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by

Ning Lu

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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HOMEOMORPHISMS OF A SOLID HANDLEBODY AND HEEGAARD SPLITTINGS OF THE 3-SPHERE S^3

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Let $\mathbf{H}_g$ be a solid handlebody of genus g, with boundary $\partial H_g = \mathbf{F}_g$. Let $\mathcal{B} = \{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ be a fixed system of basecurves based at a common basepoint O, such that the a_i's are meridian circles of $\mathbf{H}_g$. Let $\mathcal{M}g$ denote the mapping class group of the closed orientable surface $\mathbf{F}_g$. And let $\mathcal{K}_g$ denote the subgroup of $\mathcal{M}_g$ consisting of mapping classes induced by some homeomorphism of the handlebody $\mathbf{H}_g$. An element of $\mathcal{K}_g$ will be called an *extendible mapping class*.

The subgroup $\mathcal{K}_g$ plays a very important role in Heegaard splitting of 3-manifolds (Cf.[1] & [8]). In this paper, we describe this subgroup explicitly by giving a finite set of generators in the first two sections. Comparing to Suzuki's generators [7], not only is the number of generators one less, but also the expressions in the generators of the mapping class group $\mathcal{M}_g$ are quite easy. In the third section, all Heegaard splittings of the 3-sphere $\mathbb{S}^3$ are explicitly given, this was asked in Hempel's book ([3] p. 164).

1. Some Extendible Mapping Classes

First we are going to give some extendible mapping classes, show they generate the group $\mathcal{K}_g$, then reduce the number by using the technique given in the papers [4] and [5].

Recall that the mapping class group M_g is generated by three elements: *the linear cutting* L, *the normal cutting* N and *the transport* T. Algebraically, they are given by

$$L = [a_1b_1, a_2, b_2, \ldots, a_g, b_g],$$

$$N = [x\bar{a}_2b_1, \bar{a}_1, \bar{a}_1xb_2, \bar{a}_2, a_3, b_3, \ldots, a_g, b_g],$$

where $x = [a_1, b_1][a_2, b_2]$, and

$$T = [a_g, b_g, a_1, b_1, \ldots, a_{g-1}, b_{g-1}].$$

We also denote by $M = \bar{N}LN$ *the meridian cutting*, $P = LML = MLM$ *the parallel cutting*, $Q = TP\bar{T} = N^3P\bar{N}^3$ the parallel cutting of the second handle, $c_i = [a_i, b_i]$ the *waist* of the i-th handle, and $x = c_1c_2$ the waist of the first two handles.

Now we list some elementary extendible mapping classes.

1) The *meridian cutting* M, given by

$$M = [a_1, b_1\bar{a}_1, a_2, b_2, \ldots, a_g, b_g].$$

2) The *transport* T.

3) The *handle rotation* ϕ, (Figure 1.1), given by

$$\phi = [c_1\bar{a}_1, \bar{b}_1\bar{c}_1, a_2, b_2, \ldots, a_g, b_g],$$

is obtained by a 180^o-rotation of the first handle along its waist circle c_1.

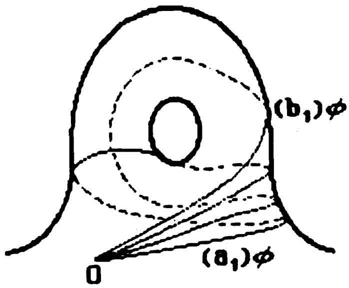


Figure 1.1 Handle rotation

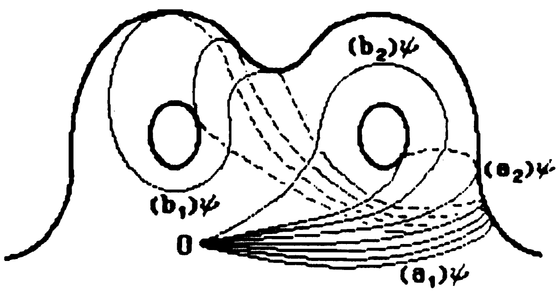


Figure 1.2 Handle switching

4) The *handle switching* ψ, (Figure 1.2), given by

$$\psi = [c_1 a_2 \bar{c}_1, c_1 b_2 \bar{c}_1, a_1, b_1, a_3, b_3, \ldots],$$

is obtained by moving the second handle around the first handle into the position in front of the first one.

5) The *handle rounding* σ, (Figure 1.3), given by

$$\sigma = [a_1, b_1 \bar{a}_1 \bar{b}_1 c_2 b_1 a_1, \bar{a}_1 c_1 a_2 \bar{c}_1 a_1, \bar{a}_1 c_1 b_2 \bar{c}_1 a_1, a_3, b_3, \ldots],$$

is obtained by moving one foot of the first handle around the second one.

