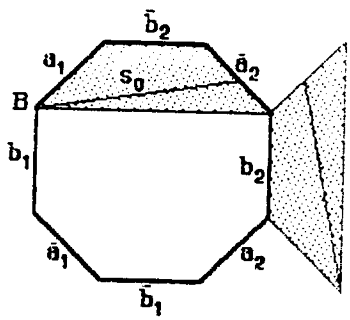


Figure 2.9 Case VI-(iii)-(a)

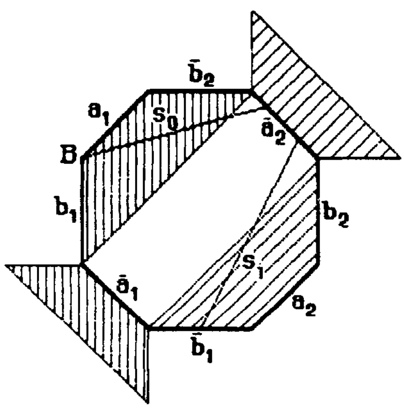


Figure 2.10 Case VI-(iii)-(b)

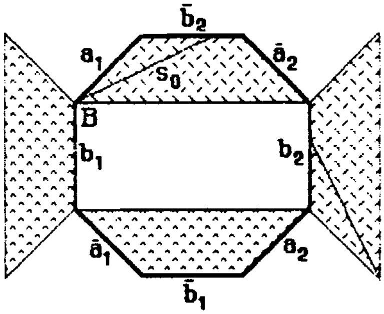


Figure 2.11 Case VI-(iv)-(a)

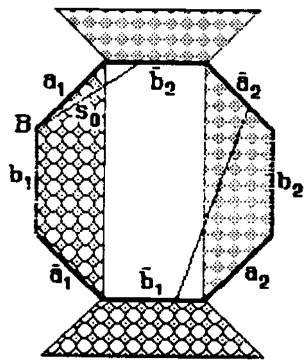


Figure 2.12 Case VI-(iv)-(b)

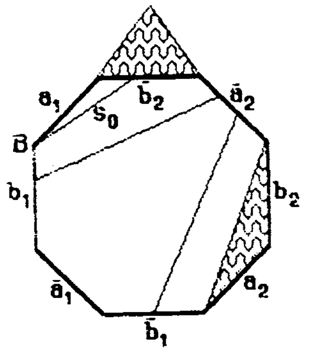


Figure 2.13 Case VI-(iv)-(c)

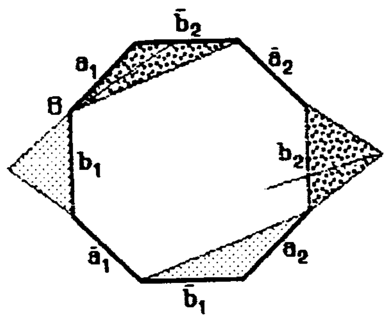


Figure 2.14 Case VI-(iv)-(d)

$$\begin{aligned}\#((l_1)h^{-1} \cap B) &= m - \#(l_1 \cap b_1) - \#(l_1 \cap b_2) + \\ &\qquad \#(l_1 \cap (b_1)h) + \#(l_1 \cap (b_2)h) \\ &= m - \#(l_1 \cap [a_1\colon \bar{b}_2]) + \\ &\qquad \#(l_1 \cap [a_2\colon b_2]) - \\ &\qquad \#(l_1 \cap [a_2\colon \bar{b}_1]) < m.\end{aligned}$$

Case VII. $P = C$.

VII-i) When $P_1 \in a_1, b_1, \bar{a}_1$ or $\bar{b}_1$, s_0 is in Cases 0-II.

VII-ii) When $P_1 \in \bar{a}_2$ or b_2, s_0 is in Case IV.

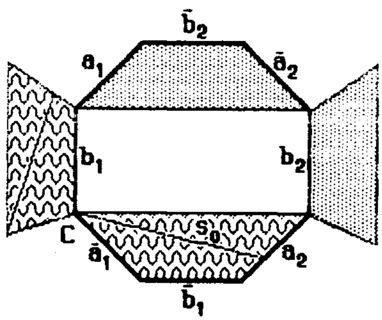


Figure 2.15 Case VII-(iii)

VII-iii) When $P_1 \in a_2$, obviously there is no arc crossing the parallel band between b_1 and b_2. Do the parallel cutting P(b), which does not change the intersection number m, and produces the situation as in Case VI-(iv), (Figure 2.15).

VII-iv) When $P_1 \in \bar{b}_2$, apply the reversion R to obtain Case VII-(iii).

Step 2. *Given* f *with* $m = \#(l_1 \cap B) = 0$, *then there exists a self-homeomorphism* h *which is a composition of elementary operations, such that* $(l_1)h^{-1} = a_1$.

