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ON THE MAPPING CLASS GROUPS OF THE CLOSED ORIENTABLE SURFACES

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In the paper [8] we gave a simple presentation of two generators for the mapping class group M_2 of the closed orientable surface of genus two, and an algorithm to write an arbitrary mapping class in those generators. Here, we will generalize them to higher genera, and we will show that,

Theorem 1.3. *The mapping class group* M_g *of the closed orientable surface is generated by three elements* L, N *and* T.

Here L and N are similar to those we gave for the genus two in [8], i.e., L is a Dehn twist along the longitude of the first handle, and N is a composition of five Dehn twists along five circles contained in the first two handles. The generator T rotates the handles.

As applications, we will study explicitly the abelianization $Ab(M_g)$ of M_g, the Torelli subgroup I_g of M_g, and the automorphism group $Aut(M_g)$ of M_g.

1. The Elemetary Mapping Classes on the Surface F_g

Let F_g be a closed orientable surface of genus g, $g \geq 3$. Let

$$B = \{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$$

be a fixed system of *basecurves* on F_g, based at a *basepoint* O, as pictured in Figure 1.1

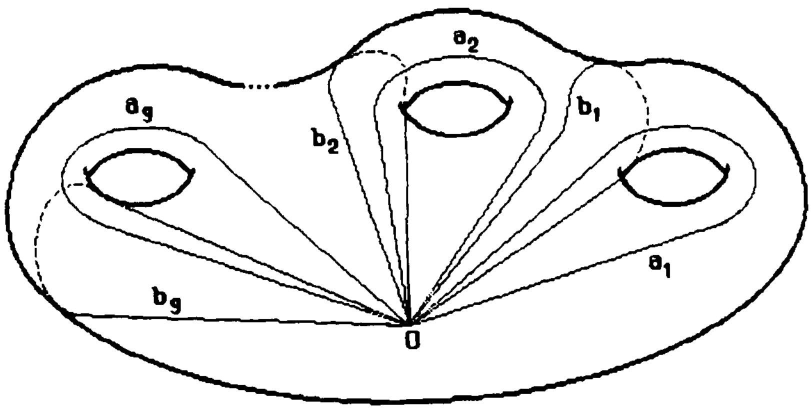


Figure 1.1

Notationally, we will not distinguish between a homeomorphism and its homeotopy class. As we did for the case of genus two, first we will list some *elementary operations* which will be described by the isotopy classes of the image of the basecurves in the fundamental group $\pi_1(F_g;O)$ of the surface F_g relative to the basepoint O, i.e., for any homotopy class f we will denote

$$f = (\mathcal{B})f = [[(a_1)f],[(b_1)f],[(a_2)f],[(b_2)f],\ldots,[(a_g)f],[(b_g)f]].$$

And conventionally, we will write the group product as the right action of basecurves, i.e., for any mapping classes f and g, for any point X from the surface, the image $(X)(f \cdot g) = ((X)f)g$.

0) The *identity* $I: F_g \to F_g$ is given by

$$I = [a_1, b_1, a_2, b_2, \ldots, a_g, b_g],$$

i.e., it is given by an isotopy deformation of the surface.

1) An orientation-reversing mapping, which flips the surface, called *reversion* $R: F_g \to F_g$, is given by

$$R = [b_1, a_1, b_g, a_g, \ldots, b_2, a_2].$$

2) An orientation-preserving mapping, which rotates the handles, called *transport* $T: F_g \to F_g$, is given by

$$T = [a_g, b_g, a_1, b_1, \ldots, a_{g-1}, b_{g-1}].$$

3) Homeotopy classes $L_j = T^{j-1}L\bar{T}^{j-1}$ and $M_j = T^{j-1}M\bar{T}^{j-1}$, $j = 1,2,\ldots,g$, are called *linear cuttings*, where L and M are the *longitude cutting* and the *meridian cutting* of the first handle, which are given by

$$L = [a_1b_1, b_1, a_2, b_2, \ldots, a_g, b_g],$$

and

$$M = [a_1, b_1\bar{a}_1, a_2, b_2, \ldots, a_g, b_g].$$

4) The *normal cutting* $N: F_g \to F_g$, similar to what we did in [8], is given by

$$N = [x\bar{a}_2b_1, \bar{a}_1, \bar{a}_1xb_2, \bar{a}_2, a_3, b_3, \ldots, a_g, b_g],$$

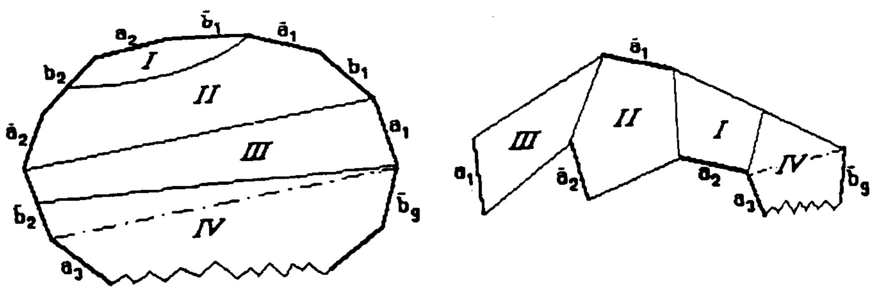


Figure 1.2

where $x = [a_1,b_1][a_2,b_2]$. Topologically, this is given by a cutting and sewing process as we draw in Figure 1.2.

Proposition 1.1.

(a) $T^g = I$, $R^2 = I$, $\overline{T} = RTR$.

(b) $M = R\overline{L}R = \overline{N}LN$, $RNR = \overline{T}\overline{N}T$, $TLT = N^3L\overline{N}^3$.

(c) $N^6L = LN^6$, $N^6R = R\overline{T}\overline{N}^6T$

Proof. All formulas may be verified directly. For example, (c), since

$$N = [x\overline{a}_2b_1,\overline{a}_1,\overline{a}_1xb_2,\overline{a}_2,a_3,b_3,\ldots,a_g,b_g],$$

$$N^2 = [x\overline{b}_2\overline{x},\overline{b}_1a_2\overline{x},\overline{b}_1,\overline{b}_2\overline{x}a_1,a_3,b_3,\ldots,a_g,b_g],$$

$$N^3 = [xa_2\overline{x},xb_2\overline{x},a_1,b_1,a_3,b_3,\ldots,a_g,b_g],$$

and $N^6 = [xa_1\overline{x},xb_1\overline{x},xa_2\overline{x},xb_2\overline{x},a_3,b_3,\ldots,a_g,b_g]$,

the formulas are obvious. In fact, L leaves the curve $x = [a_1,b_1][a_2,b_2]$ invariant, and R reverses the curve x to $([a_g,b_g][a_1,b_1])^{-1}$.

5) Finally, we denote P the *parallel cutting* $L\overline{N}LNL$. Algebraically it is given by

$$P = [a_1b_1\overline{a}_1,\overline{a}_1,a_2,b_2,\ldots,a_g,b_g].$$

Proposition 1.2

(a) $P = LML = MLM$, i.e. $L \leftrightarrow NL\overline{N}LN$.

(b) $(LN)^5 = (\overline{L}N)^{10} = N^6$.

(c) $(N^3T)^{g-1} = P^{4(g-2)}$.