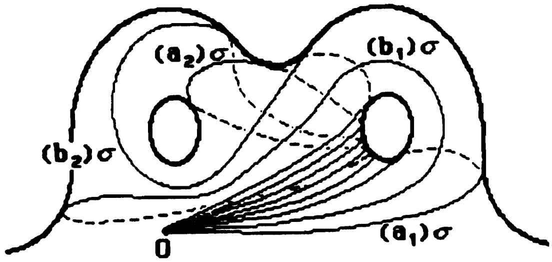


Figure 1.3 Handle rounding σ

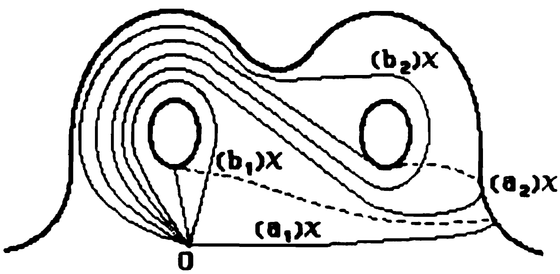


Figure 1.4 Handle crossing χ

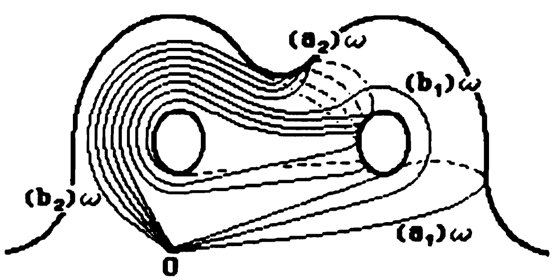


Figure 1.5 One-foot sliding ω

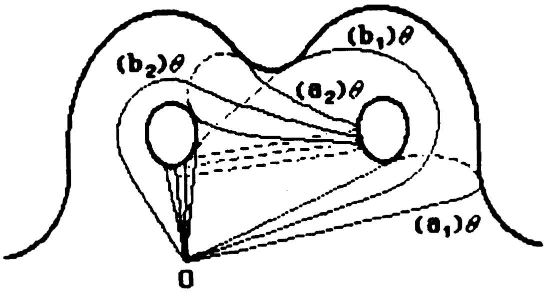


Figure 1.6 Handle knotting

6) The *handle crossing* χ, (Figure 1.4), given by

$$\chi = [c_1a_2, b_2, b_2a_1\bar{b}_2, b_2b_1\bar{b}_2, a_3, b_3, \ldots],$$

is obtained by sliding the whole first handle along the longitude circle b_2 of the second handle.

7) The *one-foot sliding* ω, (Figure 1.5), given by

$$\omega = [a_1, b_2b_1, b_2\bar{c}_1a_1\bar{a}_2\bar{a}_1c_1a_2b_2\bar{a}_1c_1\bar{b}_2, b_2\bar{c}_1a_1b_2\bar{a}_1c_1\bar{b}_2, a_3, b_3, \ldots],$$

is obtained by sliding one foot of the first handle along the longitude circle b_2 of the second handle.

8) The *one-foot knotting* θ, (Figure 1.6), given by

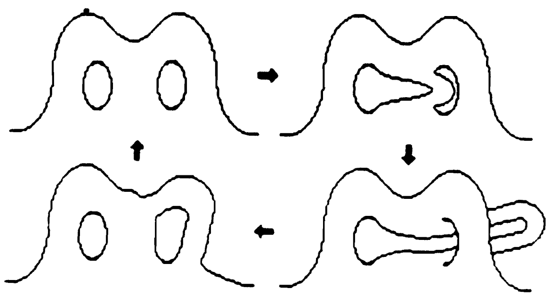
$$\theta = [a_1, \bar{a}_2b_1, \bar{a}_2\bar{c}_1a_1a_2\bar{a}_1c_1a_2, b_2\bar{a}_1c_1a_2, a_3, b_3, \ldots],$$

is obtained by moving one foot of the first handle along the meridian circle a_2 of the second handle.

9) The *handle replacing* η, (Figure 1.7), given by

$$\eta = [\bar{a}_1c_1a_2, \bar{a}_2\bar{c}_1\bar{b}_1a_2, \bar{a}_2\bar{c}_1\bar{b}_1a_2b_1c_1a_2, b_2b_1c_1a_2, a_3, b_3, \ldots],$$

is obtained by replacing the first handle with the cylinder between the first and the second handles.



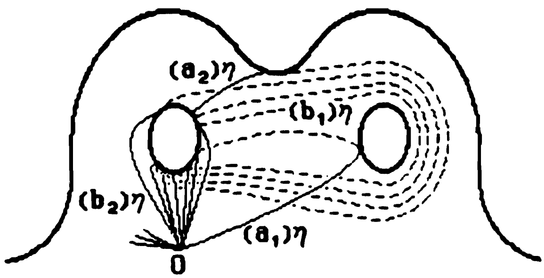


Figure 1.7 Handle replacing η

Remark. By definition, among all elementary extendible mapping classes, the operations T, ϕ, ψ, σ, χ, ω and η can be obtained by an isotopy deformation of $\mathbf{S}^3$ (i.e., obtained by moving the handlebody $\mathbf{H}_g$ inside of $\mathbf{S}^3$ without cutting it open). And the operation θ is a combination of ω and meridian twists, which can be obtained in the following way: pass the left foot of the first handle along the longitude $\bar{b}_2$ of the second handle in the anticlockwise way, twist the second handle along its meridian,

pull back the foot of the first handle along the new longitude, and adjust the longitude b_1 by a twist along the meridian of the first handle. Precisely,
$\theta = \omega \cdot T\overline{M}T \cdot \overline{\omega} \cdot M.$

Theorem 1.1. *In the mapping class group* M_g, *we have the following expressions:*

(i) $\phi = P^2 = (LM)^4,$

(ii) $\psi = \overline{P}^4N^3 = N^3\overline{Q}^4,$

(iii) $\sigma = \overline{P}(\overline{L}N)^5(L\overline{N})^5P\overline{Q}^4 = (\overline{P}QN)^2\overline{Q}^4,$

(iv) $\chi = \overline{L}(\overline{N}L)^5(N\overline{L})^5N^3 = (QN\overline{P})^2\overline{N}^3,$

(v) $\omega = Q^3\overline{NQ}^2P,$

(vi) $\theta = Q^2\overline{NQ}P = \overline{Q}\omega Q,$

(vii) $\eta = \overline{P}Q^2\overline{NQ} = \overline{P}\theta\overline{P}.$

Proof. The expressions are found by using the algorithm given in [4] and [5], which certainly was not easy. After the formulas have been discovered, the proof is just an immediate verification.