By the first step, we may suppose $\#(l_1 \cap B) = 0$, where $l_1 = (a_1)f$. Consider the curve l_1 in the presentation polygon of F_2. We will denote by $\widetilde{XY}$ the arc with ends at the vertices X and Y of the presentation polygon, for $X, Y \in \{A,B,C,D,E,F,G,H\}$.

i) If $l_1 = \widetilde{AB}$, it is done already.

ii) If $l_1 = \widetilde{AH}$, let $h = T \cdot P(a)$.

iii) If $l_1 = \widetilde{AC}$ or $\widetilde{AG}$, we may choose h to be a linear cutting.

iv) If $l_1 = \widetilde{AD}$ or $\widetilde{AG}$, we may choose h to be a parallel cutting.

v) Clearly $l_1 \neq \widetilde{AE}$, since l_1 in not null-homologous.

vi) If $l_1 = \widetilde{EX}$, where X = B,C,D,F,G and H, we do first a transport T, and the case becomes one of the first four cases.

vii) If $l_1 = \widetilde{BC}$, let $h = P(a)$.

viii) If $l_1 = \widetilde{BD}$, let $h = L(b_1,\bar{a}_1;\bar{a}_1)$.

ix) If $l_1 = \widetilde{BF}$, let $h = P(b) \cdot L(b_1,\bar{a}_1;\bar{a}_1)$.

x) If $l_1 = \widetilde{BG}$, let $h = P(b)$.

xi) If $l_1 = \widetilde{BH}$, let $h = N$.

xii) If $l_1 = \widetilde{DX}$, do first a reversion R, producing the case of $l_1 = \widetilde{BY}$. Denote by h' the map given by that case, let $h = P(a) \cdot Rh'R$.

xiii) If $l_1 = \widetilde{FX}$ or $\widetilde{HX}$, apply the transport T, producing the cases of $l_1 = \widetilde{BY}$ or $\widetilde{DY}$.

xiv) If $l_1 = \widetilde{CG}$, let $h = T \cdot P(b) \cdot L(a_1,b_1;a_1)$.

This completes Step 2.

From now on we may suppose that $l_1 = (a_1)f = a_1$. We will simplify the curve $m_1 = (b_1)f$ by using the elementary operations, and at the same time leave the curve l_1 unchanged.

Step 3. *Given* f *with* $(a_1)f = a_1$, *then there is a self-homeomorphism* h *which is a composition of elementary operations, such that* $(a_1)h = a_1$ *and* $\#((m_1)h^{-1} \cap \mathcal{B}) = 0$.

Since l_1, m_1, l_2, m_2 forms a family of basecurves, and $l_1 = a_1$, we have

$$\#(m_1 \cap a_1) = 0.$$

So, we have fewer cases. As in the last step, we denote now $m = \#(m_1 \cap \mathcal{B})$, and successively $m_1 \cap \mathcal{B} = \{O, P_1, P_2, \ldots, P_m\}$, and s_i the arc on m_1 between P_i and P_{i+1}, $i = 0, \ldots, m$, where $P_0 = P_{m+1} = O$ by convention.

Cases O-IV. *There is some* s_i *belonging to one of the following types:* $[x_k: x_k]$, $[a_2: b_2]$, $[b_2: \bar{a}_2]$, $[\bar{a}_2: \bar{b}_2]$, $[a_2: \bar{a}_2]$, $[b_k: \bar{b}_k]$, $[a_2: \bar{b}_2]$, $[b_k: \bar{b}_k]$, *where* $x = a, b, \bar{a}$ *or* $\bar{b}$ *and* $k = 1, 2$.

We can do the same operation as in Step 1, which leaves $l_1 = a_1$ unchanged.

The other cases will be studied by considering the arc s_0, as we did in Step 1.

Case V. $P_0 = A$ *or* E.

Do the same as in Case V of Step 1.

Case VI. $P_0 = F$, *and not Cases* O-IV. We have only the following two possible situations.

i) $P_1 \in b_1$. Then let $h = P(b)$.

ii) $P_1 \in \bar{b}_1$s. Then let $h = N(a,\bar{b},\bar{b})$.

Case VII. $P_0 = G$, *and not Cases* O-IV, and

i) $P_1 \in b_1$. Let $h = P(b)$.

ii) $P_1 \in \bar{b}_1$, and

(a) $\#(m_1 \cap [a_2: b_2]) < \#(m_1 \cap [a_2: \bar{b}_1])$. Let $h = N(a,\bar{b};\bar{b})$.

(b) $\#(m_1 \cap [a_2: b_2]) > \#(m_1 \cap [a_2: \bar{b}_1])$. Let $h = L(a_2,b_2;b_2)$.

(c) $\#(m_1 \cap [a_2: b_2]) = \#(m_1 \cap [a_2: \bar{b}_1])$. Apply $L(a_2,b_2;b_2)$ to obtain Case IV.

Case VIII. $P_0 = H$, *and not Cases* O-IV, and

i) $P_1 \in b_1$. Apply $L(a_1,b_1;b_1)$ to obtain Case IV.

ii) $P_1 \in \bar{b}_1$. Proceed as in Case IV.

Case IX. $P_0 = B$, *and not Cases* O-IV, and

i) $P_1 \in b_2$. Let $h = P(b)$.

ii) $P_1 \in a_2$. Since none of Cases O-IV occurs, then

$$\#(m_1 \cap b_2) = \#(m_1 \cap [b_2: \bar{a}_2]) + \#(m_1 \cap [b_2: a_2]) = \\ = \#(m_1 \cap [\bar{b}_2: \bar{a}_2]),$$

and

$$\#(m_1 \cap a_2) = \#(m_1 \cap [b_2: \bar{a}_2]) + \#(m_1 \cap [\bar{b}_2: a_2]) \neq 0$$

for $P_1 \in a_2$. Therefore, we can not have

$$\#(m_1 \cap [b_2\colon \bar{a}_2]) = \#(m_1 \cap [b_2\colon a_2]) = \#(m_1 \cap [\bar{b}_2\colon \bar{a}_2]).$$