Proof. (a) A direct verification. (b) Actually,

$$LN = [x\bar{a}_2b_1\bar{a}_1, \bar{a}_1, \bar{a}_1xb_2, \bar{a}_2, a_3, b_3, \ldots],$$

$$(LN)^2 = [x\bar{b}_2\bar{x}a_1\bar{b}_1a_2\bar{x}, a_1\bar{b}_1a_2\bar{x}, a_1\bar{b}_1, \bar{b}_2\bar{x}a_1, a_3\ b_3, \ldots],$$

$$(LN)^3 = [xb_1\bar{a}_1x\bar{b}_2\bar{x}, x\bar{a}_2b_1\bar{a}_1xb_2\bar{x}, x\bar{a}_2b_1, b_1\bar{a}_1, a_3, b_3, \ldots],$$

$$(LN)^5 = [xa_1\bar{x}, xb_1\bar{x}, xa_2\bar{x}, xb_2\bar{x}, a_3, b_3, \ldots] = N^6,$$

and

$$\bar{L}N = [x\bar{b}_2b_1a_1, \bar{a}_1, \bar{a}_1xb_2, \bar{a}_2, a_3, b_3, \ldots],$$

$$(\bar{L}N)^2 = [x\bar{b}_2\bar{a}_2b_1a_1, \bar{a}_1\bar{b}_1a_2\bar{x}, \bar{a}_1\bar{b}_1, \bar{b}_2\bar{x}a_1, a_3, b_3, \ldots],$$

$$(\bar{L}N)^3 = [xa_2\bar{b}_2\bar{a}_2b_1a_1, \bar{a}_1\bar{b}_1a_2b_2\bar{x}, \bar{a}_1\bar{b}_1a_2\bar{x}a_1, b_1a_1, a_3, b_3, \ldots],$$

$$(\bar{L}N)^5 = [x\bar{a}_1, \bar{b}_1\bar{x}, \bar{b}_1\bar{a}_1x\bar{a}_2b_1a_1, \bar{a}_1\bar{b}_1a_2\bar{b}_2\bar{a}_2b_1a_1, a_3, b_3, \ldots],$$

$$(\bar{L}N)^{10} = [xa_1\bar{x}, xb_1\bar{x}, xa_2\bar{x}, xb_2\bar{x}, a_3, b_3, \ldots] = N^6.$$

(c) Since

$$N^3T = [c_gc_1a_1\bar{c}_1\bar{c}_g, c_gc_1b_1\bar{c}_1\bar{c}_g, a_g, b_g, a_2, b_2, \ldots, a_{g-1}, b_{g-1}],$$

$$(N^3T)^{g-1} = [c_2\ldots c_gc_1^{g-1}a_1\bar{c}_1^{g-1}\bar{c}_g\ldots\bar{c}_2, c_2\ldots c_gc_1^{g-1}b_1\bar{c}_1^{g-1}\bar{c}_g\ldots\bar{c}_2, a_2, b_2, \ldots, a_g, b_g],$$ and $c_2\ldots c_gc_1 = 1$.

Remark. For the case of genus $g \geq 3$, the normal cutting N is no longer periodic. But the mapping class N^6 is still quite easy to deal with, since it is exactly the Dehn twist along the null-homologous circle $x = [a_1,b_1][a_2,b_2]$.

Now we can state our main theorem.

Theorem 1.3. *The mapping class group* M_g *of the closed orientable surface of genus* g, g $\geq$ 3, *is generated by three elements: the linear cutting* L, *the normal cutting* N *and the transport* T.

In the next section, we will give an algorithm to write an arbitrary homeotopy class in our generators. It certainly gives a direct proof of Theorem 1.3. In Section 3, we will relate them to Lickorish's set of Dehn twist generators; that produces another proof.

As a consequence, we have,

Theorem 1.4. *The homeotopy group* $\tilde{M}_g$ *of the surface* $\mathbf{F}_g$, g $\geq$ 3, *is generated by four elements:* L, N, T *and* R. *Furthermore,*

$$\tilde{M}_g = \frac{M_g * \langle R \rangle}{\{RL = \overline{N}\overline{L}NR,\ RN = \overline{T}\overline{N}TR,\ RT = \overline{T}R,\ R^2 = I\}}$$

2. Writing a Homeotopy Class in the Generators

Let f be an element of the mapping class group M_g given by the expression

$$f = [(a_1)f,(b_1)f,(a_2)f,(b_2)f,\ldots,(a_g)f,(b_g)f].$$

We are going to find an algorithm to write f in the generators introduced in the last section by assuming the existence of such an algorithm for genera less than g.

At first, we need two special kinds of mapping classes:

a) The *handle crossing* χ, (Figure 2.1), given by

$$\chi = [c_1a_2,b_2,b_2a_1\overline{b}_2,b_2b_1\overline{b}_2,a_3,b_3,\ldots],$$

is obtained by sliding the whole first handle along the longitude circle b_2 of the second handle. And

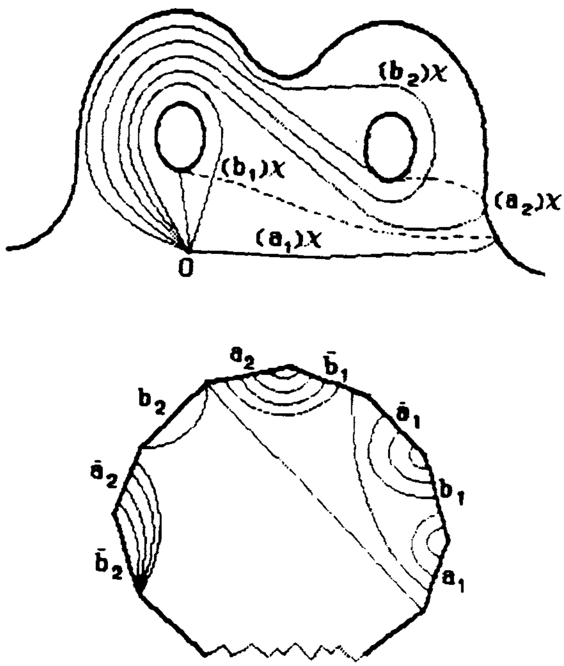


Figure 2.1

b) the *handle switchings* ψ_j, $j = 1,2,\ldots,g$, (Figure 2.2), given by

$$\psi_j = T^{j-1}N^3\bar{T}^{j-1} = [\ldots,a_{j-1},b_{j-1},xa_{j+1}\bar{x},xb_{j+1}\bar{x},$$

$$a_j,b_j,a_{j+2},b_{j+2},\ldots],$$

where $x = [a_j,b_j][a_{j+1},b_{j+1}]$,

By a direct verification, it is easy to show that,

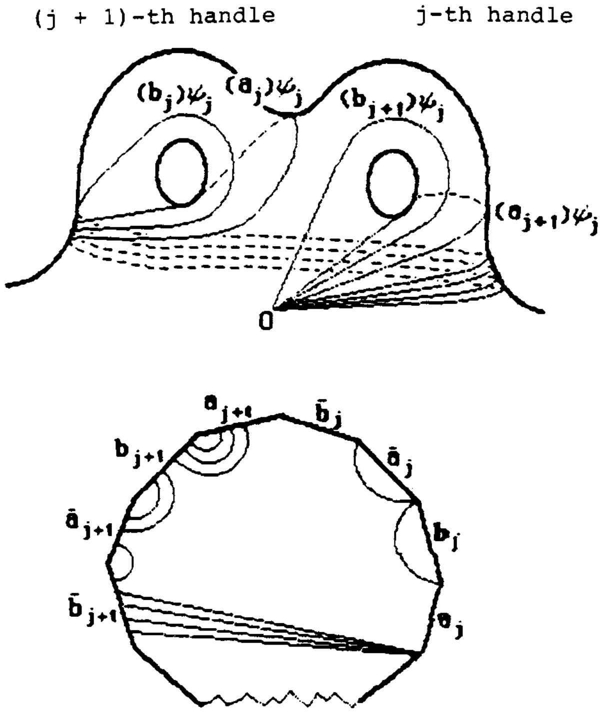


Figure 2.2

Proposition 2.1. *The handle crossing and the handle switchings are generated by the elements* L, N *and* T. *Moreover,*

$$\chi = (\bar{N}L)^5 (N\bar{L})^5 \bar{N}^3,$$

and

$$\psi_j = T^{j-1} N^3 \bar{T}^{j-1}, \quad j = 1,2,\ldots,g.$$

The parallel cutting $P_2 = TL\bar{N}LNL\bar{T}$ maps the circle b_2 to $\bar{a}_2$. Then, the mapping class $\bar{P}_2\bar{\chi}P_2$ is obtained by sliding the whole first handle along the meridian circle a_2 of the second handle. Since we may switch the second handle with any other handles by mapping classes ψ_j, the mapping classes obtained by sliding the whole first handle along some basecurve of $\mathcal{B} - \{a_1, b_1\}$ are generated by L, N and T. Therefore,

Proposition 2.2. *The mapping classes obtained by sliding the whole first handle along a closed curve in the surface* $\mathbf{F} - (a_1 \cup b_1)$ *are generated by* L, N *and* T.

Now we can start to show our algorithm.

Step I. If for some $1 \leq i \leq g$, $(a_i)f = a_i$ and $(b_i)f = b_i$, then f is generated by L, N and T.