For example, for (iii), we know that,

$$(\overline{L}N)^5 = [x\overline{a}_1,\overline{b}_1\overline{x},\overline{a}_1\overline{b}_1c_2\overline{a}_2b_1a_1,\overline{a}_1\overline{b}_1\overline{b}_2\overline{c}_2b_1a_1, a_3,b_3,\ldots],$$

thus $$(\overline{N}L)^5 = [\overline{a}_1x,\overline{xb}_1,\overline{xa}_1\overline{b}_1c_2\overline{a}_2b_1a_1x,\overline{xa}_1\overline{b}_1\overline{b}_2\overline{c}_2b_1a_1x, a_3,b_3,\ldots],$$

and $$(L\overline{N})^5 = [\overline{a}_1c_1,\overline{xb}_1,\overline{xa}_1x\overline{a}_2\overline{c}_1a_1x,\overline{xa}_1c_1\overline{b}_2\overline{x}a_1x, a_3,b_3,\ldots],$$

then $$(\overline{L}N)^5(L\overline{N})^5 = [x\overline{c}_1a_1,b_1,b_1xa_2\overline{xb}_1,b_1xb_2\overline{xb}_1, a_3,b_3,\ldots].$$

Because $P = [c_1b_1, \bar{a}_1, a_2, b_2, \ldots]$,

and $\bar{Q}^4 = [a_1, b_1, \bar{c}_2a_2c_2, \bar{c}_2b_2c_2, a_3, b_3, \ldots]$,

$$\bar{P}(\bar{L}N)^5(L\bar{N})^5P\bar{Q}^4 = \bar{P}[xb_1, \bar{a}_1, \bar{a}_1c_1a_2\bar{c}_1a_1, \bar{a}_1c_1b_2\bar{c}_1a_1, a_3, b_3, \ldots]$$
$$= [a_1, \bar{a}_1xb_1a_1, \bar{a}_1c_1a_2\bar{c}_1a_1, \bar{a}_1c_1b_2\bar{c}_1a_1, a_3, b_3, \ldots] = \sigma.$$

Also we have,

$$\bar{P}QN = [a_1, \bar{a}_1x\bar{a}_2b_1a_1, \bar{a}_1c_1c_2\bar{a}_2\bar{c}_1a_1, \bar{b}_2\bar{x}a_1, a_3, b_3, \ldots]$$

and $(\bar{P}QN)^2 = [a_1, \bar{a}_1xb_1a_1, \bar{a}_1xa_2\bar{x}a_1, \bar{a}_1xb_2\bar{x}a_1, a_3, b_3, \ldots]$,

Clearly, $\sigma = \bar{Q}^4(\bar{P}QN)^2 = (\bar{P}QN)^2\bar{Q}^4$. And similarly we can prove the other formulas.

2. Generators of the Subgroup K_g

In this section, we are going to prove that,

Theorem 2.1. *The extendible mapping class subgroup K_g of the surface mapping class group M_g is generated by five elements:*

$$M,\ T,\ N^3,\ P^2,\ \textit{and}\ \ PN^2P,$$

and also by the five elements:

$$T,\ N^3,\ \bar{N}LN,\ NLN^2\bar{L},\ \textit{and}\ L\bar{N}L^2NL.$$

Regard the handlebody H_g as the down-semispace of the Euclidean space $\mathbf{E}^3$ with g pairs of holes on its boundary F_g identified (Figure 2.1). Instead of the basecurves $\mathcal{B} = \{a_i, b_i\}_{1\leq i\leq g}$, we will study the basearcs $\mathcal{B} = \{p_i, q_i, r_i\}_{1\leq i\leq g}$, where as joining the oriented arcs, we have

$$a_i \sim p_i r_i \bar{p}_i, \quad b_i \sim q_i \bar{p}_i,$$

for $i = 1,2,\ldots,g$.

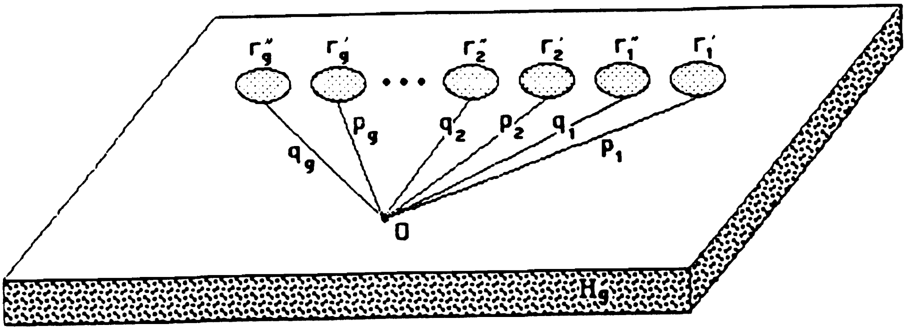
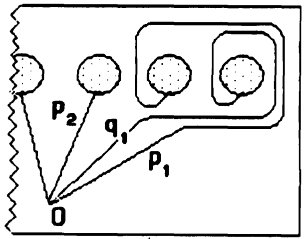


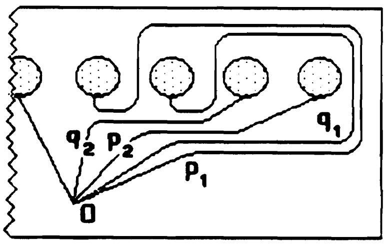
Figure 2.1

The disk holes, denoted by D_i's, will be chosen as *the meridian disks*, which form a *cutting system* of the handlebody H_g. Their boundary circles, r_i's, are the fixed *meridian circles*. In the plane in Figure 2.1, the disks D_i and the circles r_i are split in two. We will denote by D_i' and D_i'' the two copies of D_i, denote by $r_i' = \partial D_i'$ and $r_i'' = \partial D_i''$ the two copies of r_i, and call them *the cutting disks* and *cutting circles* respectively. Moreover, we also suppose that r_i' contains an endpoint of p_i and r_i'' contains one of q_i.

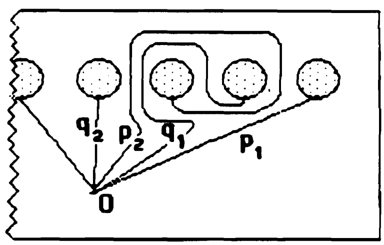
We call this new description *the planar representation of* F_g. Using it, a mapping class of the surface F_g may be drawn easily in the plane. For example, the mapping classes ϕ, ψ and θ are drawn in Figure 2.2, and it is quite easy to understand how they move the feet of handles.



