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NON-WANDERING SETS, PERIODICITY, AND EXPANSIVE HOMEOMORPHISMS

J. D. Wine

In what follows we examine the non-wandering set of a homeomorphism of a compact metric space onto itself, the ω-limit set of a point in the space, and the relationships that develop when we allow the homeomorphism to be expansive and/or the non-wandering set to be composed only of periodic points. The three possibilities examined are when the non-wandering set is pointwise periodic, the non-wandering set equals the union of ω-limit points, and when every point in the space is positively asymptotic to a point in the non-wandering set. Examples are given or cited which narrow down the possible relationships.

Preliminaries. We begin by introducing the necessary terminology and notation.

Definition 1. A homeomorphism h of a metric space (X,ρ) onto itself is *expansive* with expansive constant $\delta > 0$ if given any two distinct points x and y of X there is an integer n such that $\rho(h^n(x),h^n(y)) > \delta$.

Definition 2. If h is a homeomorphism of a metric space (X,ρ) onto itself, then a point x of X is a *non-wandering point* of h if for every open neighborhood U of x and for every positive integer N, there exists an integer $n \geq N$ such that $h^n(U) \cap U$ is nonempty.

The set of non-wandering points of h will be denoted by $\Omega(h)$. We note that $\Omega(h)$ is a closed subset of (X,ρ).

Definition 3. If h is a homeomorphism of the metric space (X,ρ) onto itself and x is a point of X, then the *orbit* of x (denoted by O(x)) is $\cup \{h^n(x) | n \text{ is an integer}\}$.

Definition 4. Let h be a homeomorphism of a metric space (X,ρ) onto itself and let x be a point of X. The *ω-limit set of* x under h (denoted by $\omega(x)$) is the set of limit points of the positive semi-orbit of x, $\cup_{i=1}^{\infty} h^i(x)$.

Note that for any point x of a metric space (X,ρ) and any self homeomorphism of (X,ρ), we have $\omega(x)$ is a subset of $\Omega(h)$.

Definition 5. Let x and y be points of the metric space (X,ρ) and let h be a homeomorphism of (X,ρ) onto itself. If for every $\varepsilon > 0$ there is an integer N such that for each integer n greater than N, $\rho(h^n(x),h^n(y)) < \varepsilon$, then x is *positively asymptotic* to y.

Definition 6. Let x be a point of the metric space (X,ρ) and let h be a homeomorphism of (X,ρ) onto itself. The homeomorphism h is *almost periodic at* x provided that for any neighborhood U of x there exists a relatively dense subset D of the integers such that $h^n(x)$ is in U when n is in D. (D is relatively dense in the integers I if I = D + K for some finite subset K of I.) If the set D is a subgroup of the integers then h is *regularly almost periodic at* x.

Definition 7. Let h be a homeomorphism of the metric space (X,ρ) onto itself. The set $\{x_\alpha | \alpha \in A\}$ is an *orbital basis* of (X,ρ) with respect to h if $\cup\{O(x_\alpha) | \alpha \in A\} = X$, and α not equal to β implies $O(x_\alpha)$ is not $O(x_\beta)$.

Results. The first theorem is known, at least when P is equal to X, and is included for completeness.

Theorem 1. *Let* (X,ρ) *be an infinite compact metric space. If* h *is a homeomorphism of* (X,ρ) *onto itself which is pointwise periodic on a dense subset* P *of* X, *then* h *is not expansive on* (X,ρ).

Proof. Let δ be an arbitrary positive real number and let $\mathcal{B} = \{x_\alpha | \alpha \in A\}$ be an orbital basis of (X,ρ) with respect to h. Let $\mathcal{G}$ be an open cover of X by $\delta/2$ neighborhoods of the points of X and let $\mathcal{H}$ be a finite subcover of $\mathcal{G}$. If $\mathcal{B}^*$ is the subset of $\mathcal{B}$ such that $x_\alpha \in \mathcal{B}^*$ implies h is periodic at x_α, then $\mathcal{B}^*$ cannot be finite and there is an open set G of $\mathcal{H}$ such that G contains two elements of $\mathcal{B}^*$, call them x_1 and x_2, with periods p_1 and p_2 respectively. If $p = p_1 p_2$, then it is clear that h^p is not expansive with expansive constant δ since $\rho(x_1,x_2) < \delta$ and they are fixed points under h^p. However, if h were expansive, h^p would be expansive for some δ since (X,ρ) is compact.

The next two results examine the structure of $\omega(x)$ for a point x of a compact metric space under a homeomorphism onto itself.

