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K-SEMIMETRICS AND 1-CONTINUOUS SEMIMETRICS

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A *distance function for* X is any nonnegative, real-valued function $d: X \times X \to \mathbb{R}$ such that $d(x,y) = d(y,x)$ and $d(x,y) = 0$ iff $x = y$ for any $x, y \in X$. We use the notation $d(x,A] = \inf\{d(x,y) \mid y \in A\}$, $d[B,A] = \inf\{d(x,A] \mid x \in B\}$ and $S_d(p,\varepsilon) = \{x \in X \mid d(p,x) < \varepsilon\}$. A distance function d is *continuous* iff, when $d(x_n,p) \to 0$ and $d(y_n,q) \to 0$, then $d(x_n,y_n) \to d(p,q)$; it is *1-continuous* iff, for any q, when $d(x_n,p) \to 0$, then $d(x_n,q) \to d(p,q)$; and it is *developable* iff, when $d(x_n,p) \to 0$ and $d(y_n,p) \to 0$, then $d(x_n,y_n) \to 0$ (or, equivalently, if $d(x_n,p) \to 0$, then $\langle x_n \rangle$ is d-Cauchy).

Any distance function d determines a topology $\mathcal{T}_d = \{A \subseteq X \mid \text{if } p \in A, \text{ then } S_d(p,\varepsilon) \subseteq A \text{ for some } \varepsilon\}$, which is called the *symmetric topology for* X. Thus, d is a *symmetric for* $(X,\mathcal{T})$ iff $\mathcal{T} = \mathcal{T}_d$. If, for each $p \in X$, the set of spheres $S_d(p,\varepsilon)$ is a neighborhood base for p in $(X,\mathcal{T})$, then we follow convention in saying that d is a *semimetric* (or an *admissible semimetric*) *for* $(X,\mathcal{T})$; a topological space $(X,\mathcal{T})$ is *semimetrizable* iff there is a semimetric for $(X,\mathcal{T})$. Clearly, if d is a semimetric for $(X,\mathcal{T})$, then $\mathcal{T} = \mathcal{T}_d$; that the converse need not hold is a well known result of Arhangel'skii (see [4]).

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Finally, when d is a distance function for X such that $\mathcal{T}_d \subseteq \mathcal{T}$ and $d[A,B] > 0$, when A and B are nonempty, disjoint compact subsets of $(X,\mathcal{T})$ (i.e., d *separates disjoint compact subsets of* $(X,\mathcal{T})$), then we say that d is a *K-distance function* on $(X,\mathcal{T})$. Similarly, we have the notion of *K-semimetric, K-developable semimetric,* etc.

1. Developable semimetrics and K-semimetrics

For semimetrizable spaces our study seeks to establish the strongest possible admissible semimetric for a space $(X,\mathcal{T})$.

First, we consider spaces which admit developable semimetrics and K-semimetrics. We note that Burke's Example [2; Example 1, p. 126], which we denote as $\mathbf{B}_2$, is developable semimetrizable, but no admissible semimetric is a K-semimetric. Borges' Example [1; Example 2.4, p. 194], which we denote as $\mathbf{B}_1$, is K-semimetrizable but no admissible semimetric is developable; see Remark 2.5.

Recall that, for any infinite, maximal family $\mathcal{R}$ of infinite almost disjoint subsets of the set $\mathbf{N}$ of natural numbers, the Isbell-Mrówka space $\Psi_{\mathcal{R}}$ is the set $\mathbf{N} \cup \mathcal{R}$ with the topology which, for each $A \in \mathcal{R}$, has the sets $U_k(A) = \{A\} \cup \{n \in A \mid k \leq n\}$, $k \in \mathbf{N}$, as a local base, and for each $n \in \mathbf{N}$, has $\{n\}$ as a local base. See [5:5I] for further details. We establish that the spaces $\Psi_{\mathcal{R}}$ admit developable semimetrics and K-semimetrics, but none that are simultaneously developable and K-semimetrics. (Note that we have shown in [3] that an analogous result holds in the case of developable semimetrics and Cauchy complete semimetrics for Ψ_R. Namely, there is a developable semimetric for Ψ_R and there is a Cauchy complete semimetric for Ψ_R; however, if d is a developable semimetric for Ψ_R, then d is not Cauchy complete.)