ϕ



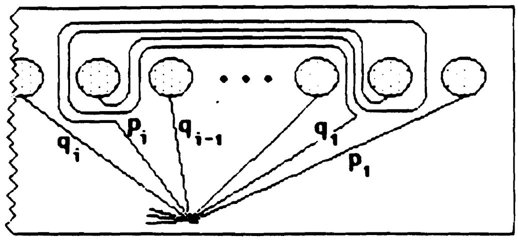
ψ



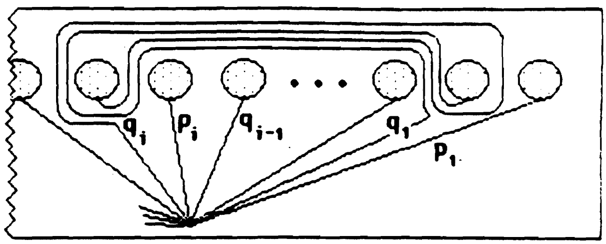
θ

Figure 2.2

Using ϕ, ψ, θ and T, we construct some more elementary movings of handle-feet. A family of mapping classes, called the *i-th foot knotting* Θ_i and the *i-th foot knotting* Θ_i', is defined by moving the foot r_1'' of the first handle along the meridian circle a_i and the meridian circle $b_i\bar{a}_i\bar{b}_i$, i.e. r_i' and r_i'', respectively, (see Figure 2.3). Therefore, $\Theta_i' = \phi_i\Theta_i\bar{\phi}_i$, where $\phi_i = T^{i-1}\phi\bar{T}^{i-1}$, for $i = 2,\ldots,g$. Precisely, we have



θ_i



θ_i'

Figure 2.3

Proposition 2.2. *The* i*-th foot knotting and* $\bar{\text{i}}$*-th foot knotting are generated by the mapping classes* T, ϕ, ψ, *and* θ. *Furthermore, they have the following expressions:*

$$\theta_i = T^{i-2}(\overline{\psi T})^{i-2}\theta(T\psi)^{i-2}\overline{T}^{i-2},$$

and

$$\theta_i' = T^{i-1}\overline{\phi T}(\overline{\psi T})^{i-2}\theta(T\psi)^{i-2}T\phi\overline{T}^{i-1}.$$

Proof. By Figure 2.3,

$$\theta_i = [a_1, c_2\cdots c_{i-1}\bar{a}_i\bar{c}_{i-1}\cdots\bar{c}_2 b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, \bar{c}_{i-1}\cdots\bar{c}_2\bar{a}_1 c_1 c_2\cdots c_{i-1}a_i, \bar{c}_{i-1}\cdots \bar{c}_2\bar{c}_1 a_1 c_2\cdots c_{i-1}, b_i\bar{c}_{i-1}\cdots\bar{c}_2\bar{a}_1 c_1 c_2\cdots c_{i-1}, a_{i+1}, b_{i+1}, \ldots].$$

$$\theta_i' = [a_1, c_2\cdots c_{i-1}c_i\bar{a}_i\bar{c}_{i-1}\cdots\bar{c}_2 b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_i, \bar{a}_i\bar{c}_{i-1}\cdots\bar{c}_2\bar{c}_1 a_1 c_2\cdots c_{i-1}a_i b_i, a_{i+1}, b_{i+1}, \ldots].$$

Then, a direct calculation implies the proposition. For example, let us compute Θ_i. Let $\psi_j = T_{j-1}\psi\bar{T}^{j-1}$ and $z = c_2c_3\ldots c_{i-1}$, then

$$\psi_2\ldots\psi_{i-1} = [a_1,b_1,za_i\bar{z},zb_i\bar{z},a_2,b_2,\ldots,a_{i-1},b_{i-1}, a_{i+1},b_{i+1},\ldots],$$

$$\theta\psi_2\ldots\psi_{i-1} = [a_1,z\bar{a}_i\bar{z}b_1,z\bar{a}_i\overline{zc}_1a_1za_i\overline{za}_1c_1z\bar{a}_i\bar{z}, zb_i\overline{za}_1c_1za_i\bar{z},a_2,b_2,\ldots,a_{i-1},b_{i-1},a_{i+1}, b_{i+1},\ldots],$$

and

$$\bar{\psi}_{i-1}\ldots\bar{\psi}_2\theta\psi_2\ldots\psi_{i-1} = [a_1,z\bar{a}_i\bar{z}b_1,a_2,b_2,\ldots,a_{i-1},b_{i-1}, \bar{a}_i\overline{zc}_1a_1za_i\overline{za}_1c_1z\bar{a}_i,b_i\overline{za}_1c_1za_2,a_2,b_2, a_{i+1},b_{i+1},\ldots].$$

Now we want to start proving that the elementary extendible mapping classes generate the group K_g.

Let f be an extendible mapping class, i.e. an element of K_g. The idea is to find another extendible mapping class g generated by our generators, such that either gf or fg becomes "simpler" than f. The process will be repeated until the identity map is obtained.

Lemma 2.3. *Let* α *be an oriented simple arc on the surface* $\mathbf{F}_g$ *from the basepoint* O *to the endpoint* Q *of* q_1 *at* r_1'', *which does not intersect any of the meridian circles* r_i *for all* i, *and does not intersect any of the arcs* p_j *and* q_j *for* $j \geq s$. *Then, there exists a self-homeomorphism* g *whose homeotopy class is generated by the classes* T, M, ϕ, ψ, *and* θ, *such that* $(q_1)g = \alpha$, $(r_i)g = r_i$, *for any* $i \geq 1$.

Furthermore, $(p_j)g = p_j$ *if* $\alpha \cap p_j = \{0\}$, *and* $(q_j)_g = q_j$ *if* $\alpha \cap q_j = \{0\}$, *for any* $j \geq 1$.

Proof. Suppose the arc α intersects q_1 transversally, and denote by k the number of points of the intersection $\alpha \cap q_1$ other than O and Q. When $k = 0$, the union of these two curves becomes a simple closed curve $\gamma = \bar{\alpha}q_1$.