If either $\#(m_1 \cap [b_2\colon ba_2]) \neq \#(m_1 \cap [b_2\colon a_2])$ or $\#(m_1 \cap [b_2\colon ba_2]) \neq \#(m_1 \cap [\bar{b}_2\colon \bar{a}_2])$, a suitable linear cutting reduces the intersection number m.

iii) $P_1 \in \bar{b}_2$. The discussion is similar to what we did in Case VI-(iv) of Step 1, in which a_1 was left unchanged.

iv) $P_1 \in \bar{a}_2$. Evidently $\#(m_1 \cap b_2) = \#(m_1 \cap [\bar{a}_2\colon \bar{b}_2])$, doing $L(\bar{a}_2,\bar{b}_2,\bar{b}_2)$ it becomes the above situation (iii).

Case X. $P_0 = D$, *and not Cases* O-IV, and

i) $P_1 \in a_2$, and

(a) $\#(m_1 \cap [a_2\colon b_2]) \neq \#(m_1 \cap [\bar{a}_2\colon b_2])$. Do a linear cutting.

(b) $\#(m_1 \cap [a_2\colon b_2]) = \#(m_1 \cap [\bar{a}_2\colon b_2]) = \mu = \frac{1}{2}\#(m_1 \cap b_2)$. We do first the operation $h = LN^2LN$ as pictured in Figure 2.16. Clearly

$$\#(m_1 \cap h(B)) = \#(m_1 \cap B) - \#(m_1 \cap b_1) - \#(m_1 \cap a_2) + \#(m_1 \cap h(a_1)) + \#(m_1 \cap h(b_2))$$
$$= m - \mu_a - \mu_b + \mu_c + \mu_d, \qquad (*)$$

since $h(a_2) = \bar{a}_1$ and $h(b_1) = b_2$, where we write $\mu_a = \#(m_1 \cap a_2)$, $\mu_b = \#(m_1 \cap b_1)$, $\mu_c = \#(m_1 \cap h(a_1))$ and $\mu_d = \#(m_1 \cap h(b_2))$. If we suppose $P_{m+1} = D$ or C, (otherwise, we may consider first s_m instead of s_0,) and denote $\mu_1 = \#(m_1 \cap [b_1\colon a_2])$, $\mu_2 = \#(m_1 \cap [b_1\colon \bar{a}_2])$ and $\mu_3 = \#(m_1 \cap [b_1\colon \bar{b}_2])$, then

$$\mu_a = \#(m_1 \cap [\bar{b}_1 : a_2]) + \mu_1 + \mu$$
$$= \varepsilon + \mu_b + \mu_1 + \mu,$$
$$\mu_b = -\varepsilon' + \mu_1 + \mu_2 + \mu_3,$$
$$\mu_c = \mu + \mu_2 + \mu_3,$$

and $$\mu_d = |\mu_1 - \mu_2| + \mu_3,$$

where $\varepsilon = \#(\{P_1, P_m\} \cap a_2) \in \{1,2\}$ and $\varepsilon' = \#(P_{m+1} \cap C) \in \{0,1\}$. Therefore, the formula (*) becomes

$$\#(m_1 \cap h(\beta)) = m - (\varepsilon - \varepsilon' + 3\mu_1 + \mu_2 - |\mu_1 - \mu_2|), \qquad (**)$$

which is strictly less than m except when $\varepsilon = \varepsilon' = 1$ and $\mu_1 = 0$.

(c) When $\varepsilon = \varepsilon' = 1$ and $\mu_1 = 0$, we have $P_{m+1} = C$ and $P_m \notin a_2$. And

(1) if $P_m \in b_2$ or $\bar{a}_2$, let h be the parallel cutting P(b) or P($\bar{a}$) respectively.

(2) if $P_m \in \bar{b}_2$, then $\mu_1 = \mu_2 = 0$, (Figure 2.17), and a discussion similar to that in Case III-(ii) of the first step leads to a contradiction.

ii) $P_1 \in b_2$, and

(a) $\#(m_1 \cap [a_2 : \bar{b}_1]) > \#(m_1 \cap [a_2 : b_2])$. Let $h = N(a)$.

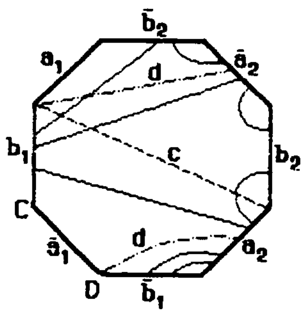


Figure 2.16 Case X-(i)-(b)

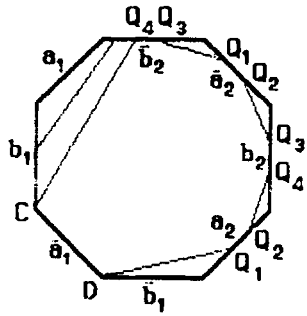


Figure 2.17 Case X-(i)-(c)-(2)

(b) $\#(m_1 \cap [a_2: \bar{b}_1]) \leq \#(m_1 \cap [a_2: b_2])$. First let $h = L(a_2,b_2;b_2)$, which either reduces the intersection number or produces case (i) above.

iii) $P_1 \in \bar{a}_2$, and

(a) $\#(m_1 \cap [a_2: \bar{b}_1]) > \#(m_1 \cap [a_2: b_2])$. Let $h = N(a)$.