Theorem 2. *Let* (X,ρ) *be a compact metric space and let* h *be a homeomorphism of* (X,ρ) *onto itself. If* x *is a point in* X, *then there is no decomposition of* $\omega(x)$ *into unions of orbits of* h *that have positive separation.*

Proof. For x a point of X, suppose $\omega(x)$ is the union of the families $A = \cup\{O(y_i)|i \in P\}$ and $B = \cup\{O(Y_j)|j \in Q\}$, $Q \cap P = \emptyset$. Further, suppose that the distance between A and B is $L = \inf\{\rho(u,v)|u \in A,\ v \in B\}$ and that L is not zero. Since h is uniformly continuous there is a positive real number ε such that $\varepsilon < L/3$ and $\rho(u,v) < \varepsilon$ implies $\rho(h(u),h(v)) < L/3$ for every u and v in X. Let A* and B* be ε-neighborhoods of A and B respectively. By construction, the intersection of A* and B* is empty.

For the same point x in X, now let

$$\mathcal{A} = \{k|k \text{ is a positive integer, } h^k(x) \in A^*\}$$

$$\mathcal{B} = \{k|k \text{ is a positive integer, } h^k(x) \in B^*\}$$

$$\mathcal{C} = \{i|i \text{ is a positive integer, } h^i(x) \notin A^*\}$$

and $\mathcal{H} = \{\hat{k}|\hat{k} = \max\{i|i \in \mathcal{C},\ i < k\ ,\ k \in \mathcal{A}\}.$

The cardinality of $\mathcal{H}$ is infinite since such is true for $\mathcal{B}$ and $\mathcal{A}$. The intersection of $\mathcal{H}$ and $\mathcal{B}$ is empty since $n \in \mathcal{H} \cap \mathcal{B}$ implies $h^n(x) \in B^*$ and hence $\rho(h^{n+1}(x),u) < L/3$ for some u in B, but $h^{n+1}(x)$ is in A*.

Since (X,ρ) is compact, the set $S = \{h^{\hat{k}}(x)|\hat{k} \in \mathcal{H}\}$ has a cluster point p. Thus p is in $\omega(x)$, but p is not in $A \cup B$. A contradiction has been reached.

In the next result, for the first time, we require that h be an expansive homeomorphism of a compact metric space onto itself.

Theorem 3. *Let* (X,ρ) *be a compact metric space and let* h *be an expansive homeomorphism of* (X,ρ) *onto itself. If* $\Omega(h)$ *is pointwise periodic, then for every point* x *of* X, $\omega(x)$ *is either a single orbit or* $\omega(x)$ *is empty.*

Proof. The set $\omega(x)$ is closed and invariant under the homeomorphism h. Since $\omega(x)$ is a subset of $\Omega(h)$, $\omega(x)$ must be periodic, and since h restricted to $\omega(x)$ is expansive, the cardinality of $\omega(x)$ is finite. However, if $\omega(x)$ is finite and has more than one orbit, then any decomposition of $\omega(x)$ into unions of orbits would have a positive separation of components. This is in contradiction to the preceding theorem. Therefore, $\omega(x)$ can have at most one orbit.

Theorem 4. *Let* (X,ρ) *be a compact metric space and let* h *be an expansive homeomorphism of* (X,ρ) *onto itself. If* $\Omega(h)$ *is pointwise periodic, then every point of* X *is positively asymptotic to a point of* $\Omega(h)$.

Proof. Let x be a point in X. Since (X,ρ) is compact, $\omega(x)$ is not empty unless x is in $\Omega(h)$. If x is in $\Omega(h)$, then x is positively asymptotic to $h^p(x)$, where p is the period of x.

If $\omega(x)$ is not empty, let $\omega(x)$ consist of $O(y)$, then y is periodic and by Lemma 3 in [1] it must be the case that there is a point z of $O(y)$ such that x and z are positively asymptotic.

Theorem 11 of Bryant and Walters in [3] is now a corollary of Theorem 4.

Corollary 4A. *Let* (X,ρ) *be a compact metric space and* h *be an expansive homeomorphism of* (X,ρ) *onto itself. If* $\Omega(h)$ *is finite, then every point of* X *is positively asymptotic to a point in* $\Omega(h)$.

Proof. If $\Omega(h)$ is finite, it is pointwise periodic.

As noted earlier, for any homeomorphism h of a metric space (X,ρ) onto itself and for any point x of X we have $\omega(x) \subset \Omega(h)$. We now give sufficient conditions for equality to exist.