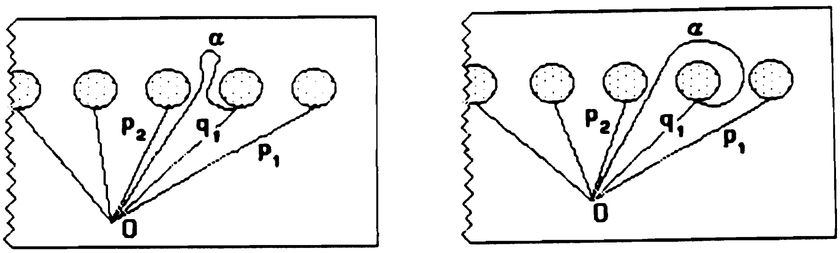


Figure 2.4

If γ does not intersect any of the arcs p_i and q_i other than q_1, we may let g either be an isotopy if γ does not separate the circles r_1' and r_1'', or a meridian twist from $M^{\pm 1}$ if the disk area bounded by γ includes r_1'' (Figure 2.4). Otherwise, let P be an intersection point closest to Q along α. If $P \in p_1$ we may use the mapping class ϕ to remove it, and if $P \in p_i$ or q_i, for some $i \geq 2$, we may use the mapping class Θ_i' or Θ_i given in Proposition 2.2 to remove it. Actually, g will be the mapping which moves the cutting circle r_1'' along the curve γ, its explicit expression in mappings Θ_i's, Θ_i''s and ϕ may be easily found from the intersection set $\gamma \cap (\cup(p_i \cup q_i))$ along the curve γ. This clearly leaves the unintersected p_i' and q_i's unchanged.

Suppose $k \geq 1$, and let P be the intersection point of α and q_1 closest to the point O along α. Let $\beta = \alpha \,|_{OP}\, q_1 \,|_{PQ}$. After an isotopy deformation, we have the intersection numbers $k(q_1,\beta) = 0$ and $k(\beta,\alpha) \leq k - 1$. And clearly β does not intersect other p_i's and q_i's more than α does, since we have

$$\beta \cap (\cup(p_i \cup q_i)) = \alpha \,|_{OP} \cap(\cup(p_i \cup q_i)) \subset \alpha \cap (\cup(p_i \cup q_i)),$$

(Figure 2.5). By induction, we have g_1 and g_2 generated by T, M, ϕ, ψ, θ and η, such that $(q_1)g_1 = \beta$ and $(\beta)g_2 = \alpha$. Then, take $g = g_1 g_2$.

Lemma 2.4. *Let* f *be an arbitrary self-homeomorphism of the handlebody* H_g, *such that* $(r_i)f = r_i$, *for all* i. *Then, the homeotopy class of* f *is generated by the classes* T, M, ϕ, ψ, *and* θ.

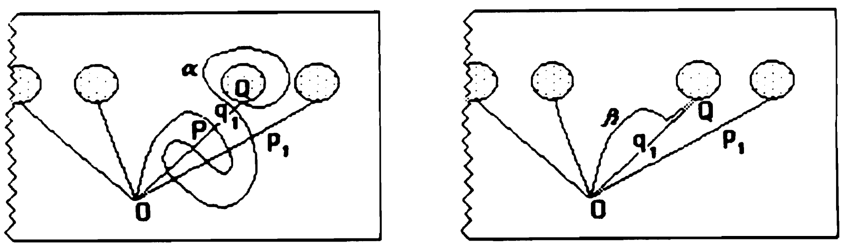


Figure 2.5

Proof. This is a direct consequence of Proposition 2.2 and Lemma 2.3. Inductively, suppose we have $(p_i)f = p_i$ and $(q_i)f = q_i$ for $i \leq s - 1$, for some s.

Rotate the handles until (p_s, q_s) is in the first position, switch p_s and q_s by ϕ, simplify p_s by using Lemma 2.3, then switch back to simplify q_s in the same way, and finally rotate it back. Again by Lemma 2.3, all p_i's and q_i's for $i \leq s - 1$ are unchanged.

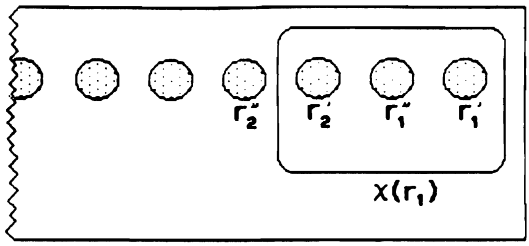


Figure 2.6

$(r_2)\chi = r_1$, $(r_i)\chi = r_i$ for $i \geq 3$

By Lemma 2.4, from now on, it is sufficient to study the image of the cutting system r_i's of an extendible class. Thus, we first discuss some extendible mapping classes which change the cutting system. For example, the images of the cutting system of the mapping classes χ, ω and η are drawn in Figures 2.6-8.

Lemma 2.5. *Let* γ *be an oriented simple closed curve on the surface* $\mathbf{F}_g$, *which does not intersect any of the meridian circles* $\mathbf{r}_i$, *and whose homology class in* $H_1(\mathbf{F}_g, \mathbb{Z})$ *relative to the meridian circle* $\mathbf{r}_i$ *is nontrivial (i.e.,* γ *separates* r_1' *and* r_1'' *in two sides in the planar representation). Then, there exists a self-homeomorphism* g *whose homeotopy class is generated by the classes* T, M, ϕ, ψ, θ *and* η, *such that* $(\gamma)g = \mathbf{r}_1$, *and* $(\mathbf{r}_i)g = \mathbf{r}_i$, *for any* $i \geq 2$.

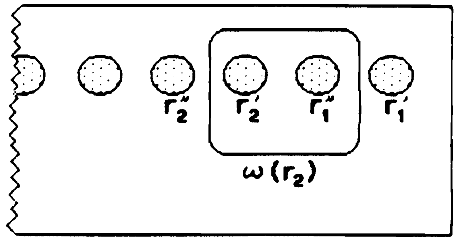


Figure 2.7

$(r_i)\omega = r_i$ for $i \neq 2$

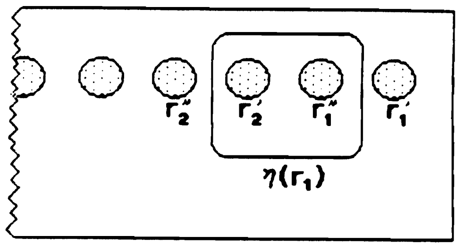


Figure 2.8

$(r_i)\eta = r_i$ for $i \geq 2$

Proof. Denote by k the number of cutting circles in the disk area Δ bounded by γ in the planar representation of F_g. The lemma will be proved by induction on k.