(b) $\#(m_1 \cap [a_2: \bar{b}_1]) \leq \#(m_1 \cap [a_2: b_2])$. Let $h = L(a_2,b_2;b_2)$.

Case XI. $P_0 = C$. Applying $L(\bar{a}_1,\bar{b}_1;\bar{b}_1)$ yields Case X.

Step 4. *Given* f *with* $(a_1)f = a_1$ *and* $m = \#(m_1 \cap \beta) = 0$, *then, there is a self-homeomorphism* h *which is a composition of elementary operations, such that* $(a_1)h = a_1$ *and* $(m_1)h^{-1} = b_1$.

The proof of this step is quite different from the above. It is more topological.

First, we consider two based simple closed curves l and m, and we say they are cobasic, if there are other curves l' and m' such that the set l, m, l', m' forms a system of basecurves on the surface F_2.

Proposition 2.3. *The curves* l *and* m *are cobasic, if and only if*

$$F_2 - \{l,m\} \cong F_2 - \{a_1,b_1\} \cong F_1 - B^2.$$

Proof. If l and m are cobasic, the formula obviously holds.

If $F_2 - \{l,m\} \cong F_1 - B^2$, we consider its boundary circle $\mathbf{S}^1$ which obviously may be written in a word of l and m, i.e.

$$\mathbf{S}^1 = l^{i_1}m^{j_1}l^{i_2}m^{j_2}\ldots l^{i_k}m^{j_k},$$

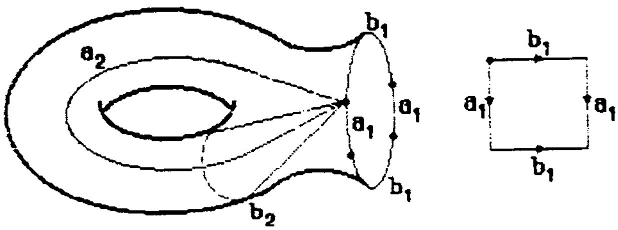


Figure 2.18 $F_2 - \{a_1,b_1\}$

for some integers $i_1,j_1,i_2,j_2,\ldots,i_k,j_k \in \mathbb{Z}$. Since the orientable surface F_2 is obtained from this surface by gluing along the curves l and m, we have that

$$\sum_{p=1}^{k} |i_p| = \sum_{p=1}^{k} |j_p| = 2 \text{ and } \sum_{p=1}^{k} i_p = \sum_{p=1}^{k} j_p = 0.$$

Thus, the only possibilities $\mathbb{S}^1 = lm\overline{lm}$ and $\mathbb{S}^1 = l\overline{ml}m$ determine the surface F_2, and in the both cases l and m are cobasic. Morover, for orientation-preserving homeomorphisms, only the first one is possible, and the orientation-reversing operation

$$RP = [\bar{a}_1, b_1, \bar{a}_1 b_2 a_1, \bar{a}_1 a_2 a_1]$$

interchanges these two situations.

By the previous proposition, this step can be done easily in the following way. It is enough to find elementary operation h such that $(a_1)h = a_1$, and $(m_1)h = b_1^{\pm 1}$. We show it by listing all possible cases under the assumption $\#(m_1 \cap \mathcal{B}) = 0$.

i) First we claim that m_1 can not be one of the following types: $\widetilde{AA}$, $\widetilde{AB}$, $\widetilde{AE}$, $\widetilde{AF}$, $\widetilde{AG}$, $\widetilde{AH}$, $\widetilde{BB}$, $\widetilde{BE}$, $\widetilde{BF}$, $\widetilde{BG}$, $\widetilde{BH}$ $\widetilde{CC}$, $\widetilde{CD}$, $\widetilde{CF}$. $\widetilde{CG}$, $\widetilde{CH}$, $\widetilde{DD}$, $\widetilde{DF}$, $\widetilde{DG}$, $\widetilde{DH}$, $\widetilde{EE}$, $\widetilde{EF}$, $\widetilde{EG}$, $\widetilde{EH}$, $\widetilde{FF}$, $\widetilde{FG}$, $\widetilde{FH}$, $\widetilde{GG}$, $\widetilde{GH}$ or $\widetilde{HH}$, since the curve m_1 is not null-homologous, is not homotopic to any power of a_1, and is cobasic with a_1 (i.e. $F_2 - m_1$ must be homeomorphic to the bounded surface in Figure 2.19.

ii) If m_1 is of the type $\widetilde{BC}$ or $\widetilde{DE}$, a small isotopic deformation of F_2 may turn m_1 into b_1.

iii) If m_1 is of the type $\widetilde{AC}$, $\widetilde{BD}$ or $\widetilde{CE}$, we need only one more linear cutting.

iv) If m_1 is of the type $\widetilde{AD}$, just do the parallel cutting $P(b)$.

Step 5. *Given* f *with* $(a_1)f = a_1$ *and* $(b_1)f = b_1$, *there is a self-homeomorphism* h *which is a composition of* *elementary operations, such that* $(a_1)h = a_1$, $(b_1)h = b_1$ *and* $(l_2)h^{-1} = a_2$.