Theorem 5. *Let* (X,ρ) *be a compact metric space, let* h *be an expansive homeomorphism of* (X,ρ) *onto itself with* $\Omega(h)$ *pointwise periodic, and for each* x *in* $\Omega(h)$ *let there be a* y *in* X *such that* y *is positively asymptotic to* x, $y \neq x$. *Then* $\Omega(h) = \cup\{\omega(x) \mid x \in X\}$.

Proof. Let x be a point of the non-wandering set, $\Omega(h)$, with period p, let y be the point in X which is positively asymptotic to x, and let an arbitrary $\varepsilon > 0$ be given. There is a positive integer N such that for all $n > N$, $\rho(h^n(x),h^n(y)) < \varepsilon$. Therefore, there is a positive integer k such that $\rho(x,h^{(k+m)p}(y)) = \rho(h^{(k+m)p}(x), h^{(k+m)p}(y)) < \varepsilon$ for all positive integers m. Hence x is in $\omega(y)$ and $\Omega(h) \subset \cup\{\omega(x) \mid x \in X\}$ and we have equality of the two sets.

Examples. The first example illustrates that in Theorem 5 some hypothesis beyond h being expansive and $\Omega(h)$ pointwise periodic is needed for the equality in the conclusion.

Example 1. We use an example given by Bryant and Walters [3, page 65], among others. The space (X,ρ) is a subspace of the real numbers where

$$X = \{0,1\} \cup \{x = \frac{1}{n} | n = 2,3,4,\cdots\} \cup \{x = 1 - \frac{1}{n} | n = 3,4,5,\cdots\},$$

$$h(x) = \begin{cases} x, \text{ if } x \text{ is } 0 \text{ or } 1 \\ \hat{x}, \text{ otherwise} \end{cases}$$

and $\hat{x}$ is the point immediately to the right of x. Equality between $\Omega(h)$ and $\cup\{\omega(x) | x \in X\}$ does not hold.

We now give an example of a compact metric space which has an expansive homeomorphism possessing the following properties:

(1) the non-wandering set is equal to the union of the ω-limit points, but

(2) the non-wandering set is not pointwise periodic, and

(3) there is a point which is not positively asymptotic to any point in the non-wandering set.

Example 2. Define the following subsets of Euclidean 3-space where the points are given in cylindrical coordinates

$C = \{(1,\theta,0) \mid \theta = 0,\ \pi/n,\ \text{or}\ \frac{2n-1}{n}\pi$ for n a positive integer$\}$

$$C_k = \{(\tfrac{k}{k+1},\theta(M_k),\tfrac{1}{2+k}) \mid \theta(M_k) = \pi + \frac{M_k\pi}{|M_k|+1},$$

$$-k \le M_k \le k,\ M_k \text{ an integer}\},\ k = 0,1,2,\cdots$$

$$L = \{(0,0,\tfrac{k}{k+1}) \mid k = 2,3,4,\cdots\} \cup \{(0,0,1)\}$$

$T = C$ translated by the vector $(-1,0,1)$.

Let (X,ρ) be the metric space which is obtained by considering the union of the above sets to be a subspace of Euclidean 3-space (see diagram). The space (X,ρ) is compact.

We now define the function h taking (X,ρ) onto itself by the following.

(1) For points in L,

$h((0,0,1)) = (0,0,1)$

$h((0,0,\frac{k}{k+1})) = (0,0,\frac{k-1}{k}) \quad k = 2,3,4,\cdots$

(2) For points r in C_k, let $\hat{r}$ be the point in C_k with next larger angular coordinate, and let q_k be the point in C_k with smallest angular coordinate, and define

$$h(r) = \begin{cases} \hat{r}, & r \text{ not having largest angular coordinate in } C_k \\ q_{k+1}, & \text{otherwise} \end{cases}$$

(3) For points r in C,

$h((1,0,0)) = (1,0,0)$

$h(r) = \hat{r}$ otherwise

where $\hat{r}$ is defined the same as for C_k.

(4) For points in T,

define h as for analogous points in C.

We observe the following concerning h:

(i) h is a self homeomorphism of (X,ρ)

(ii) h is expansive with expansive constant of 1/6

(iii) the non-wandering set of h is composed of $\{(0,0,1)\} \cup C$

(iv) the union of the ω-limit points of h is equal to the non-wandering set,

(v) the non-wandering set is not pointwise periodic, and

(vi) the point (0,0,1/2) is not positively asymptotic to any point in X.