For k = 1, the cutting circle in Δ must be either r_1' or r_1''. If γ is oriented in the same way as this cutting circle, we may let g be an isotopy deformation, which deforms γ into r_1. If γ is oriented in the opposite way, follow the isotopy by the operation ϕ, which reverses the orientation of r_1.

For k = 2, by some handle switchings and rotations, i.e. a mapping class generated by ϕ, ψ and T, we may suppose that the two cutting circles in Δ are r_1'' and r_2'. Connecting the point $P = p_2 \cap r_2'$ and the point $Q = q_1 \cap r_1''$ by a simple arc δ in Δ which intersects neither q_1 nor p_2 (Figure 2.9).

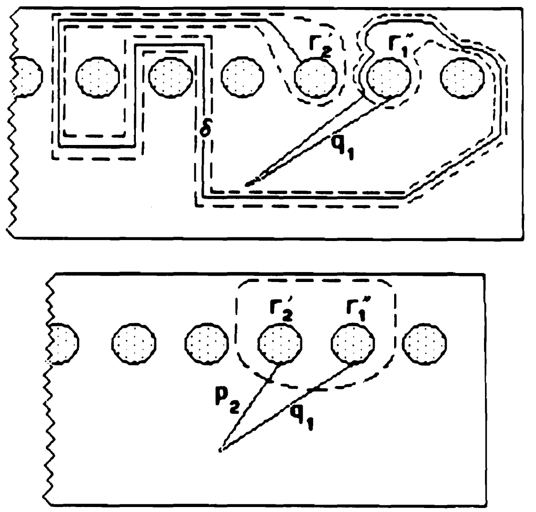


Figure 2.9

If δ does not intersect any other p_i's and q_i's, the disk Δ is isotopic to a neighborhood of $r_1'' \cup \delta \cup r_2'$ whose boundary is exactly the circle $(r_1)\eta$ as shown in Figure 2.9. Thus, the lemma is done.

If δ does intersect some p_i's or q_i's, we may simplify the intersection by the method we did in Lemma 2.3. Actually, letting $\alpha = p_2\delta$, apply Lemma 2.3 to reduce to the previous case.

For $k \geq 3$, by some handle switchings and rotations, i.e. a mapping class generated by ϕ, ψ and T, we may suppose again that the cutting circles r_1'' and r_2' are in the

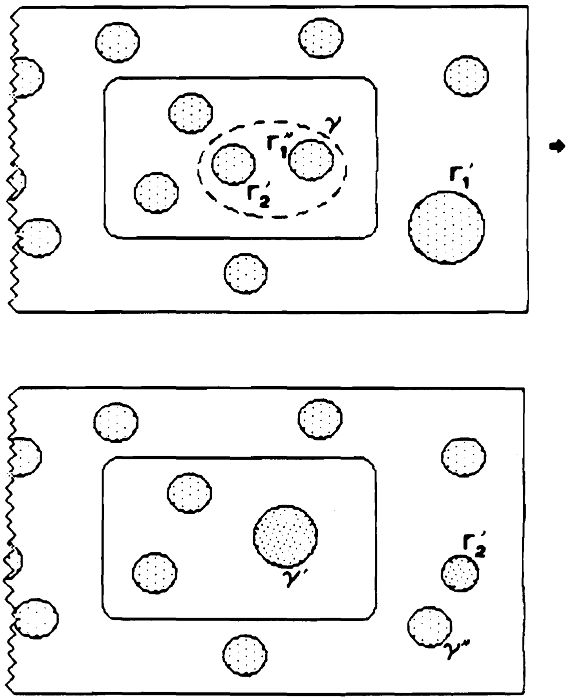


Figure 2.10

domain Δ. Connecting the point $P = p_2 \cap r_2'$ and the point $Q = q_1 \cap r_1''$ by a simple arc δ in Δ which intersects neither q_1 nor p_2 (Figure 2.10), we may choose a disk neighborhood Δ' of $r_1'' \cup \delta r_2'$ contained in the interior of Δ but including no other cutting circles. Applying the case of $k = 2$ to the disk Δ', the original Δ will be reduced to the case of $k - 1$.

Applying Lemma 2.5 repeatedly, we have the following immediate consequence.

Lemma 2.6. *Let* f *be an arbitrary self-homeomorphism of the handlebody* H_g, *with the property that,* $(r_i)f \cap r_j = \emptyset$, *for all* $i,j = 1,2,\ldots,g$. *Then, there exists another self-homeomorphism* g *whose homeotopy class is generated by the classes* T, ϕ, ψ, θ *and* η, *such that* $(r_i)g = (r_i)f$, *for* $i = 1,2,\ldots,g$.

Lemma 2.7. *Let* f *be an arbitrary self-homeomorphism of the handlebody* H_g, *then there exists another self-homeomorphism* g *whose homeotopy class is generated by the classes* T, ϕ, ψ, θ *and* η, *such that*

$$\left(\bigcup_{i=1}^{g} (r_i)f\right) \cap \left(\bigcup_{i=1}^{g} (r_i)g\right) = \emptyset.$$

i.e. none of the circles $(r_i)\bar{g}f$'s *intersects a meridian circle of* r_i's.

Proof. Denote by k_i, for $i = 1,2,\ldots,g$, and k the numbers of intersection points given by

$$k_i = \#\left((r_i)f \cap \left(\bigcup_{j=1}^{g} r_j\right)\right) \text{ and}$$

$$k = \sum_{j=1}^{g} k_i = \#\left(\bigcup_{i=1}^{(} r_i)gf \cap \left(\bigcup_{j=1}^{g} r_j\right)\right).$$

For $k = 0$, take g to be the identity.

For $k \geq 1$, we may suppose $k_1 \neq 0$, i.e. $(r_1)f \cap (\cup_j r_j) \neq 0$. Consider the meridian disks D_i bounded by the r_i in the solid handlebody H_g, which have nonempty intersection with the disk $(D_1)f$. By an isotopy deformation, we can suppose the set $(D_1)f \cap (\cup_j D_j)$ is a collection of disjoint arcs in $(D_1)f$. Thus, there is a disk component of $(D_1)f - (\cup_j D_j)$ whose boundary circle is formed exactly by one arc α from $(r_1)f$ and one arc β from $(D_1)f \cap D_s$ for some s (Figure 2.11). In the planar representation of H_g, the disk D_s and the arc β have two copies D_s', D_s'', and β' and β'' for each of them, and one of the arcs β' and β'', e.g. β', together with the arc α forms a simple closed curve.