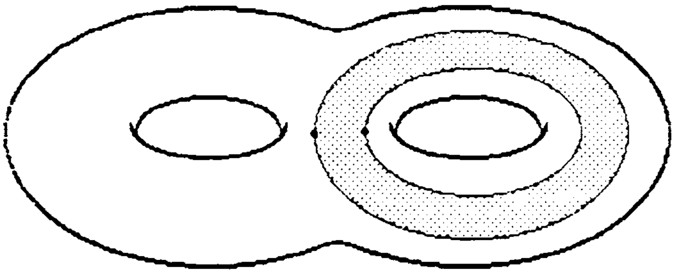


Figure 2.19 $F_2 - m_1$

Denote $m = \#(l_2 \cap B)$, where $l_2 = (a_2)f$.

Case I. $m > 0$.

We do the same thing as in the first step to reduce the number m.

I-i) If an arc of l_2 in the presentation polygon is of the types in Cases 0-III of the first step, do the same operations as there. Since all possible situations involve only the second handle, the operations leave the basecurves a_1 and b_1 unchanged.

Therefore, from now on we will suppose that Case I-(i) does not occur for any arc of l_2. We will denote

$$\begin{aligned}\lambda_a &= \#(l_2 \cap a_2)\\ &= \#(l_2 \cap [b_2\colon a_2]) + \#([\{B,C,D\}\colon a_2])\\ &= \#(l_2 \cap [b_2\colon \bar{a}_2]) + \#(l_2 \cap [\bar{b}_2\colon \bar{a}_2]) +\\ &\qquad \#([\{B,C,D\}\colon \bar{a}_2])\end{aligned}$$

and

$$\begin{aligned}\lambda_b &= \#(l_2 \cap b_2)\\ &= \#(l_2 \cap [\bar{a}_2\colon \bar{b}_2]) + \#([\{B,C,D\}\colon \bar{b}_2])\\ &= \#(l_2 \cap [a_2\colon b_2]) + \#(l_2 \cap [\bar{a}_2\colon b_2]) +\\ &\qquad \#([\{B,C,D\}\colon b_2]),\end{aligned}$$

where $\#([\{B,C,D\}\colon c_2])$ is the number of arcs of l_2 in the presentation polygon with one endpoint from the set $\{B,C,D\}$ and the other on the edge c_2. Obviously

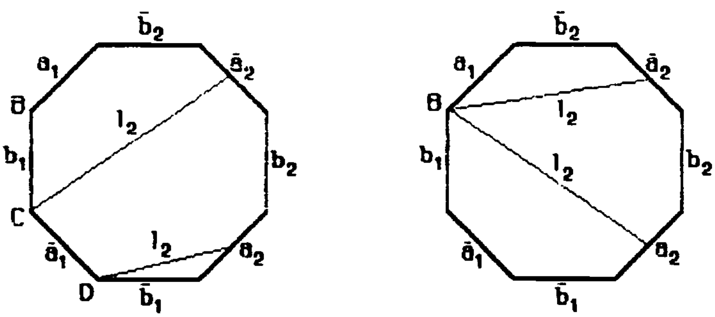
$$\#([\{B,C,D\}\colon a_2]) + \#([\{B,C,D\}\colon b_2]) + \#([\{B,C,D\}\colon \bar{a}_2]) + \#([\{B,C,D\}\colon \bar{b}_2]) \le 2.$$

The assumption $m > 0$ implies that $\lambda_a + \lambda_b > 0$. Thus, we may suppose $\lambda_a > 0$ (or $\lambda_b > 0$ similarly). Then,

I-ii) if $\#(l_2 \cap [a_2\colon b_2]) > \#([\{B,C,D\}\colon a_2])$ or $\#(l_2 \cap [\bar{a}_2\colon \bar{b}_2]) > \#([\{B,C,D\}\colon \bar{b}_2])$, a suitable linear cutting on the second handle makes m smaller.

I-iii) if not (ii) and $\#([\{B,C,D\}\colon \bar{b}_2]) = 0$, then $\lambda_b = 0$, and this implies that

$$\lambda_a = \#([\{B,C,D\}\colon a_2]) = \#([\{B,C,D\}\colon \bar{a}_2]) = 1.$$



(a) $P_0 = C$ and $P_2 = D$ (b) $P_0 = P_2 = B$

Figure 2.20 Case I-(iii)

(Figure 2.20 shows two of the possible situations.) We will show that this is impossible.

In fact, we consider the presentation annulus of the surface F_2 bounded by the basecurves $\{a_1,b_1,a_2\}$, whose one boundary circle is a_2 containing one basepoint and the other is $a_1b_1\bar{a}_1\bar{b}_1a_2$ containing five basepoints, (Figure 2.21). Under the given homeomorphism, for the basecurves $\{a_1,b_1,l_2,m_2\}$, the presentation annulus of F_2 bounded by the system $\{a_1,b_1,l_2\}$ also has one boundary circle containing five basepoints and the other containing only one.

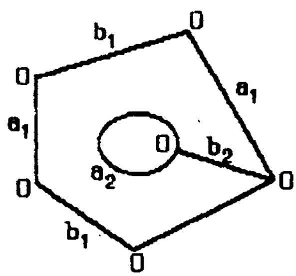


Figure 2.21 $F_2 - \{a_1,b_1,a_2\}$

In the case $P_0 = C$ and $P_2 = D$ we have one circle with four basepoints and the other with two (Figure 2.22). The same thing happens for the case $P_0 = B$ and $P_2 = C$. When $P_0 = B$ and $P_2 = D$, both circles have three basepoints. All these cases are impossible. When $P_0 = P_2 = B$ (or C or D, similarly), the circle bounded by five basepoints is $b_1\bar{a}_1\bar{b}_1a_1l_2$, which can not be given by a homeomorphism keeping a_1 and b_1 fixed, (Figure 2.23).