Theorem 1.1. *There is a developable semimetric for* Ψ_R *and there is a* K-*semimetric for* Ψ_R*; however, if* d *is a developable semimetric for* Ψ_R*, then* d *is not a* K-*semimetric.*

Proof. The distance function for $\mathbf{N} \cup R$ with $d(x,y) = d(y,x) = 2^{-x}$, when $x \in y \in R$, and, otherwise, $d(x,y) = 1$, when $x \neq y$, is a K-semimetric for Ψ_R. Note that d is not developable since each $A \in R$, viewed as an increasing sequence in $\mathbf{N} \subseteq \mathbf{N} \cup R$, converges to $A \in \Psi_R$, but is not Cauchy. On the other hand, if we modify this distance function so that $d(x,y) = |2^{-x} - 2^{-y}|$ for $x,y \in \mathbf{N}$, then we have a developable semimetric for Ψ_R.

Finally, we show that a developable semimetric for Ψ_R cannot be a K-semimetric. Suppose that d is any developable semimetric for Ψ_R. For any positive ε, there are at most finitely many $A \in R$ such that $d[A, \mathbf{N}\setminus A] \geqq \varepsilon$. (Otherwise, choose a sequence $\langle A_n \rangle$ of distinct members of R such that $d[A_n, \mathbf{N}\setminus A_n] \geqq \varepsilon$. Now, for each i, choose $a_i \in A_i \setminus \cup \{A_j \mid j < i\}$. Note that, for $i \neq j$, $d(a_i, a_j) \geqq \varepsilon$. The maximality of R implies that the sequence $\langle a_n \rangle$ in $\mathbf{N}$ has a subsequence $\langle b_n \rangle$ which converges to some $B \in R$.

Thus, $\langle b_n \rangle$ is a convergent sequence which is not d-Cauchy. This contradicts that d is developable.) Now, since $\mathcal{R}$ is uncountable, choose $A \in \mathcal{R}$ such that $d[A, \mathbb{N} \setminus A] = 0$. There is an increasing sequence $\langle x_n \rangle$ in $\mathbb{N} \setminus A$ such that $d(x_n, A] \to 0$. Again, from the maximality of $\mathcal{R}$, we obtain a subsequence $\langle b_n \rangle$ of $\langle x_n \rangle$ which converges to some $B \in \mathcal{R}$. It follows that $A \cup \{A\}$ and $(B \setminus A) \cup \{B\}$ are disjoint compact sets in $\Psi_{\mathcal{R}}$ which are not separated by d.

Remark 1.2. The critical factor in establishing our result is the failure of $\Psi_{\mathcal{R}}$ to have a regular G_δ-diagonal. This becomes apparent in our next theorem. We show that $(X,\mathcal{T})$ admits a K-developable semimetric iff $(X,\mathcal{T})$ is a $w\Delta$-space with a regular G_δ-diagonal. On the other hand, McArthur [7] has shown that any pseudocompact, completely regular, Hausdorff space $(X,\mathcal{T})$ with a regular G_δ-diagonal is metrizable. It follows that $\Psi_{\mathcal{R}}$ does not have a regular G_δ-diagonal and, therefore, can not admit a K-developable semimetric.

Recall that X has a G_δ-*diagonal* iff the diagonal of X, $\Delta_X = \{(x,x) \mid x \in X\}$, is a G_δ-set in the product; X has a *regular* G_δ-*diagonal* [8] iff Δ_X is a countable intersection of regular closed neighborhoods. $(X,\mathcal{T})$ is a $w\Delta$-*space* iff there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that, if $\langle x_n \rangle$ is a sequence such that, for some $p \in X$, $x_n \in st(p, \mathcal{G}_n)$, then $\langle x_n \rangle$ has a cluster point in $(X,\mathcal{T})$; in this case, we say (following Hodel) that $\langle \mathcal{G}_n \rangle$ is a $w\Delta$-*sequence* for $(X,\mathcal{T})$.

Theorem 1.3. *A topological space admits a* K-*developable semimetric iff it is a* wΔ-*space with a regular* G_δ-*diagonal.*

Proof. Suppose that d is a K-developable semimetric for $(X,\mathcal{T})$. Let $\mathcal{G}_n$ be the set of open sets G in $\mathcal{T}$ which have d-diameter less than 2^{-n}. The set of spheres centered at p is a neighborhood base for p in $(X,\mathcal{T})$; moreover, since d is developable, there are spheres of arbitrarily small diameter centered at p. Consequently, $\mathcal{G}_n$ is a cover of X.