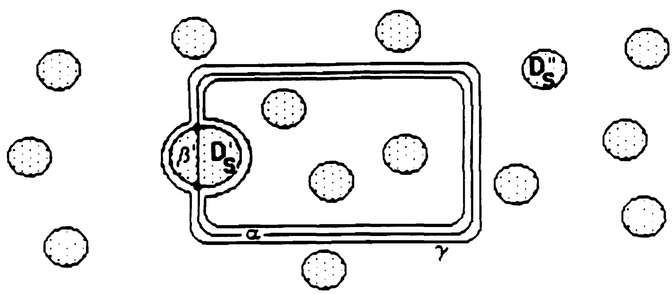


Figure 2.11

Consider the two boundary circles of an annular neighborhood of $D_s' \cup \alpha$ in the representation plane, there is one and only one of them, denoted by γ, separating D_s' and D_s'' in two parts. By Lemma 2.5, we may replace r_s by γ without changing other r_i's by composing some mapping classes generated by M, T, ϕ, ψ, θ and η. Since $\#(\gamma \cap (r_1)f) \leq \#(r_s \cap (r_1)f) - 2$, and $\#(\gamma \cap (r_j)f) \leq \#(r_s \cap (r_j)f)$, for $j \geq 2$, the number k has been reduced by at least two. This completes our lemma.

From Lemmas 2.4, 2.6 and 2.7, we conclude that, *Theorem* 2.8. *The subgroup* K_g *is generated by* M, T, ϕ, ψ, θ *and* η.

Proof of Theorem 2.1. All we need is to give the relations between the generators claimed in Theorem 2.1 and the mapping classes M, T, ϕ, ψ, θ and η. By Theorem 1.1 and using some relations from the paper [5], we have the following equations:

$$M = \overline{N}LN,$$

$$p^2 = M \cdot L\overline{N}L^2NL \cdot M,$$

$$\phi = p^2,$$

$$\psi = p^4N^3,$$

$$\theta = TP^2\overline{T} \cdot (PQN)^{-1} \cdot p^2,$$

$$\eta = TP^2\overline{T} \cdot (QNP)^{-1},$$

$$PQN = \overline{N}^3 \cdot PN^2P \cdot \overline{P}^2 \cdot PNPN,$$

$$QNP = N^3p^2 \cdot (PN^2P)^{-1} \cdot p^2,$$

$$PN^2P = L\overline{N}LNLN^2L\overline{N}LNL = L\overline{N}LN^3L\overline{N}^2LNLNL = L\overline{N}LN^3LN^3\overline{LNLN} =$$

$$= LN^2LN^2\overline{LN} = L\overline{N}^2\overline{L}N^4LN = (NLN^2\overline{L})^{-1} \cdot N^6M,$$

and

$$PNPN = M \cdot L\overline{N}LNLNL\overline{N}LN^2 = M \cdot L\overline{N}^2\overline{LNLN}^2LN^2 \cdot N^6 =$$

$$M \cdot L\overline{N}^2\overline{LN} \cdot N^3 \cdot NLN^2\overline{L}.$$

By Theorem 2.8 and by the above formulas, Theorem 2.1 is obvious.

Remark. The topological explanation of the generators of K_g is very clear. M is the 360°-twist along the meridian circle a_1, P^2 and N^3 are the 180°-twists along the circles $[a_1,b_1]$ and $[a_1,b_1][a_2,b_2]$ respectively, T totates the handles, and $\bar{N}^3PN^2P = \bar{N}^3LN^3 \cdot N\bar{L}\bar{N} \cdot M$ is a composition of Dehn twists along the curves a_1, $b_1\bar{a}_1\bar{b}_1a_2b_2\bar{a}_2$ and b_2, and is also obtained by sliding one foot of the first handle around the longitude b_2 of the second handle (Figure 2.12).

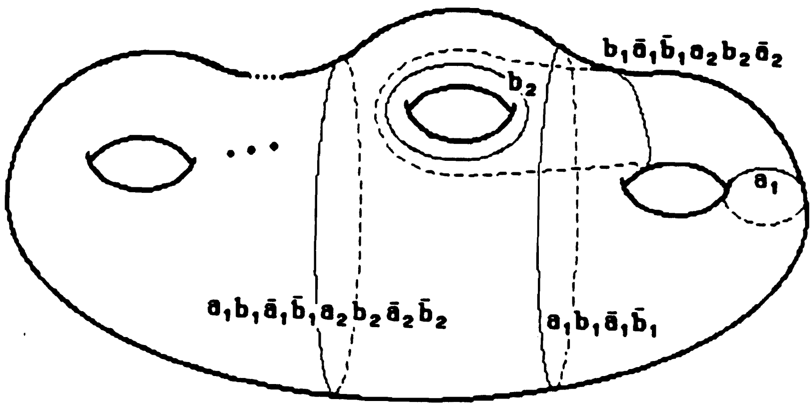


Figure 2.12

3. Heegaard Splitting of the 3-Sphere S^3

Let $\mathbf{F}_g$ be the closed orientable surface of genus g embedded unknottedly in $\mathbf{S}^3$ and bounding two handlebodies $\mathbf{H}_g$ and $\mathbf{H}'_g$. Let $\mathcal{B} = \{a_1,b_1,a_2,b_2,\ldots,a_g,b_g\}$ be a system of *basecurves* on the surface $\mathbf{F}_g$ based at a basepoint O, such that a_i's are meridians of the handlebody $\mathbf{H}_g$, and b_i's are meridians of the handlebody $\mathbf{H}'_g$. Let K_g and K'_g denote the subgroups of the group M_g formed by the mapping classes which can extend to the solid handlebodies $\mathbf{H}_g$ and $\mathbf{H}'_g$

respectively. For any mapping class f of $\mathcal{M}g$, we will denote by

$$M_f = H'_g \underset{f}{\cup} H_g$$

the closed 3-manifold associated by f, formed by identifying each point X of $\partial H'_g$ with the point (X)f of ∂H_g. It is easy to see,

Proposition 3.1. *For any mapping classes* $f \in \mathcal{M}_g$, $h \in \mathcal{K}_g$ *and* $h' \in \mathcal{K}'_g$,

$$M_f = M_{h'fh}.$$

In particular, Waldhausen ([8]) proved that,

Theorem 3.2. *Any genus-*g *Heegaard splitting of the 3-sphere is an element of the semiproduct of subgroups,* $\mathcal{K}'_g \cdot \mathcal{K}_g$.