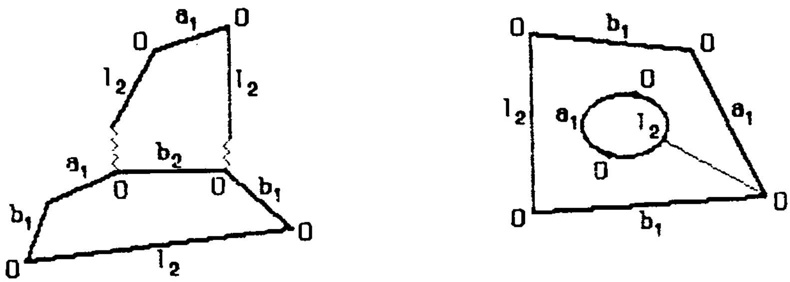


Figure 2.22 Case I-(iii) for P_0 = C and P_2 = D

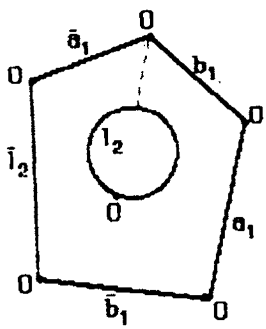


Figure 2.23 Case I-(iii) for $P_0 = P_2$ = B

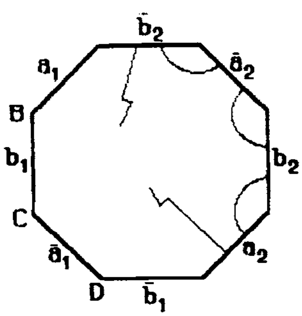


Figure 2.24 Case I-(iv)

I-iv) if $\#([\{B,C,D\}: \bar{b}_2]) = \#([\{B,C,D\}: a_2]) = 1$, then we have $\#([\{B,C,D\}: \bar{a}_2]) = \#([\{B,C,D\}: b_2]) = 0$. The only case is when $\#(l_2 \cap [a_2: b_2]) \neq \#(l_2 \cap [\bar{a}_2: b_2])$, which can be simplified by a linear cutting. Indeed, if $\#(l_2 \cap [a_2: b_2]) = \#(l_2 \cap [\bar{a}_2: b_2]) = \#(l_2 \cap [\bar{a}_2: \bar{b}_2])$ (= 1, if not (ii)), the curve l_2 is homotopic to a word of a_1 and b_1 (Figure 2.24), which is impossible.

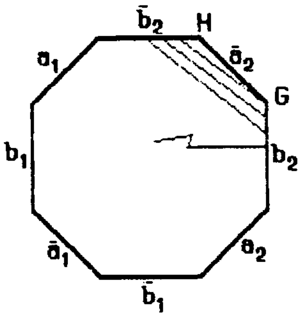
Case II. *If* $m = 0$. An if one endpoint is one of B, C or D, the discussion similar to Case I-(iii) shows the impossibility. All remaining cases except $\widetilde{AE}$, may be done easily by the linear cuttings.

Step 6. *Given* f *with* $f(a_1) = a_1$, $f(b_1) = b_1$ *and* $f(a_2) = a_2$, *then there exists a self-homeomorphism* h *which is a composition of elementary operations, such that* $h(a_1) = a_1$, $h(b_1) = b_1$, $h(a_2) = a_2$ *and* $h^{-1}(f(b_2)) = b_2$.

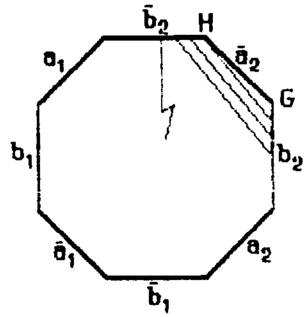
This is the last step of the algorithm.

i) If $m = \#(m_2 \cap \mathcal{B}) > 0$, then one of the cases pictured in Figure 2.25(a) & (b) must occur. Their intersection number m can be reduced by the linear cuttings $L(\bar{a}_2,\bar{b}_2;\bar{b}_2)$ and $L(\bar{a}_2,b_2;b_2)$ respectively.

ii) If $m = 0$, at least one endpoint must be F, G or H, by considering the homotopy class of m_2. And it is sufficient to discuss when $P_1 = H$ (Figure 2.26), since F is symmetric with H, and since the linear cutting $L(\bar{a}_2,\bar{b}_2;\bar{b}_2)$ transfers the case $P_1 = G$ to that of $P_1 = H$.

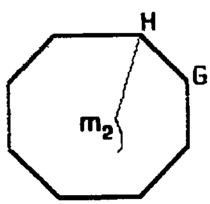


(a)

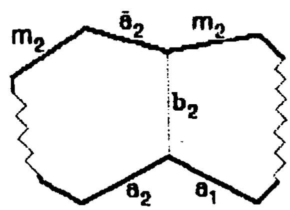


(b)

Figure 2.25 Step 6-(i)



(a)



(b)

Figure 2.26 Step 6-(ii)

By a discussion as before, listing all possible cases of P_0 and cutting along m_2 and gluing along b_2, the only cases that may happen are $P_0 = A$ and $P_0 = F$, which can be done either by some isotopic deformation or by the linear cutting $L(\bar{a}_2, b_2; b_2)$. This completes our algorithm.