Composing the handle switchings, we may let $i = g$. Since $(a_g)f = a_g$ and $(b_g)f = b_g$, we assume that the restriction of f in the last handle is the identity map, in particular f leaves the waist curve $c_g = [a_g, b_g]$ fixed. Thus, letting $\mathbf{F}'$ be a closed surface of genus $g - 1$ obtained by cutting off the last handle of $\mathbf{F}$ along the circle c_g and filling by some disk D so that $\partial D = c_g$, the mapping class f induces a unique mapping class f' of M_{g-1} of the surface $\mathbf{F}'$, which will be called the *restriction of* f *in* $\mathbf{F}'$. Clearly,

Proposition 2.3. *Let* f_1 *and* f_2 *be two elements of* M_g, *such that both leave the waist curve* $c_g = [a_g, b_g]$ *fixed. Then, their composition* $f_1 f_2$ *also leaves* c_g *fixed, and the restriction of their composition in* $\mathbf{F}'$ *is equal to the composition of their restrictions in* $\mathbf{F}'$, *i.e.*

$$(f_1 f_2)' = f_1' f_2'.$$

Denote by $L^{(g-1)}$, $N^{(g-1)}$ and $T^{(g-1)}$ the elementary generators of the group M_{g-1}, it is obvious that,

$$L^{(g-1)} = L', \ N^{(g-1)} = N', \text{ and } T^{(g-1)} = (\psi_g T)'.$$

Thus, by induction f' can be written as a word in them, i.e.,

$$f' = F'(L',N',(\psi_g T)').$$

Let's define

$$\tilde{f} = F'(L,N,\psi_g T).$$

Clearly, it is an element of M_g generated by L, N and T, and its restriction in F' is equal to that of f, i.e. $\tilde{f}' = f'$ by Proposition 2.3. Since $(f\tilde{f}^{-1})' = f'\tilde{f}'^{-1} = 1$, we may consider $f\tilde{f}^{-1}$ instead of f, or equivalently, we may assume $f' = 1$ from now on.

Let f be a self-homeomorphism of **F**, such that $(a_g)f = a_g$, $(b_g)f = b_g$ and $f' = 1$, i.e. its restriction in the last handle is the identity map, and its restriction in **F**' - D extends to some f' which is isotopic to the identity map in **F**'. If the isotopy between f' and the identity map leaves the disk D fixed, then the map f itself must be isotopic to the identity map of **F**. In general, the isotopy of f' can be decomposed into two operations. One is to slide the disk D around some closed curve γ in **F**' - D, and the other is to do some Dehn twists along the boundary of D. Thus f is obtained from the identity map of **F** by sliding the last handle along the same curve γ in the inverse way, and by doing some Dehn twists along the waist circle $c_g = [a_g,b_g]$ of the last handle. The first one is generated by the elementary generators L, N and T according to Proposition 2.2. And so is the second one, since the Dehn twist along c_g has the following expression:

$$\overline{T}P^4T = \overline{T}(L\overline{N}LN)^6T =$$

$$= [a_1,b_1,\ldots,a_{g-1},b_{g-1},c_g a_g \bar{c}_g, c_g b_g \bar{c}_g].$$

Step II. If for some $1 \leq i, j \leq g$, $(a_i \cup b_i) \cap ((a_j)f \cup (b_j)f) = \emptyset$, then f is generated by L, N and T.

Indeed, we may assume $i = j = 1$ by composing the transport T. Since $(a_1 \cup b_1) \cap ((a_1)f \cup (b_1)f) = \emptyset$, we may construct a system of basecurves

$$\mathcal{B}' = \{a_1,b_1,(a_1)f,(b_1)f,a_3',b_3',\ldots,a_g',b_g'\},$$

for some suitable circles $a_3',b_3',\ldots,a_g',b_g'$. Since between any two systems of base curves $\mathcal{B}$ and $\mathcal{B}'$ there always exists a unique mapping class h such that $(\mathcal{B})h = \mathcal{B}'$, among the systems $\mathcal{B}$, $\mathcal{B}'$ and $(\mathcal{B})f$ of basecurves we have two mapping classes f_1 and f_2 such that $(\mathcal{B})f_1 = \mathcal{B}'$ and $(\mathcal{B}')f_2 = (\mathcal{B})f$, furthermore $f = f_1f_2$.

Since $(a_1)f_1 = a_1$ and $(b_1)f_1 = b_1$, by Step I, the mapping class f_1 is generated by L, N and T. Since

$$(a_2)(\overline{\Psi}_1 f_2 \overline{f}_2) = (a_1)f_2\overline{f}_1 = ((a_1)f)\overline{f}_1 = a_2$$

and $$(b_2)(\overline{\Psi}_1 f_2 \overline{f}_1) = (b_1)f_2\overline{f}_1 = ((b_1)f)\overline{f}_1 = b_2$$

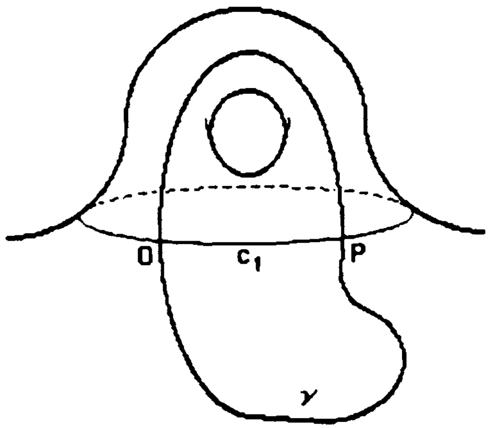
again by Step I, the mapping class $\overline{\Psi}_1 f_2 \overline{f}_1$ is generated by L, N and T. Thus also the mapping class f_2 is generated by L, N and T by Proposition 2.1. Then clearly the mapping class $f = f_1f_2$ is too.

Step III. For any self-homeomorphism f of **F**, there is a self-homeomorphism h whose mapping class is generated by L, N and T, such that

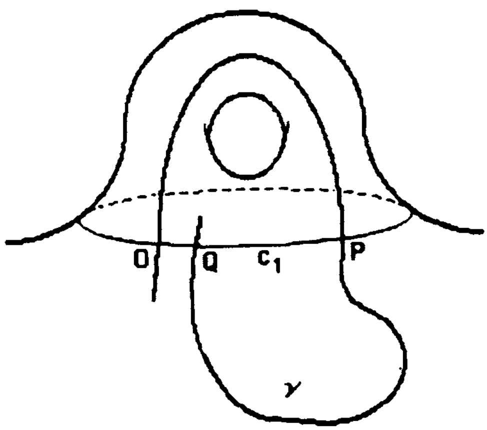
$$(a_1 \cup b_1) \cap ((a_1)fh \cup (b_1)fh) = \emptyset.$$

Denote by m_a the number of arc components of the circle $(a_1)f$ located in the first handle, and denote by m_b the number of arc components of the circle $(b_1)f$ located in the first handle. First let $\gamma = (a_1)f$, thus the intersection number between the circles γ and $c_1 = [a_1, b_1]$ is equal to $2m_a$. Suppose $m_a > 0$. Let $\gamma_0 = \gamma|_{OP}$ be an arc starting from O and ending at P in $\gamma \cap c_1$. Let $\gamma_1 = \gamma|_{PQ}$ denote the arc of γ from the point P to the next point Q of $\gamma \cap c_1$, (where Q = O when $m_a = 1$). And let c_{10}, c_{11} and c_{12} be the three arc components of $c_1 - \{O,P,Q\}$ starting at O, P and Q and ending at P, Q and O respectively, (where $c_{12} = \emptyset$ when $m_a = 1$), (Figure 2.3).

Considering the circle $\delta = \bar{c}_{11}\gamma_1$, we choose arbitrarily a self-homeomorphism h_1 of **F** which leaves the first handle fixed and has the property that $(\delta)h_1 \cap (a_g \cup b_g) = \emptyset$. In fact, if δ is not null-homologous we may choose h_1 so that $(\delta)h_1 = a_2$ since $g \geq 3$, if δ is null-homologous we may choose h_1 so that $(\delta)h_1 = c_1c_2 \ldots c_k$, where $k < g$ is the genus of the component $\mathbf{F} - \delta$ which contains the first handle and $c_j = [a_j, b_j]$ for $j = 1, \ldots, k$. By Step I, h_1 is generated by L, N and T.



(a) $m_a = 1$



(b) $m_a > 1$

Figure 2.3

If $m_a = 1$, we have either $(\gamma)h_1 = a'a_2$ when δ is not null-homologous, or $(\gamma)h_1 = a'c_1c_2...c_k$ otherwise, where a' is some word in a_1 and b_1 and $k < g$. Hence $(\gamma)h_1 \cap (a_g \cup b_g) = \emptyset$ for $g \geq 3$. Therefore, there is some self-homeomorphism h_2 of F which leaves the last handle (a_g,b_g) fixed so that

$$(\gamma)(h_1h_2) = ((\gamma)h_1)h_2 = a_1.$$

By Step I, h_2 is generated by L, N and T. Thus, we have reduced to the case $m_a = 0$.