We obtained a specific description of $\mathcal{K}_g$ in the last section, now we need one for $\mathcal{K}'_g$. In fact, if φ is a homeotopy class induced by a homeomorphism from the handlebody H'_g onto the handlebody H_g, then $\mathcal{K}'_g = \varphi \mathcal{K}_g \bar{\varphi}$. We will call such a homeotopy class *a transfer operation*. For example,

Examples 3.3.

(1) the reversion map R is a transfer operation, since

$(a_i)R = b_{g-i+2 \pmod g}$, and $(b_i)R = a_{g-i+2 \pmod g}$,

for any $i = 1,2,\ldots,g$.

(2) the homeotopy class $\pi = (PT)^g\overline{T}^g = P_1P_2\ldots P_g$ is a transfer operation, where $P_i = T^{i-1}P\overline{T}^{i-1}$, since

$$(a_i)\pi = a_ib_i\overline{a}_i, \text{ and } (b_i)\pi = \overline{a}_i,$$

for any $i = 1,2,\ldots,g$.

Using the homeotopy class π, we have,

Proposition 3.4. *The subgroup* $K'_g = \pi K_g\overline{\pi}$ *is generated by the mapping classes* T, N^3, P^2, PN^2P *and* L.

Proof. The proposition is an obvious consequence of the following formulas:

$$\pi M\overline{\pi} = PM\overline{P} = LP\overline{P} = L,$$

$$\pi T\overline{\pi} = P_1P_2\ldots P_gT\overline{P}_g\ldots\overline{P}_2\overline{P}_1 = P_1P_2\ldots P_g\overline{P}_1\overline{P}_g\ldots\overline{P}_3\overline{P}_2T = T,$$

$$\pi N^3\overline{\pi} = P_1P_2N^3\overline{P}_1\overline{P}_2 = PN^3P\overline{N}^3N^3\overline{PN}^3\overline{P}N^3 = N^3,$$

$$\pi P^2\overline{\pi} = P_1 \cdot P^2 \cdot \overline{P}_1 = P^2,$$

and $$\pi PN^2P\overline{\pi} = P_1P_2 \cdot PN^2P \cdot \overline{P}_2\overline{P}_1 = P^2 \cdot P_2N^2\overline{P}_2$$

$$= P^2N^3 \cdot PN^2P \cdot \overline{P}^2\overline{N}^3.$$

Denote by N the subgroup of M_g generated by the elements T, P^2, N^3 and PN^2P, which obviously is a subgroup of $K'_g \cap K_g$. Using a result of Powell [6] that the subgroup $K'_g \cap K_g$ is generated by T, N^3, P^2, ω and η, we have the following consequence:

Corollary 3.5.

$$N = K'_g \cap K_g.$$

Theorem 3.6. *The associated* 3-*manifold* M_f *of a mapping class* f *is the* 3-*sphere* $\mathbf{S}^3$ *if and only if* f *is an element of the set*

$\langle L, \mathcal{N} \rangle \cdot \langle \mathcal{N}, M \rangle$.

Before we end this section, we discuss some more relations among the mapping classes in those subgroups.

Let $L_i = T^{i-1}L\overline{T}^{i-1}$, and $M_i = T^{i-1}M\overline{T}^{i-1}$, for $i = 1,2,\ldots,g$. Let $\mathcal{L}$ and $\mathcal{M}$ denote the abelian subgroups of rank g generated by the L_i's and M_i's respectively.

Proposition 3.7. *For any* $i = 1,2,\ldots,g$,

(a) $L_iT = TL_{i-1}$, $M_iT = TM_{i-1}$,

(b) $L_iP^2 = P^2L_i$, $M_iP^2 = P^2M_i$,

(c) $L_iN^3 = N^3L_i$, $M_iN^3 = N^3M_i$, for $i \neq 1,2$,

$L_1N^3 = N^3L_2$, $M_1N^3 = N^3M_2$,

$L_2N^3 = N^3L_1$, $M_2N^3 = N^3M_1$,

(d) $L_iPN^2P = PN^2PL_i$, $M_iPN^2P = PN^2PM_i$, for $i \neq 1,2$, and $L_1PN^2P = PN^2PL_2$, $M_2PN^2P = PN^2PM_1$.

Proof. Since

$$T = [a_g,b_g,a_1,b_1,\ldots,a_{g-1},b_{g-1}],$$

$$N^3 = [xa_2\overline{x},xb_2\overline{x},a_1,b_1..a_3,b_3,\ldots,a_g,b_g],$$

$$P^2 = [c_1\overline{a}_1,\overline{b}_1\overline{c}_1,a_2,b_2,\ldots,a_g,b_g],$$

and $$PN^2P = [x\overline{b}_2\overline{x}a_1a_2b_2\overline{x},xb_2\overline{x},a_1,\overline{b}_2\overline{c}_2b_1,a_3,b_3,\ldots,a_g,b_g],$$

the proposition is clear.

Proposition 3.8. $L \cap N = 1$, *and* $M \cap N = 1$.

Proof. Consider the image of L and N in Siegel's modular group ([2]). For any element $f \in N \cdot f$ leaves the subspace $\mathbb{Z}^g$ generated by the a^i's in $H_1(F_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ invariant, by looking at the expressions in the proof of the last proposition. But the only element of L having this property is the identity. Therefore $L \cap N = 1$. And analogously, $M \cap N = 1$.

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Rice University

Houston, Texas 77251