3. A Presentation of the Mapping Class Group $\mathcal{M}_2$

The presentation of the group M_2 first was given by Birman ([1]) in Lickorish's generators.

Theorem 3.1. *(Birman) The mapping class group M_2 of the closed orientable surface of genus two is presented by five Dehn twists D_1, D_2, D_3, D_4 and D_5 as generators, and following relations:*

(1.a) $D_i \leftrightarrow D_j$, *for* $|i - j| \geq 2$;

(1.b) $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$, *for* $1 \leq i \leq 4$;

(1.c) $(D_1 D_2 D_3 D_4 D_5)^6 = 1$;

(1.d) $(D_1 D_2 D_3 D_4 D_5 D_5 D_4 D_3 D_2 D_1)^2 = 1$;

(1.e) $(D_1 D_2 D_3 D_4 D_5 D_5 D_4 D_3 D_2 D_1) \leftrightarrow D_i$, *for* $i = 1,2,3,4,5$.

Where $D_1 = B_1$, $D_2 = A_1$, $D_3 = C$, $D_4 = A_2$ *and* $D_5 = B_2$ *are Dehn twists along the curves* b_1, a_1, c, a_2 *and* b_2 *respectively.*

Now we will write a simple presentation in the generators L and N.

First we observe that, for any given mapping class f, we may replace the family of generators $\{D_i\}$ by the family $\{fD_i\bar{f}\}$ in Theorem 3.1. i.e., we may suppose that where D_1, D_2, D_3, D_4 and D_5 are Dehn twists along five arbitrary curves γ_1, γ_2, γ_3, γ_4 and γ_5 which may be identified with b_1, a_1, c, a_2 and b_2 by some mapping class f (Figure 1.6).

Considering $\Gamma_i = N^i L \bar{N}^i$, $i = 0, 1, 2, 3, 4, 5$. They are Dehn twists along the curves $\gamma_i = N^i(b_1)$, $i = 0, 1, 2, 3, 4, 5$, which are b_1, $\bar{a}_1$, $\bar{b}_1 a_2$, b_2, $\bar{a}_2$ and $\bar{b}_2 a_1$ as pictured in Figure 3.1. By the observation, any five of them may be chosen as a family of generators in Theorem 3.1. Therefore, it is natural to choose that $D_i = \Gamma_{i-1} = N^{i-1}L\bar{N}^{i-1}$, $i = 1, \ldots, 5$, and then to substitute them in the formulas (1.a) - (1.e). Using this idea, a presentation of M_2 in the generators L and N will be nicely given.

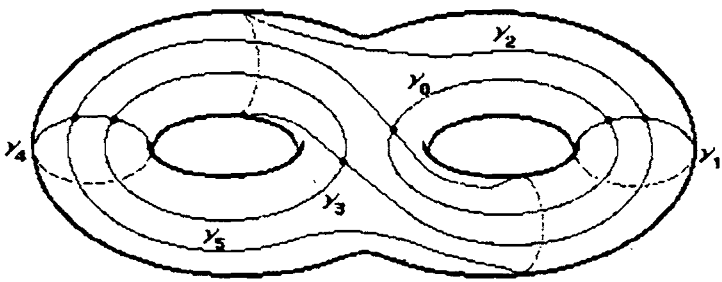


Figure 3.1 Twist curves of Γ_i's

Theorem 3.2. *The surface mapping class group* M_2 *is finitely presented by a family of two generators* L *and* N, *and six relators:*

(2.a) $N^6 = 1$,

(2.b) $(LN)^5 = 1$,

(2.c) $(L\bar{N})^{10} = 1$,

(2.d) $L \leftrightarrow N^2LN^4$,

(2.e) $L \leftrightarrow N^3LN^3$,

(2.f) $L \leftrightarrow (L\bar{N})^5$.

The relations in Theorem 3.2 were certainly not easily found. But the proof is just a straightforward verification. As useful facts, we show some of the calculations below.

$$N = [\bar{a}_2b_1, \bar{a}_1, \bar{a}_1b_2, \bar{a}_2],$$
$$N^2 = [\bar{b}_2, \bar{b}_1a_2, \bar{b}_1, \bar{b}_2a_1],$$
$$N^3 = [a_2, b_2, a_1, b_1],$$
$$N^4 = [\bar{a}_1b_2, \bar{a}_2, \bar{a}_2b_1, \bar{a}_1],$$
$$N^5 = [\bar{b}_1, \bar{b}_2a_1, \bar{b}_2, \bar{b}_1a_2],$$

$$LN = [\bar{a}_2b_1\bar{a}_1, \bar{a}_1, \bar{a}_1b_2, \bar{a}_2],$$
$$(LN)^2 = [\bar{b}_2a_1\bar{b}_1a_2, a_1\bar{b}_1a_2, a_1\bar{b}_1, \bar{b}_2a_1],$$
$$(LN)^3 = [b_1\bar{a}_1b_2, \bar{a}_2b_1\bar{a}_1b_2, \bar{a}_2b_1, b_1\bar{a}_1],$$
$$(LN)^4 = [\bar{b}_1, \bar{b}_2a_1\bar{b}_1, \bar{b}_2, \bar{b}_1a_2],$$