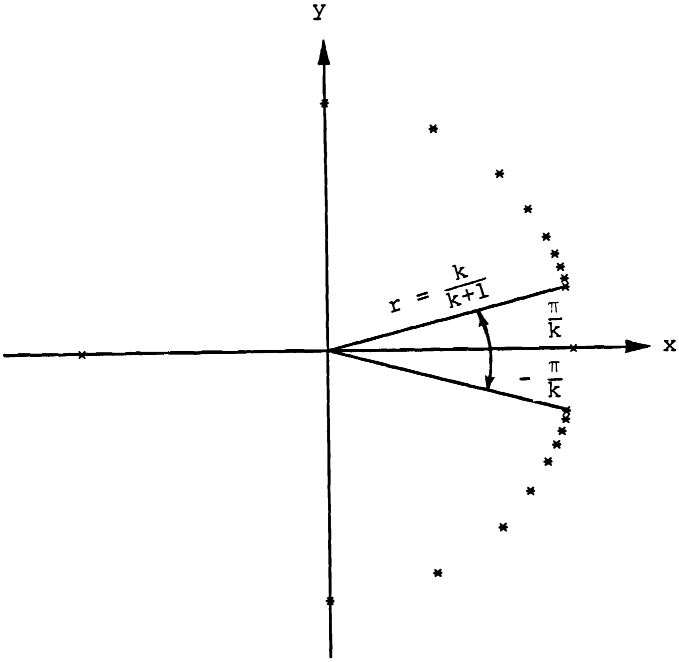
If we let P, Q, and R be the following properties of a self homeomorphism h

P: the non-wandering set of h is pointwise periodic,

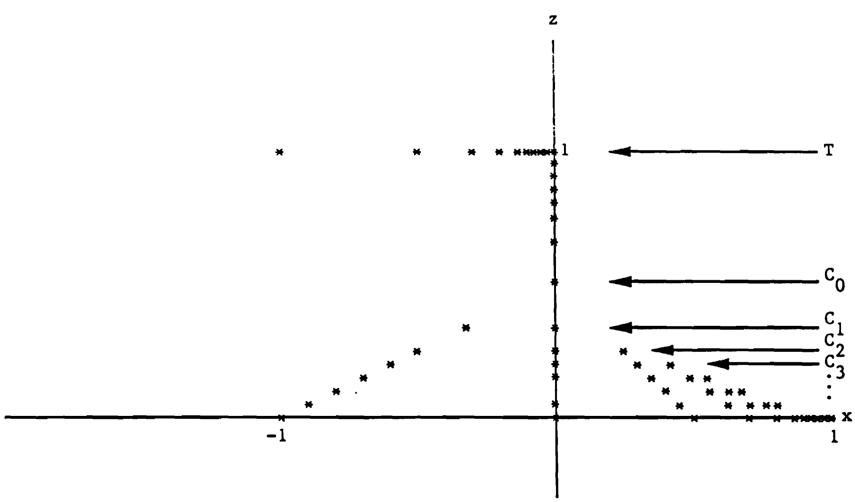
Q: the non-wandering set of h equals the union of ω-limit points,

R: every point in X is positively asymptotic to a point in the non-wandering set,

then for (X,ρ) compact and h expansive, besides the implications shown we have that Example 2 shows that property Q implies neither property P nor property R.



C_k viewed from positive z axis



(X, ρ) viewed from negative y axis

There is an example [4, page 342] of a nonexpansive homeomorphism on a compact metric space in which property R is true but property P is not. Whether for expansive homeomorphisms of compact metric spaces property R implies anything about properties P or Q, appears to be an open question.

References

[1] B. F. Bryant, *On expansive homeomorphisms*, Pacific J. Math 10 (1986), 1163-1167.

[2] __________, *Expansive self-homeomorphisms*, Amer. Math. Monthly 69 (1962), 386-391.

[3] B. F. Bryant and P. Walters, *Asymptotic properties of expansive homeomorphisms*, Math. Systems Theory 3 (1969), 60-66.

[4] W. H. Gottschalk, *Minimal sets: an introduction to topological dynamics*, Bull. Amer. Math. Soc. 64 (1958), 336-351.

[5] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. 1 (1950), 769-774.

[6] R. K. Williams, *On expansive homeomorphisms*, Amer. Math. Monthly 76 (1969), 176-178.

[7] __________, *Some results on expansive mappings*, Proc. Amer. Math. Soc. 26 (1970), 655-663.

[8] __________, *Further results on expansive mappings*, Proc. Amer. Math. Soc. 58 (1976), 284-288.

[9] J. D. Wine, *Extending expansive homeomorphisms*, Proc. Amer. Math. Soc. 86 (1982), 531-534.

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