Since $st(p,\mathcal{G}_n) \subseteq S_d(p,2^{-n})$, we conclude that $\langle \mathcal{G}_n \rangle$ is a wΔ-sequence. Finally, letting $U_n = \cup \{G \times G | G \in \mathcal{G}_n\}$, we claim that the intersection of the closures of U_n is the diagonal of X. Otherwise, there are distinct p and q such that, for each n, there is $(x_n,y_n) \in G \times G$, for some $G \in \mathcal{G}_n$, such that $(x_n,y_n) \in S_d(p,2^{-n}) \times S_d(q,2^{-n})$. But, $(X,\mathcal{T})$ is Hausdorff, since it is K-semimetrizable. Hence, we may choose disjoint open sets U and V with $p \in U$ and $q \in V$. Now, choose $m \in \mathbf{N}$ so that $x_n \in U$ and $y_n \in V$ for all $n \geq m$. It follows that $\{x_n | n \geq m\} \cup \{p\}$ and $\{y_n | n \geq m\} \cup \{q\}$ are disjoint compact sets that are not separated by d.

Conversely, suppose that $\langle \mathcal{W}_n \rangle$ is a wΔ-sequence and that $\langle U_n \rangle$ is a decreasing sequence of open sets in $X \times X$ such that $\Delta_X = \cap \{U_n | n \in \mathbf{N}\} = \cap \{\overline{U}_n | n \in \mathbf{N}\}$. Let $\mathcal{U}_n = \{G \in \mathcal{T} | G \times G \subseteq U_n\}$; note that $\mathcal{U}_n$ is a cover of X. With appropriate finite intersections of sets from these covers we may construct a sequence $\langle \mathcal{G}_n \rangle$ of open covers such that $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$, for each n, and $\mathcal{G}_n$ refines both $\mathcal{W}_n$ and $\mathcal{U}_n$. (Note that $\langle \mathcal{G}_n \rangle$ is also a wΔ-sequence.)

There is a distance function $d: X \times X \to \mathbb{R}$ such that, if $x \neq y$, then $d(x,y) = 2^{-n}$, where n is the first positive integer such that $x \notin st(y,\mathcal{G}_n)$. Note that $S_d(p,2^{-n}) = st(p,\mathcal{G}_n)$ so that $\mathcal{T}_d \subseteq \mathcal{T}$. Furthermore, we claim that $\{S_d(p,2^{-n}) \mid n \in \mathbb{N}\}$ is a neighborhood base for p in $(X,\mathcal{T})$. Otherwise, obtain $G \in \mathcal{T}$ and a sequence $\langle a_n \rangle$ such that $p \in G$, $a_n \in S_d(p,2^{-n})$, but $a_n \notin G$. Since $\langle \mathcal{G}_n \rangle$ is a wΔ-sequence, it must be that $\langle a_n \rangle$ clusters at a point q ($\neq$p) and, since $\langle \mathcal{G}_n \rangle$ refines $\langle \mathcal{U}_n \rangle$, there is $V \in \mathcal{T}$ such that $q \in V$ and $V \cap st(p,\mathcal{G}_n) = \emptyset$ for some n. This contradicts that q is a cluster point of $\langle a_n \rangle$. Thus, d is an admissible semimetric for $(X,\mathcal{T})$; moreover, d is developable since each open set in $\mathcal{G}_n$ has d-diameter less than 2^{-n}.

Finally, we show that d is a K-semimetric. If A and B are compact sets such that $d[A,B] = 0$, then choose sequences $\langle a_n \rangle$ in A and $\langle b_n \rangle$ in B such that $d(a_n,b_n) \to 0$, $d(a_n,p) \to 0$ and $d(b_n,q) \to 0$, for some point $p \in A$ and some point $q \in B$. We claim that $p = q$ which completes the proof. Otherwise, there are m and k such that $S_d(p,2^{-k}) \times S_d(q,2^{-k}) \cap U_m = \emptyset$. Now, there are $a_n \in S_d(p,2^{-k})$ and $b_n \in S_d(q,2^{-k})$ such that $d(a_n,b_n) < 2^{-m}$;

hence, $a_n \in st(b_n,\mathcal{G}_m)$ which contradicts that $S_d(p,2^{-k}) \times S_d(q,2^{-k}) \cap U_m = \emptyset$.

Our approach also provides an easy proof to an analogous theorem of Hodel. A topological space X *has a* G_δ^*-*diagonal* [6] iff there is a G_δ^*-*diagonal sequence for* X, that is, there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that $\{p\} = \cap \{\overline{st(p,\mathcal{G}_n)} \mid n \in N\}$. This definition parallels the well known result that X has a G_δ-diagonal iff there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that $\{p\} = \cap \{st(p,\mathcal{G}_n) \mid n \in N\}$.