If $m_a > 1$, let c' be a circle whose homotopy class is equal to $c_{10}\gamma_1c_{12}\gamma_0c_{11}\bar{\gamma}_1\bar{\gamma}_0$ as pictured in Figure 2.4.

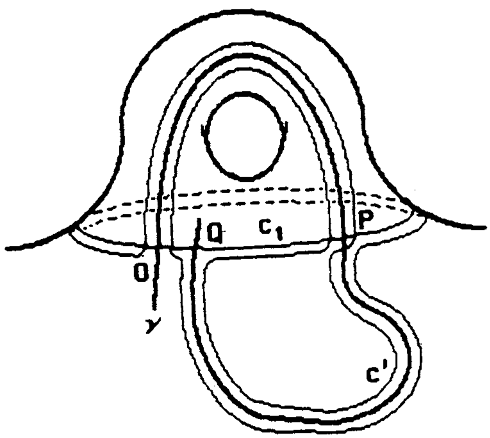


Figure 2.4

Thus, we have the following properties: $(c')h_1 \cap (a_g \cup b_g) = \emptyset$, c' is null-homologous, c' separates the surface **F** in two components, the component that does not contain the last handle has genus one, the cardinality of the set $c' \cap \gamma$ is less than $2m_a$, and moreover if the intersection point $(a_1)f \cap (b_1)f$ is not contained in the arc $\gamma|_{OQ} = \gamma_0\gamma_1$ the cardinality of the set $c' \cap (b_1)f$ remains unchanged. Therefore, there exists a self-homeomorphism h_2 of **F**, which leaves the last handle so that

$$(c')(h_1h_2) = ((c')h_1)h_2 = c_1.$$

By Step I, the mapping class of h_2 is generated by L, N and T. Instead of f we are going to study fh_1h_2, and clearly the cardinality of the set $(a_1)fh_1h_2 \cap c_1$ is the same as the set $(a_1)f \cap c'$, which is less than $2m_a$.

If $m_a = 0$ and $m_b > 1$, we may do the same process by taking $\gamma = f(b_1)$ as we did for $m_a > 1$. Since $m_b > 1$, i.e. there is more than one component of γ in the first handle, we may choose the points O, P and Q such that the intersection point $(a_1)f \cap (b_1)f$ is not located in $\gamma|_{OQ}$. As we mentioned before, the number $m_a = 0$ remains unchanged.

If $m_a = 0$ and $m_b = 1$, and if $(a_1)f$ is contained in the first handle, we may let $\gamma = (b_1)f$ and construct the same h_1 as we did before. Since $\gamma_1 = \gamma|_{PQ}$ must be the only part of the set $(a_1 \cup b_1)f$ outside of the first handle, we have $((a_1 \cup b_1)f)h_1 \cap (a_g \cup b_g) = \emptyset$. Therefore fh_1 is generated by L, N and T by Step II.

If $m_a = 0$ and $m_b = 1$, and if $(a_1)f$ is not contained in the first handle, let $\gamma = (b_1)f$ and construct the same circle δ as we did before. Then δ intersects $(a_1)f$ transversally at one point. Thus, δ is not null-homologous, and we may choose h_1 such that $(\delta)h_1 = a_2$ and $((a_1)f)h_1 = b_2$. Again, we have $((a_1 \cup b_1)f)h_1 \cap (a_g \cup b_g) = \emptyset$ since $g \geq 3$. Therefore fh_1 is generated by L, N and T by Step II.

Finally, if $m_a = m_b = 0$, we may apply Step II directly.

3. Relation with Lickorish's Generators

Lickorish [7] found a finite set of generators of the mapping class group M_g, which are Dehn twists along the simple closed curves a_i's, b_i's and z_i's, $i = 1,2,\dots,g$, as pictured in Figure 3.1. They will be denoted by A_i's, B_i's and Z_i's respectively. Humphries [6] reduced the set of generators to only 2g+1 of them. They are related to our generators L, N and T in the following way:

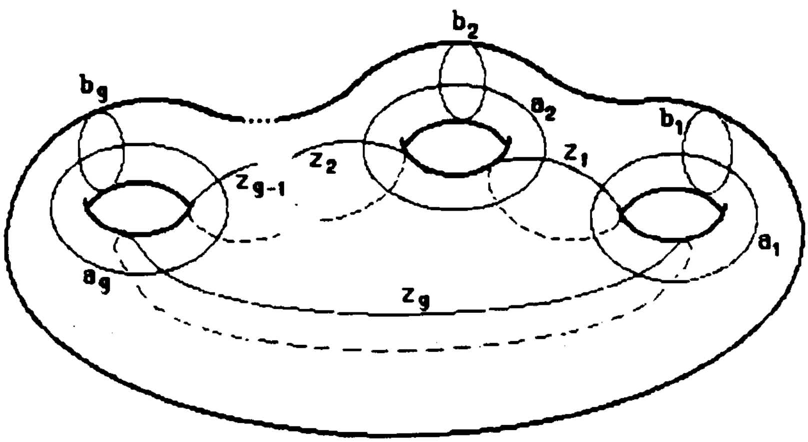


Figure 3.1 Lickorish's generators

Theorem 3.1.

(a) $A_i = T^{i-1}M\bar{T}^{i-1}$, *where* $M = \bar{N}LN$,

(b) $B_i = T^{i-1}L\bar{T}^{i-1}$,

(c) $Z_i = T^{i-1}Z\bar{T}^{i-1}$, *where* $Z = \overline{ML}NL\bar{N}LM$,

for $i = 1,2,\dots,g$.

Proof. The only expression we need to prove is the last one $Z_1 = Z$. Indeed,

$$Z_1 = [a_1z_1, \bar{z}_1b_1z_1, \bar{z}_1a_2, b_2, a_3, b_3, \dots],$$

where $z_1 = a_2\bar{b}_2\bar{a}_2b_1$. And

$$Z = \overline{M}\overline{L}NL\overline{N}LM = \overline{M}\overline{L}NL\overline{N} \cdot [a_1b_1\bar{a}_1, b_1\bar{a}_1, a_2, b_2, \ldots]$$

$$= \overline{M}\overline{L}NL \cdot [a_1\bar{b}_1, \bar{b}_2\bar{x}a_1b_1\bar{a}_1, \bar{b}_2, \bar{x}a_1\bar{b}_1a_2, \ldots],$$

$$x = [a_1,b_1][a_2,b_2]$$

$$= \overline{M}\overline{L}N \cdot [a_1\bar{b}_1\bar{b}_2\bar{c}_2b_1, \bar{b}_2\bar{c}_2b_1, \bar{b}_2, \bar{x}a_1\bar{b}_1a_2, \ldots],$$

$$c_2 = [a_2,b_2].$$

$$= \overline{M}\overline{L} \cdot [c_1b_1, \bar{b}_1c_2b_2b_1\bar{a}_1, \bar{b}_1a_2b_2, b_2, \ldots],$$

$$c_1 = [a_1,b_1].$$

$$= [a_1\bar{b}_2\bar{c}_2b_1, \bar{b}_1c_2b_2b_1\bar{b}_2\bar{c}_2b_1, \bar{b}_1a_2b_2, b_2 \ldots] = z_1.$$

Reciprocally, we also may write L, N and T in Lickorish's Dehn twists.

Theorem 3.2.

(a) $L = B_1$,

(b) $N = A_1 \cdot B_1 \cdot B_1A_1Z_1\bar{A}_1\bar{B}_1 \cdot A_2 \cdot B_2$,

(c) $T = \bar{P}_1^4N_1^3\bar{P}_2^4N_2^3\ldots\bar{P}_{g-1}^4N_{g-1}^3P_g^4$

$$= \bar{P}^{4(g-2)}N_1^3N_2^3\ldots N_{g-1}^3,$$

where

$$P_i = T^{i-1}P\bar{T}^{i-1} = A_iB_iA_i = B_iA_iB_i,$$

and

$$N_i = T^{i-1}NT^{i-1} = A_i \cdot B_i \cdot B_iA_iZ_i\bar{A}_i\bar{B}_i \cdot A_{i+1} \cdot B_{i+1},$$

for $i = 1,2,\ldots,g$.

Proof. (a) It is obvious.