$$L\bar{N} = [\bar{b}_1\bar{b}_2a_1, \bar{b}_2a_1, \bar{b}_2, \bar{b}_1a_2],$$
$$(L\bar{N})^2 = [\bar{a}_1b_2\bar{a}_2b_2\bar{a}_1, \bar{a}_2b_2a_1, \bar{a}_2b_1, \bar{a}_1],$$
$$(L\bar{N})^3 = [\bar{a}_1b_2b_1a_1\bar{b}_1, b_2\bar{a}_2\bar{b}_2a_1, a_1, \bar{a}_1b_2b_1],$$
$$(L\bar{N})^4 = [a_2\bar{b}_2\bar{b}_1a_2b_2\bar{a}_2, \bar{b}_2a_1\bar{b}_1a_2b_2\bar{a}_2, \bar{b}_2a_1\bar{b}_1, a_2],$$
$$(L\bar{N})^5 = [\bar{b}_2\bar{a}_2\bar{a}_1b_2a_2, \bar{a}_2\bar{b}_2\bar{b}_1a_2b_2, \bar{a}_2, \bar{b}_2],$$

and $N^6 = (LN)^5 = (L\bar{N})^{10} = 1.$

We will prove the theorem, after several lemmas.

Lemma 3.3. *The following relations may be obtained by the formulas* (2.a) - (2.f):

(a) $L \leftrightarrow N^iL\bar{N}^i$, *for* $i = 2,3,4$,

(b) $L \leftrightarrow N^iL\bar{N}^iLN^i$, *for* $i = 1,5$.

Proof. (a) When $i = 2$, it is the formula (2.d). When $i = 3$, it is (2.e). And when $i = 4$, we have

$$\begin{aligned} N^4LN^2 \cdot L &= N^4 \cdot L \cdot N^2LN^4 \cdot N^2, && \text{by (2.a)}, \\ &= N^4 \cdot N^2LN^4 \cdot L \cdot N^2, && \text{by (2.d)}, \\ &= L \cdot N^4LN^2, && \text{by (2.a)}. \end{aligned}$$

(b) This is obtained from (2.b), (2.d) and (2.e). Indeed,

$$\bar{N}LNL\bar{N} = \bar{N}^2\overline{LNLNLN}^2, \text{ by (2.b)},$$
$$= (N^4L\bar{N}^4)^{-1}(N^3L\bar{N}^3)^{-1}(N^2L\bar{N}^2)^{-1}.$$

This implies the case $i = 5$. The case when $i = 1$ is equivalent to that $i = 5$, since

$$LNL\bar{N}LN = N \cdot \bar{N}LNL\bar{N} \cdot LN.$$

Lemma 3.4. *The relations* (1.a) *and* (1.b) *are consequences of the formulas in Lemma* 3.3.

Proof. Since $D_i = N^{i-1}L\bar{N}^{i-1}$, this lemma is evident.

Theorem 3.5.

(a) $N = D_1D_2D_3D_4D_5$;

(b) $D_5D_4D_3D_2D_1 = \bar{N}(\bar{N}L)^5$;

(c) $D_1D_2D_3D_4D_5D_5D_4D_3D_2D_1 = (\bar{N}L)^5$.

Proof. The proof is straightforward, since

$$D_1D_2D_3D_4D_5 = LNLNLNLNL\bar{N}^4 = (LN)^5\bar{N}^5 = N.$$

Similarly, we can easily prove the formula (b), and the formula (c) is just a product of the formulas (a) and (b).

Conjugating the formula (a) by a power of N, it follows that,

$$N = \Gamma_i\Gamma_{i+1}\Gamma_{i+2}\Gamma_{i+3}\Gamma_{i+4},$$

for any $i = 0, 1, \ldots, 5$, where $\Gamma_{6+j} = \Gamma_j$ by convention.

Proof of Theorem 3.2. Since we have Lemma 3.3, Lemma 3.4 and Theorem 3.5, our relations imply those in the Birman's theorem.

In fact, the relations of (1.a) and (1.b) have been shown in Lemma 3.3, and (1.c) is exactly $N^6 = 1$ by Theorem 3.5(a). Relations (1.d) and (1.e) are equivalent to the other two formulas since we have the formula in Theorem 3.5(c). As a useful fact, we give here two more relations:

Proposition 3.6.

(a) $L^2 = (N\overline{LNL}N)^4$;

(b) $T \leftrightarrow P^4$,

where $T = N^3$ *and* $P = L\overline{N}LNL$, *and moreover* $P^4 = (L\overline{N}LN)^6$.

The proof is straightforward.

As a consequence, for the homeotopy group $\tilde{M}_2$ we have the theorem:

Theorem 3.7. *The homeotopy group* $\tilde{M}_2$ *is finitely presented by three generators: the linear cutting* L, *the normal cutting* N, *and the reversion* R, *and nine relations: six from Theorem* 3.2 *and three more*

(3.g) $R^2 = I$,

(3.h) $NR = R\overline{N}$,

(3.i) $LR = R\overline{NL}N$.

An interesting observation is the following.

Corollary 3.8. *Both the mapping class group* M_2 *and the homeotopy group* $\tilde{M}_2$ *are generated by some periodic elements. Actually, the elements* N, LN, *and* R *are periodic of orders* 6, 5, *and* 2 *respectively.*

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