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Given a measure space X and a self map $T: X \rightarrow X$ one can show the existence of a T -invariant measure (cf. [W]). However, in this context, little can be said about the specific properties of the measure. Knowledge of these properties would prove invaluable in the study of discrete dynamical systems. In related work ([BBS], [BS], [PS], [S1], [S2] and [S3]), we have focused upon area preserving, orientation preserving homeomorphisms of orientable two manifolds. On a list of questions from the 1986 Spring Topology Conference, Mort Brown [STC] suggested the following question: Does every Brouwer homeomorphism (i.e., an orientation preserving homeomorphism which satisfies the conclusions of the Brouwer Lemma) of the plane preserve some nice measure? The word "nice" makes this somewhat vague, but should certainly mean a measure which is similar to area, and require at least that the measure be nonatomic, finite on bounded sets, and positive on open sets. Given that "nice" should include at least these properties, the question needs to be modified somewhat to make it non-trivial, as it is easily seen that the function $f(x,y) = (x/2,y)$ is a Brouwer homeomorphism which preserves no measure having the above properties. This example leaves the entire y -axis fixed, but a nontrivial example having only a single fixed point (of index -1) was constructed

in [PS]. Thus, additional hypotheses are required to get an affirmative answer. We show that the answer is yes if the additional hypothesis "fixed-point-free" is added, showing that the constructed measure can be made to have the additional property that Lebesgue measure is absolutely continuous with respect to the constructed measure. Our proof is topological and rests almost exclusively upon the Brouwer Lemma (see below). Finally, we give an example to show that the absolute continuity cannot be reversed, i.e., in general there does not exist a nice measure μ which is preserved by f and absolutely continuous with respect to Lebesgue measure.

Definition. Let λ be Lebesgue measure on \mathbb{R}^2 . A measure μ on \mathbb{R}^2 will be called a nice measure iff μ is (the completion of) a countably additive measure on the set of all Borel subsets of \mathbb{R}^2 satisfying the following additional properties:

- (a) If X is a bounded subset of \mathbb{R}^2 , then $\mu(X)$ is finite.
- (b) If U is a nonempty open subset of \mathbb{R}^2 , then $\mu(U) > 0$.
- (c) If X is any subset of \mathbb{R}^2 satisfying the additional property that $\lambda(h(X)) = 0$ for every homeomorphism h of \mathbb{R}^2 , then $\mu(X) = 0$.
- (d) Lebesgue measure λ is absolutely continuous with respect to μ (i.e., if $\mu(X) = 0$, then X is Lebesgue measurable and $\lambda(X) = 0$).

Our main tool for constructing the desired measure is the Brouwer Lemma (see [B], [Br], or [F] for a proof).

Brouwer Lemma. If f is an orientation preserving, fixed-point free homeomorphism of \mathbb{R}^2 , and U is any connected open subset of \mathbb{R}^2 such that $f(U) \cap U = \emptyset$, then $f^n(U) \cap U = \emptyset$ for all nonzero integers n .

Definition. If f is a homeomorphism of \mathbb{R}^2 , and U is an open subset of \mathbb{R}^2 , then U is a Brouwer neighborhood (with respect to f) if $f^n(U) \cap U = \emptyset$ for all nonzero integers n .

Lemma. Let f be an orientation preserving, fixed-point free homeomorphism of \mathbb{R}^2 , and let $V \subseteq \mathbb{R}^2$ be a Brouwer neighborhood (with respect to f) with $\lambda(V) \leq 1$. Then there is a countably additive measure μ on \mathbb{R}^2 satisfying (a) and (c) in the definition of nice, and the following properties in addition:

- (1) $\mu(U) \leq 1$ for every Brouwer neighborhood U with respect to f .
- (2) λ and μ agree on all subsets of V .
- (3) μ is invariant with respect to f , i.e. $\mu(f(X)) = \mu(X)$ for all μ -measurable subsets X of \mathbb{R}^2 .

Proof. Define the measure μ by:

$$\mu(X) = \sum_{i=-\infty}^{\infty} \lambda(f^i(X) \cap V)$$

Let a set be μ -measurable iff every term on the right hand side is well-defined, with $\mu(X)$ infinite if the series fails to converge. That μ is a countably additive measure, and satisfies (c) in the definition of nice, and (2) and (3) in the conclusion of the Lemma, is easy to check (for

(2), note that $f^n(V) \cap V = \emptyset$ for all nonzero integers n , by the Brouwer Lemma). To see (1), note that if X is a Brouwer neighborhood, then the sum is over a pairwise disjoint collection of subsets of V . Getting (a) in the definition of nice is now routine. For, suppose X is bounded with $\mu(X)$ infinite. By successively dividing X into bounded subsets of smaller diameter with infinite μ -measure, we would get a point x such that $\mu(U)$ was infinite for all open U containing x . But this is clearly impossible, since the hypotheses of the theorem implies that x has a Brouwer neighborhood. *qed.*

Main Theorem. *If f is an orientation-preserving, fixed-point-free homeomorphism of the plane, then there is a nice measure on the plane which is invariant with respect to f .*

Proof. The Brouwer Lemma easily implies that every point has a Brouwer neighborhood of Lebesgue measure no more than 1. Let $\{V_1, V_2, V_3, V_4, \dots\}$ be an open cover of \mathbb{R}^2 by such neighborhoods and define measures μ_n on \mathbb{R}^2 satisfying the conclusions of the lemma. Define the measure μ by:

$$\mu(X) = \sum_{i=1}^{\infty} 2^{-n} \mu_n(X)$$

Then μ is bounded on Brouwer neighborhoods, and therefore (by the same argument as the lemma) on bounded sets, and the remaining properties are easy to check.

If the function f satisfies a reasonable growth condition (for example, if there are positive $c < d$ such that $c < |f(x) - f(y)|/|x-y| < d$ for all x, y in \mathbb{R}^2), then it is easy to check that the measure μ is absolutely continuous with respect to Lebesgue measure. However, the conclusion of the main theorem cannot, in general, be improved to a measure μ which is absolutely continuous with respect to Lebesgue measure, as the following example on \mathbb{R} shows:

Example. Let $\{C_1, C_2, C_3, C_4, \dots\}$ be a nested ($C_n \subseteq C_{n+1}$) sequence of Cantor subsets of $[0,1]$ such that $\{0,1\} \subseteq C_1$, and $\lambda(C_n)$ converges to 1. Define the orientation preserving homeomorphism f on \mathbb{R} by first defining $f|_{[n,n+1]}$ by induction on n . Let f take $[0,1]$ to $[1,2]$ in such a way that C_1 is mapped to a Lebesgue measure 0 Cantor subset of $[1,2]$, and if $f|_{[n-1,n]}$ has been defined mapping $[n-1,n]$ onto $[n,n+1]$, let g map $[0,1]$ to $[n+1,n+2]$ such that C_{n+1} is mapped to a Lebesgue measure 0 Cantor subset of $[n+1,n+2]$ by g . Then let $f|_{[n,n+1]}$ be $gf^{-n}|_{[n,n+1]}$. This defines f on the nonnegative reals, and define f on the negative numbers by $f(x) = x + 1$. It is easy to check that the only countably additive f -invariant measure which is absolutely continuous with respect to λ is the identically zero measure.

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