(b) Since $(LN)^5 = N^6$, we have

$$N = \bar{N}LN \cdot L \cdot NL\bar{N} \cdot N^2L\bar{N}^2 \cdot N^3L\bar{N}^3$$

Therefore, the formula is immediate.

(c) Since

$$P^4 = [c_1a_1\bar{c}_1, c_1b_1\bar{c}_1, a_2, b_2, a_3, b_3, \ldots],$$

$$\bar{P}^4N^3 = [c_1a_2\bar{c}_1, c_1b_2\bar{c}_1, a_1, b_1, a_3, b_3, \ldots],$$

and since $P_iN_i^3 = N_i^3P_{i+1}$, the formula is easy. Indeed, this is a consequence of the formulas (1.1.a) $T^g = 1$ and (1.2.c) $(N^3T)^{g-1} = P^{4(g-2)}$.

A new presentation of the mapping class group M_g can be found by plugging our generators into the presentation given by Hatcher-Thurston [5] and Wajnryb [11]. Now we recall their result.

Theorem 3.3. ([5] & [11]) *The mapping class group* M_g *has a presentation with* 2g+1 *generators* $A_1, A_2, \ldots, A_g, B_1, B_2, Z_1, Z_2, \ldots, Z_{g-1}$, *and the following relations:*

(A) (1) $A_iA_j = A_jA_i$, (2) $B_iB_j = B_jB_i$, (3) $Z_iZ_j = Z_jZ_i$,

(4) $A_iB_j = B_jA_i$, *if* $j \neq i$, (5) $A_iB_iA_i = B_iA_iB_i$,

(6) $A_iZ_j = Z_jA_i$, *if* $j \neq i, i-1$,

(7) $A_iZ_jA_i = Z_jA_iZ_j$, *if* $j = i, i+1$

(8) $B_iZ_j = Z_jB_i$, *for all* $i, j = 1, 2, \ldots.$

(B) $(B_1A_1Z_1)^4 = B_2\bar{A}_2\bar{Z}_1\bar{A}_1\bar{B}_1^2\bar{A}_1\bar{Z}_1\bar{A}_2B_2A_2Z_1A_1B_1^2A_1Z_1A_2.$

(C) $B_2 \cdot t_2B_2\bar{t}_2 \cdot t_1t_2B_2\bar{t}_2\bar{t}_1 \cdot \bar{Z}_2 \cdot \bar{Z}_1 \cdot \bar{B}_1 =$

$$\bar{A}_3\bar{Z}_2\bar{A}_2\bar{Z}_1\bar{A}_1\bar{u}vuA_1Z_1A_2Z_2A_3,$$

where

$$t_1 = A_1B_1Z_1A_1,\quad t_2 = A_2Z_1Z_2A_2,\quad u = Z_2A_3t_2B_2\bar{t}_2\bar{A}_3\bar{Z}_2,$$

and

$$v = B_1A_1Z_1A_2B_2\bar{A}_2\bar{Z}_1\bar{A}_1\bar{B}_1.$$

(D) $B_g \leftrightarrow A_g Z_{g-1} A_{g-1} \cdots Z_1 A_1 B_1^2 A_1 Z_1 \cdots A_{g-1} Z_{g-1} A_g$,

where

$$B_g = \overline{u}_{g-1} \cdots \overline{u}_2 \overline{u}_1 B_1 u_1 u_2 \cdots u_{g-1},\ t_1 = A_1 B_1 Z_1 A_1,\ v_1 = B_2,$$

$$u_1 = A_1 Z_1 A_2 v_1 \overline{B}_1 \overline{A}_1 \overline{Z}_1 \overline{A}_2,\ t_i = A_i Z_{i-1} Z_i A_i,$$

$$v_i = t_{i-1} t_i v_{i-1} \overline{t}_i \overline{t}_{i-1},$$

and

$$u_i = A_i Z_i A_{i+1} v_i \overline{Z}_{i-1} \overline{A}_i \overline{Z}_i \overline{A}_{i+1},\ \textit{for } i = 2\ldots,g-1.$$

Now we begin to simplify these relations by using our new generators.

Proposition 3.4.

(a) $L \leftrightarrow T^i L \overline{T}^i$, $i = 1,2,\ldots,g-1$.

(b) $L \leftrightarrow T^i N \overline{T}^i$, $i = 1,2,\ldots,g-2$.

(c) $N \leftrightarrow T^i N \overline{T}^i$, $i = 2,3,\ldots,g-2$.

(d) $TM\overline{T} = N^3 M \overline{N}^3 = N^2 L \overline{N}^2$.

(e) $L \leftrightarrow \overline{N} T Z \overline{T} N$.

(f) $TN^3 \overline{T} N T \overline{N}^3 \overline{T} = N^3 T N \overline{T} \overline{N}^3$.

(g) $L \leftrightarrow N^2 L \overline{N}^2,\ N^3 L \overline{N}^3,\ N^4 L \overline{N}^4$.

(h) $L \leftrightarrow NL\overline{N}LN,\ \overline{N}LNL\overline{N}$.

Proof. The main part of the proposition is proven by direct calculation of the image of basecurves.

Remark. Among the above relations, the formulas (g) and (h) are consequences of

$$L \leftrightarrow N^6,\ L \leftrightarrow T\overline{N}LN\overline{T} = N^2 L \overline{N}^2,\ L \leftrightarrow TL\overline{T} = N^3 L \overline{N}^3.$$

Indeed, the formulas of (g) are evident, and thos of (h) can be read from the following equality:

$$\bar{N}LNL\bar{N} = \bar{N}^2\overline{LNLNLN}^2, \text{ by (1.2.b)},$$
$$= (N^4L\bar{N}^4)^{-1}(N^3L\bar{N}^3)^{-1}(N^2L\bar{N}^2)^{-1}.$$

In our later calculation, the formulas (g) and (h) will be used very frequently.

Proposition 3.5. *The relation* (A) *of Theorem* 3.3 *is a consequence of the relations in Propositions* 1.1 *and* 3.4.

Proof. Without loss of generality, we assume i = 1.

(A-1) If j ≠ 2 and g, this is clear from (3.4.a) and (3.4.b). If j = 2,

$$A_1A_2 = MTM\bar{T} = \bar{N}LN\cdot N^2L\bar{N}^2 = \bar{N}LTL\bar{T}N = \bar{N}TL\bar{T}LN = N^2L\bar{N}^3LN = A_2A_1,$$

by (1.1.b), (3.4.a) and (3.4.d). And if j = g,

$$A_1A_g = \bar{T}\cdot A_2A_1 \cdot T = \bar{T}\cdot A_1A_2 \cdot T = A_gA_1.$$

(A-2) It is equivalent to (3.4.a).

(A-3) If j ≠ 2 and g, it is obvious from (3.4.a-c). and if j = 2 or g, it is a consequence of (3.4.e).

(A-4) If j ≠ 2, it is obvious from (3.4.a) and (3.4.b). And if j = 2, it follows from (3.4.c) and the result for the case j ≠ 2, indeed,

$$A_1B_2 = MN^3L\bar{N}^3 = N^3A_2B_1\bar{N}^3 = N^3TA_1B_g\overline{TN}^3 =$$
$$N^3TB_gA_1\overline{TN}^3 = B_2A_1.$$

(A-5) It is exactly the formula (1.2.a), or equivalently (3.4.h).

(A-6) If j ≠ 2, again it is evident. If j = 2,

$$A_1Z_2 = MTZ\bar{T} = \bar{T}(TM\bar{T}\cdot T^2Z\bar{T}^2)T = \bar{T}(N^2L\bar{N}^2T^2Z\bar{T}^2)T$$
$$= \bar{T}(T^2Z\bar{T}^2N^2L\bar{N}^2)T = TZ\bar{T}M = Z_2A_1.$$

by (3.4.a-d).

(A-7) If j = 1, we have

$$
\begin{aligned}
Z_1A_1Z_1 &= \bar{M}\bar{L}NL\bar{N}LM \cdot M \cdot \bar{M}\bar{L}NL\bar{N}LM = \bar{M}\bar{L}\bar{M}NL\bar{N}MLM\bar{L}\bar{M}NL\bar{N}MLM \\
&= \bar{L}\bar{M}\bar{L}NL\bar{N}LNL\bar{N}LML = \bar{L}\bar{M}\bar{L}NLML\bar{N}LML = \bar{L}\bar{M}\bar{L}NMLM\bar{N}LML \\
&= \bar{L}\bar{M}NL\bar{N}L^2ML = \bar{L}\bar{N}\bar{L}N^2L\bar{N}L^2\bar{N}LNL = \bar{L}NL\bar{N}^2\bar{L}NL\bar{N}LNL\bar{N}LN \\
&= \bar{L}NL\bar{N}L\bar{N}L^2N = \bar{L}NL\bar{N}LM^2 = A_1Z_1A_1.
\end{aligned}
$$

by using the relations (3.4.g) and (3.4.h).