Theorem 1.4. [6] *A Hausdorff space admits a developable semimetric iff it is a* wΔ-*space with a* G_δ^*-*diagonal.*

Proof. Suppose that d is a developable semimetric for the Hausdorff space $(X,\mathcal{T})$. If $\mathcal{G}_n$ is the set of open sets in $\mathcal{T}$ which have d-diameter less than 2^{-n}, then $\langle \mathcal{G}_n \rangle$ is easily a wΔ-sequence for $(X,\mathcal{T})$ such that $st(p,\mathcal{G}_n) \subseteq S_d(p,2^{-n})$. For $q \neq p$, there is an open neighborhood G of p such that $q \notin \overline{G}$, from which it follows that $\langle \mathcal{G}_n \rangle$ is also a G_δ*-diagonal sequence. The converse follows easily using the construction of our Theorem 1.3.

2. Developable Semimetrics and 1-Continuous Semimetrics

Any metric for X is a continuous distance function; any continuous distance function for X is both developable and 1-continuous. As in the case of metrics, when d is a 1-continuous distance function for X, $\mathcal{T}_d$ is a topology for X for which the set $\{S_d(p,\varepsilon) \mid p \in X, \varepsilon > 0\}$ of spheres is a base; thus, d is a symmetric for (X,T) iff d is a semimetric for (X,T).

Theorem 2.1. *For any separable space* (X,T), *if there is a* 1-*continuous distance function* d *for* X *such that* $T_d \subseteq T$, *then* (X,T) *is submetrizable (i.e., there is a metric* ρ *for* X *such that* $T_\rho \subseteq T$*).*

Proof. Suppose that d is a 1-continuous distance function on (X,T) and that $A = \{a_n \mid n \in \mathbf{N}\}$ is a countable dense subset. For each n, there is a pseudometric ρ_n on (X,T) such that $\rho_n(x,y) = \min\{2^{-n}, |d(x,a_n) - d(y,a_n)|\}$. Since A is dense, the pseudometric $\rho = \Sigma\{\rho_n \mid n \in N\}$ is a metric for X, and, since d is 1-continuous on $X \times X$, it follows that $T_\rho \subseteq T_d \subseteq T$.

Corollary 2.2. Ψ_R *is not* 1-*continuously semimetrizable.*

Proof. As we have indicated in Remark 1.2, Ψ_R does not have a regular G_δ-diagonal. Consequently, Ψ_R is not submetrizable. Since Ψ_R is separable, we conclude from Theorem 2.1 that any admissible semimetric for Ψ_R can not be 1-continuous.

Theorem 2.3. *If* (X,T) *is a semimetrizable space with a zero set diagonal, then* (X,T) *is* K-*semimetrizable.*

Proof. Suppose that d is a semimetric for (X,T) and $\alpha\colon X \times X \to [0,1]$ is a continuous function whose zero set is the diagonal. If d_1 is the distance function for X with $d_1(x,y) = \min\{\alpha(x,y), \alpha(y,x)\}$, then d_1 separates disjoint compact subsets of (X,T). It follows that $d + d_1$ is a K-semimetric for (X,T).

Corollary 2.4. *If* (X,T) *is a separable* 1-*continuously semimetrizable space, then* (X,T) *is* 1-*continuously* K-*semimetrizable.*

Proof. This follows easily by applying, first, Theorem 2.1 and then, using the construction of our proof of Theorem 2.3.

Remark 2.5. *Concerning admissible semimetric types*

For any R, Ψ_R is developable semimetrizable, but not 1-continuously semimetrizable (Corollary 2.2); it is K-semimetrizable, but not K-developable semimetrizable (Theorem 1.1).

Borges' example B_1 is 1-continuously K-semimetrizable (from [1] and Corollary 2.4, since it is separable); it is not developable semimetrizable [1].

Burke's example B_2 is developable semimetrizable, but not K-semimetrizable [2]; it is not 1-continuously semimetrizable (from Corollary 2.4, since it is separable).

Remark 2.6. *Concerning* G_δ-*diagonal types*

The Isbell-Mrówka spaces Ψ_R and Burke's example B_2 are separable $w\Delta$-spaces which have G_δ*-diagonals, but do not have regular G_δ-diagonals; see Theorems 1.3 and 1.4.

Borges' example $\mathbf{B}_1$ is a separable space with a zero set diagonal; it is not a wΔ-space.

Remark 2.7. *Concerning the Normal Moore Space Conjecture*

Borges' example is normal because it is regular and Lindelöf. It is not continuously semimetrizable because it is not developable semimetrizable. Thus, a normal 1-continuously semimetrizable space need not be continuously semimetrizable. This is of some interest because of its relationship to the Normal Moore Space Conjecture. Note that this result does not require additional set-theoretic axioms.

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