If j = g, we have

$$
\begin{aligned}
Z_gA_1Z_g &= \bar{T}\bar{M}\bar{L}NL\bar{N}LMT \cdot M \cdot \bar{T}\bar{M}\bar{L}NL\bar{N}LMT = \bar{T}\bar{M}\bar{L}NLNL\bar{N}L\bar{N}LMT \\
&= \bar{T}\bar{N}\bar{L}N\bar{L}N^2L\bar{N}LNL\bar{N}^2L\bar{N}LNT = \bar{T}\bar{N}\bar{L}N^3L\bar{N}^2\bar{L}NL\bar{N}LN^2L\bar{N}^3LNT \\
&= \bar{T}N^2L\bar{N}^3\bar{L}N\bar{L}NL\bar{N}L\bar{N}LN^3L\bar{N}^2T \\
&= \bar{T}N^2L\bar{N}^2T \cdot \bar{T}\bar{N}\bar{L}N\bar{L}NL\bar{N}L\bar{N}LNT \cdot \bar{T}N^2L\bar{N}^2T \\
&= M \cdot \bar{T}ZT \cdot M = A_1Z_gA_1.
\end{aligned}
$$

(A-8) If j ≠ 1 and g, it is easy from the relations of Proposition 3.4. If j = 1,

$$
B_1Z_1 = L\bar{M}\bar{L}NL\bar{N}LM = L\bar{M}\bar{L}\bar{M}NL\bar{N}MLM = \bar{M}\bar{L}NL\bar{N}MLM = Z_1B_1,
$$

by (1.2.a) and $M \leftrightarrow NL\bar{N}$. And if j = g,

$$
\begin{aligned}
B_1Z_g &= L\bar{T}\bar{M}\bar{L}NL\bar{N}LMT = \bar{T}N^3L\bar{N}^3\bar{M}\bar{L}NL\bar{N}LMT = \bar{T}\bar{M}\bar{L}N^3L\bar{N}^2L\bar{N}LMT \\
&= \bar{T}\bar{M}\bar{L}NLN^2L\bar{N}^3LMT = \bar{T}\bar{M}\bar{L}NL\bar{N}LMN^3L\bar{N}^3T = Z_gB_1.
\end{aligned}
$$

Actually, we have more interesting relations:

Proposition 3.6. *The following formulas are the consequences of the relations given in Propositions* 1.1, 1.2 *and* 3.4:

(a) $(NL\bar{N}L\bar{N}LN)^4 = N^3L^2(\bar{L}N)^5(L\bar{N})^5\bar{N}^3$.

(b) $(\bar{N}LNL\bar{N})^4 = (N\bar{L})^5(\bar{N}L)^5\bar{L}^2$.

(c) $L \leftrightarrow (\bar{L}N)^5(N\bar{L})^5$.

(d) $TL\bar{T} \leftrightarrow (\bar{L}N)^5$, *and* $TM\bar{T} \leftrightarrow (\bar{L}N)^5$.

Proof. (d)

$$TL\bar{T}(L\bar{N})^5 = \bar{N}^3LN^3 \cdot L \cdot \bar{N}LN \cdot \bar{N}^2L\bar{N}LN \cdot \bar{N}^2LN^2 \cdot \bar{N}^3$$
$$= L\bar{N}L\bar{N}^2LNL\bar{N}LN \cdot \bar{N}^2LN^2 \cdot \bar{N}^3$$
$$= L\bar{N}L\bar{N}L\bar{N}LNL \cdot \bar{N}^2LN^2 \cdot \bar{N}^3 = (L\bar{N})^3N^3L\bar{N}^3 =$$
$$(L\bar{N})^5TL\bar{T}.$$

And similarly we may obtain the other one.

(b) & (c) We show both at the same time. Since

$$(\bar{N}LNL\bar{N})^4 = (\bar{N}L)NL(\bar{N}^2LN^2)\bar{N}L\bar{N}^2L\bar{N}(N^2L\bar{N}^2)LN(L\bar{N})$$
$$= (\bar{N}L)^2N^2L\bar{N}L(\bar{N}^2L\bar{N}LN^3)\bar{N}(L\bar{N})^2$$
$$= (\bar{N}L)^2N^2L(\bar{N}^3L\bar{N}LN^4)\bar{N}(L\bar{N})^3$$
$$= (\bar{N}L)^4N^4(L\bar{N})^4$$
$$= \bar{L}(L\bar{N})^5N^6(\bar{N}L)^5\bar{L}$$
$$= \bar{L}(N\bar{L})^5(\bar{N}L)^5\bar{L}, \text{ by (1.2.b)},$$

since $L \leftrightarrow \bar{N}LN\bar{N}$ by (1.2.a), the formula (c) is evident. And the above calculation together with (c) shows directly the formula (b).

(a) By the formula (b), it is enough to show that,

$$(NL\bar{N}L\bar{N}LN)^4 = N^3(N\overline{\bar{L}N\bar{L}}N)^4\bar{N}^3.$$

Actually,

$$N^3(N\overline{\bar{L}N\bar{L}}N)^4\bar{N}^3 = N^3(\bar{N}^4LNLNLN^2)^4\bar{N}^3$$
$$= \bar{N}LNLNL\bar{N}^2LNLNL\bar{N}^2LNLNL\bar{N}^2LNLNL\bar{N}$$
$$= \bar{N}LNL\bar{N}LN^2L\bar{N}L\bar{N}LN^2L\bar{N}L\bar{N}LN^2L\bar{N}LNL\bar{N}$$
$$= MLM(NL\bar{N}L\bar{N}LN)^3\bar{N}\bar{L}N^2L\bar{N}$$
$$= MLM(NL\bar{N}L\bar{N}LN)^3NL\bar{N}^2\bar{L}N$$
$$= MLM(NL\bar{N}L\bar{N}LN)^4\overline{MLM}$$
$$= (NL\bar{N}L\bar{N}LN)^4, \text{ by (1.2.a) and (3.4.a - c)}.$$

Proposition 3.7. *The relation* (B) *of Theorem* 3.3 *is a consequence of the relations in Propositions* 1.1, 1.2 *and* 3.4.

Proof. Since $B_1A_1Z_1 = L \cdot M \cdot \overline{M}\overline{L}NL\overline{N}LM = NL\overline{N}L\overline{N}LN$, and

$$A_2Z_1A_1B_1^2A_1Z_1A_2 = N^3M\overline{N}^3 \cdot \overline{M}\overline{L}NL\overline{N}LM \cdot M \cdot L^2 \cdot M \cdot \overline{M}\overline{L}NL\overline{N}LM \cdot N^3M\overline{N}^3$$

$$= \overline{M}\overline{L} \cdot N^2L\overline{N}^2 \cdot NL\overline{N} \cdot LM^2L \cdot NL\overline{N} \cdot N^2L\overline{N}^2 \cdot LM, \text{ by (3.4.d)}$$

$$= \overline{M}\overline{L}(N^2L\overline{N}L\overline{N}L\overline{N}L^2NLNLNL\overline{N}^2)LM$$

$$= \overline{M}\overline{L} \cdot N^3((\overline{N}L)^4(LN)^4)\overline{N}^3 \cdot LM, \text{ by (3.4.a-c)}$$

$$= \overline{M}\overline{L} \cdot N^3((\overline{N}L)^4\overline{N}\overline{L}N^6)\overline{N}^3 \cdot LM, \text{ by } (LN)^5 = N^6.$$

$$= \overline{M}\overline{L} \cdot N^3((\overline{N}L)^5\overline{L}^2)\overline{N}^3 \cdot N^6 \cdot LM$$

$$= N^3(\overline{N}L)^5\overline{L}^2N^3, \text{ by (3.6.d)}.$$

The proposition is clear by comparing with (3.6.a).

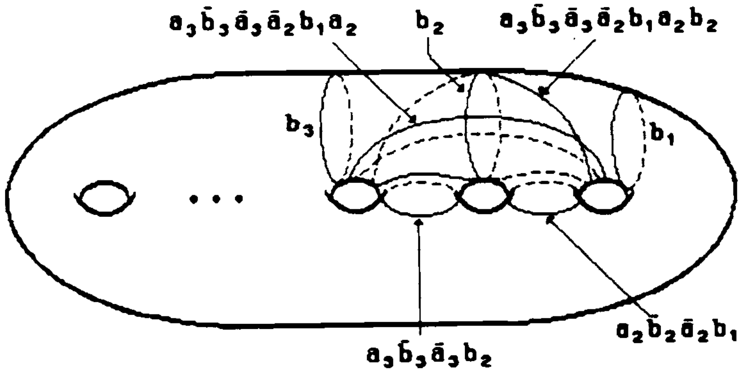


Figure 3.2

Now we introduce a notation. We denote the conjugate $\beta\alpha\beta^{-1}$ simply by $\langle\beta\rangle\alpha$. The relation (C), called *the lantern law*, is a special relation for the mapping class group M_g of genus $g \geq 3$, and is a composition of seven Dehn twists, which can be chosen arbitrarily up to conjugacy. Here, we let the seven twist curves be the following:

$$b_1,\ a_2\bar{b}_2\bar{a}_2b_1,\ a_3\bar{b}_3\bar{a}_3b_2,\ b_3,\ b_2,\ a_3\bar{b}_3\bar{a}_3\bar{a}_2b_1a_2,$$
$$a_3\bar{b}_3\bar{a}_3\bar{a}_2b_1a_2b_2,$$

as pictured in Figure 3.2. Since

$$\overline{NT}(N\overline{LM})^4\bar{N}^2P(b_1) = a_3\bar{b}_3\bar{a}_3\bar{a}_2b_1a_2,$$

and

$$\overline{NT}NP(b_1) = a_3\bar{b}_3\bar{a}_3\bar{a}_2b_1a_2b_2,$$

the formula (C) is equivalent to

$$\langle T\rangle L\cdot\langle\bar{P}N^2(ML\bar{N})^3TN\rangle L\cdot\langle\overline{PN}TN\rangle L =$$
$$L\cdot\langle\bar{P}N\rangle L\cdot\langle T\bar{P}N\rangle L\cdot\langle T^2\rangle L.$$

Since $P = LML$ commutes with $TL\bar{T}$, $T^2L\bar{T}^2$, $TM\bar{T}$ and $TN\bar{T}$ by the formulas (3.4.a-d), and since $PL\bar{P} = M = \bar{N}LN$, conjugating the above formula by P, we have

Proposition 3.8. *The lantern law* (C) *is equivalent to*

$$\langle T\rangle L\cdot\langle N^2(ML\bar{N})^3TN\rangle L\cdot\langle\bar{N}TN\rangle L =$$
$$\langle\bar{N}\rangle L\cdot\langle N\rangle L\cdot\langle T\bar{P}N\rangle L\cdot\langle T^2\rangle L.$$

According to Wajnryb's work ([11]), the relation (D) is special for a closed surface. Moreover, we may consider L, N and T as mapping classes of the orientable surface $F_{g,1}$ of genus g with one boundary component

$\beta = c_1 c_2 \ldots c_g$, as pictured in Figure 3.3, which form a system of generators of the group $M_{g,1}$, and relations (A), (B) and (C) give a complete presentation of it. Therefore, we may replace (D) by any relations which generate the kernel of the quotient map from $M_{g,1}$ to M_g.

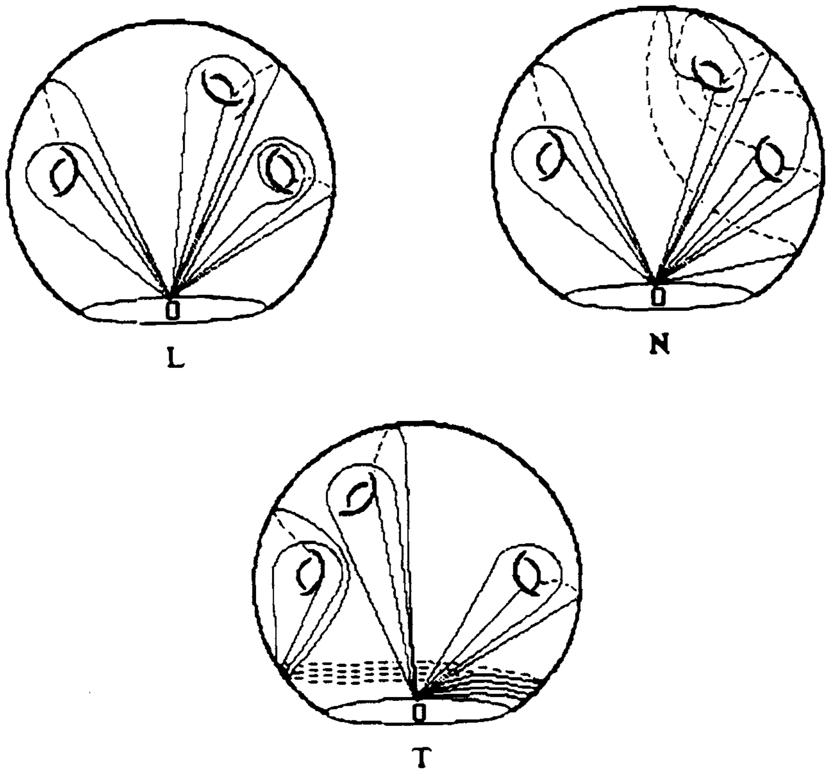


Figure 3.3

The Dehn twist along the curve β is clearly equal to T^g. And sliding β along some nonseparating curve, e.g. b_1, is given by $(\bar{T}N^3L^3(\bar{N}LNL\bar{N})^4)^{g-1}$, since

$$\bar{T}N^3(N\bar{L})^5(\bar{N}L)^5L = \bar{T}N^3L^3(\bar{N}LNL\bar{N})^4$$

$$= [\bar{b}_1\bar{c}_2b_1a_1, b_1, a_3, b_3, \ldots, a_g, b_g, \bar{b}_1a_2b_1, \bar{b}_1b_2b_1],$$

$$(\bar{T}N^3L^3(\bar{N}LNL\bar{N})^4)^{g-1}$$
$$= [\bar{b}_1\bar{c}_g\ldots\bar{c}_2\bar{c}_1a_1b_1, b_1, \bar{b}_1a_2b_1, \bar{b}_1b_2b_1, \ldots, \bar{b}_1a_gb_1, \bar{b}_1b_gb_1].$$

These two classes form a set of normal generators for the kernel of the quotient map from $M_{g,1}$ onto M_g, since all mapping classes obtained by sliding β along some nonseparating simple closed curve are conjugate each other, and since all mapping classes obtained by sliding β along some separating simple closed curve are composed by those along nonseparating curves, in particular along the basecurves, it is enough to replace the formula (D) by the formulas

$$T^g = 1, \text{ and } (\bar{T}N^3L^3(\bar{N}LNL\bar{N})^4)^{g-1} = 1.$$

Putting all together we have a new presentation of the surface mapping class group in our generators:

Theorem 3.9. *The mapping class group* M_g *of the closed orientable surface of genus* g, $g \geq 3$, *has a presentation of three generators: the linear cutting* L, *the normal cutting* N *and the transport* T, *and* 3g + 4 *relations:*

(I) $L \leftrightarrow \langle T^i \rangle L$, $i = 1,2,\ldots,g-1$,
$L \leftrightarrow \langle T^i \rangle N$, $i = 1,2,\ldots,g-2$,
$L \leftrightarrow N^6$, $L \leftrightarrow \langle \bar{N}T\bar{N}(\bar{L}N)^2 \rangle L$,

(II) $N \leftrightarrow \langle T^i \rangle N$, $i = 2,3,\ldots,g-2$,

(III) $\langle T \rangle L = \langle N^3 \rangle L$, $\langle T\bar{N} \rangle L = \langle N^2 \rangle L$,

(IV) $(LN)^5 = N^6$, $(\bar{L}N)^{10} = N^6$,

(V) $\langle T\rangle L\cdot\langle N^2(\bar{N}LNL\bar{N})^3TN\rangle L\cdot\langle\bar{N}TN\rangle L =$

$= \langle\bar{N}\rangle L\cdot\langle N\rangle L\cdot\langle T\bar{N}(\bar{L}N)^2\rangle L\cdot\langle T^2\rangle L,$

(VI) $T^g = 1,\ (N^3T)^{g-1} = (L\bar{N}LN)^{6(g-2)},$

and $(\bar{T}N^3L^3(\bar{N}LNL\bar{N})^4)^{g-1} = 1.$

All above formulas are collected from earlier discussions, though they may be slightly different, in fact,

$$\langle\bar{N}T\bar{N}(\bar{L}N)^2\rangle L = \langle\bar{N}T\overline{ML}N\rangle L,$$

and $M = \bar{N}LN$.

4. Some Applications

By using the new system of generators, some of the properties of the mapping class group M_g can be easily shown.

i) *Abelianization* $\mathrm{Ab}(M_g)$ of M_g

Denote by $\mathrm{Ab}(M_g)$ the abelianization of the mapping class group M_g, for $g \geq 1$, which was determined first by Birman [1] and Powell [10]. Here we may reprove their result easily from the relations we have got.

Theorem 4.1. $\mathrm{Ab}(M_1) = \mathbb{Z}_{12}$, $\mathrm{Ab}(M_2) = \mathbb{Z}_{10}$, *and* $\mathrm{Ab}(M_g) = 0$, *for* $g \geq 3$.

Proof. When genus $g = 1$, the mapping class group $M_1 = SL_2(\mathbb{Z})$ has a presentation

$$M_1 = \langle L,N;\ LN = N^2\bar{L},\ \text{and } N^6 = 1.\rangle,$$

where the operation N is slightly different, defined as $N = [b_1, b_1\bar{a}_1]$, since we do not have the second handle. Thus,

$$\mathrm{Ab}(M_1) = \langle L,N;\ L^2 = N,\ \text{and } N^6 = 1.\rangle = \mathbb{Z}_{12}.$$

When $g = 2$, we recall the presentation of M_2 given in [8],

$$M_2 = \left\langle L,N;\ \begin{matrix}(LN)^5 = 1,\ (L\bar{N})^{10} = 1,\ N^6 = 1,\ \text{and} \\ \text{some commutativity relations.}\end{matrix} \right\rangle.$$

Then,

$$\begin{aligned} Ab(M_2) &= \langle L,N;\ L \leftrightarrow N,\ L^5 = N^5,\ L^{10} = N^{10},\ \text{and } N^6 = 1.\rangle \\ &= \langle L,N;\ L^5 = N,\ \text{and } N^2 = 1.\rangle = \mathbb{Z}_{10}. \end{aligned}$$

And for $g \geq 3$, the formula $(LN)^5 = N^6$ implies $N = L^5$ in $Ab(M_g)$, the formulas $T^g = 1$ and $(N^3T)^{g-1} = (L\bar{N}LN)^6 g - 2$ imply $T = L^{15(g-1)-12(g-2)} = L^{3g-9}$ in $Ab(M_g)$, and the lantern law (3.9.V) implies $L = 1$ in $Ab(M_g)$. Thus $Ab(M_g) = 1$.

Similarly, for the homeotopy groups, we have that,

$$Ab(\tilde{M}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ Ab(\tilde{M}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \text{ and}$$
$$Ab(\tilde{M}_g) = \mathbb{Z}_2,\ g \geq 3,$$

by using the relations $RLR = \overline{NLN}$, $RNR = \overline{TNT}$ and $R^2 = 1$.

ii) *The Torelli subgroup I_g of M_g*

Let $\lambda: M_g \to Sp(2g,\mathbb{Z})$ be the natural homeomorphism defined so that, for each mapping class f of M_g, the element $\lambda(f)$ is the automorphism of the group $H_1(F_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ induced by f. We will call the normal subgroup $I_g = \ker\lambda$ the Torelli subgroup of M_g.

The first set of normal generators of the group I_g was given by Birman [2] in Lickorish's Dehn twists. Powell reduced to three maps: a Dehn twist along a null-homologous curve which splits one handle from the others, a Dehn twist along a null-homologous curve which splits two handles from the others, and twists along a pair of disjoint homologous (not homotopic) curves in which none of them is null-homologous in the surface.

Here we may write them easily in our generators.

Theorem 4.2. *The Torelli group* I_g *is a normal subgroup generated by* $P^4 = (L\bar{N}LN)^6$, N^6, *and* $(N\bar{L})^5(\bar{N}L)^5$.

Proof. The mapping classes P^4 and N^6 are exactly the Dehn twists along the curves $c_1 = [a_1,b_1]$ and $x = [a_1,b_1][a_2,b_2]$. And we may write the last one in the following way,

$$(N\bar{L})^5(\bar{N}L)^5 = L \cdot (\bar{L}N)^5\bar{L}(\bar{N}L)^5,$$

and clearly it is a composition of the Dehn twists along the circle b_1 and the circle $(\bar{L}N)^5(b_1) = \bar{b}_1\bar{x}$.

iii) *The automorphism group* $\mathrm{Aut}(M_g)$ *of* M_g

Let $\mathrm{Aut}(M_g)$ and $\mathrm{Aut}(\tilde{M}_g)$ denote the automorphism groups of the mapping class group M_g and the homeotopy group $\tilde{M}_g$ respectively. Let $\mathrm{Inn}(M_g)$ and $\mathrm{Inn}(\tilde{M}_g)$ denote their corresponding inner-automorphism normal subgroups. And let $\mathrm{Out}(M_g)$ and $\mathrm{Out}(\tilde{M}_g)$ be their quotients. McCarthy and Ivanov [9] proved that,

Theorem 4.3. *The short exact sequences*

$$1 \to \mathrm{Inn}(M_g) \to \mathrm{Aut}(M_g) \to \mathrm{Out}(M_g) \to 1,$$

and

$$1 \to \mathrm{Inn}(\tilde{M}_g) \to \mathrm{Aut}(\tilde{M}_g) \to \mathrm{Out}(\tilde{M}_g) \to 1,$$

are split, and

i) $\mathrm{Out}(M_g) = \mathbb{Z}_2$, *and* $\mathrm{Out}(\tilde{M}_g) = 1$, *for* $g \geq 3$,

ii) $\mathrm{Out}(M_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, *and* $\mathrm{Out}(\tilde{M}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Here we give those outer-automorphisms explicitly.

Proposition 4.4. i) *The group* $\mathrm{Out}(M_2)$ *is generated by,*

$\rho: M_2 \to M_2$, $\rho(L) = RLR = \bar{M}$, *and* $\rho(N) = RNR = \bar{N}$,

and $\kappa: M_2 \to M_2$, $\kappa(L) = LK$, *and* $\kappa(N) = NK$, *where* $K = T\bar{P}^2TP^2$.

(Remark: $L \nleftrightarrow K$, $N \nleftrightarrow K$, and $K^2 = 1$.)

ii) *The group* $\mathrm{Out}(\tilde{M}_2)$ *is generated by,*

$\kappa_1: \tilde{M}_2 \to \tilde{M}_2$, $\kappa_1(L) = LK$, $\kappa_1(N) = NK$, *and* $\kappa_1(R) = R$,

and $\kappa_2: \tilde{M}_2 \to \tilde{M}_2$, $\kappa_2(L) = LK$, $\kappa_2(N) = NK$, *and* $\kappa_2(R) = RK$.

iii) *The group* $\mathrm{Out}(M_g)$, *for* $g \geq 3$, *is generated by,*

$\rho: M_g \to M_g$, $\rho(L) = RLR = \bar{M}$, $\rho(N) = RNR = \bar{N}$, *and*

$\rho(T) = RTR = \bar{T}$.

The idea to prove Theorem 4.3 is to show that, any automorphism of the mapping class group M_g maps a Dehn twist to a Dehn twist, for $g \geq 3$, by using the result of Birman-Lubotzky-McCarthy [3] about abelian subgroups of M_g. Thus, we can have only one nontrivial outer automorphsm ρ (modulo inner automorphisms) which maps a Dehn twist to some Dehn twist with reversing twist orientation, in particular ρ can be chosen as in Proposition 4.4.

When $g = 2$, it is slightly different, we have one element of order two $K = \bar{N}^3\bar{P}^2N^3P^2 \in [M_2, M_2]$ commuting with all mapping classes. Since $\mathrm{Ab}(M_2) = \mathbb{Z}_{10}$, multiplying K to the mapping classes whose image in the abelianization is odd, we obtain a nontrivial outer automorphism κ. Since $Ab(\tilde{M}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $R\kappa = \kappa R$, we have two different extensions κ_1 and κ_2 of κ as shown in Proposition 4.4. According to McCarthy and Ivanov, there is no more.

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