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1. Introduction

Scattered spaces have been studied by several authors (see [6], [7], [8], [11], [12], [13], [14], [15], [16] and [17]). Recently, in [11], [15] and [17] some generalizations of scattered spaces have been considered and have been extensively studied. Our interest in this topic was stimulated by some questions in [8] and some of the results obtained in [11], [15] and [18].

In this paper, we introduce the concept of ω-scattered spaces as a natural generalization of the concept of scattered spaces. It is proved that in the class of compact Hausdorff spaces the concept of ω-scatteredness of the space coincides with scatteredness. It is noted that ω-scattered need not be scattered in general. Also, the C-scattered spaces introduced in [15] are not comparable with the ω-scattered spaces. We start out by giving a characterization of ω-scattered spaces. Then, a relationship between ω-scatteredness of the space and scatteredness of some extensions is established. This relationship helps us to prove that Lindelöf P*-spaces are functionally countable and Lindelöf ω-scattered spaces are functionally countable. Later on, we show that for a compact Hausdorff space X, (i) X is scattered, (ii) X is ω-scattered and (iii) X is functionally countable are equivalent. Finally, some product theorem for a class of Lindelöf spaces have been established, and it is proved that a T_3, first countable, paracompact, and ω-scattered space is metrizable. The last result improves a result of Wicke and Worrell in [18].

2. Preliminaries

In this section some essential definitions are introduced, notations are explained and some basic facts which are essential in obtaining the main results are stated.

Throughout this paper X denotes a T_1 space. The symbol ω and c denote the cardinal number of integers and reals respectively. The cardinality of any set A is denoted by $|A|$.

Definition 2.1 [9]. A function $f: X \to Y$ is called *barely continuous* if, for every non-empty closed $A \subseteq X$, the restriction $f_{|A}$ has at least one point of continuity.

Definition 2.2 [8]. A space X is called *functionally countable* if every continuous real valued function on X has a countable image.

Definition 2.3 [8]. Given a topological space (X,T), $b(X,T)$ will represent the set X with the topology generated by the G_δ-sets of (X,T). Sometimes T is not mentioned and bX is written instead of $b(X,T)$.

Definition 2.4 [3]. A space X is called a P-space if the intersection of countably many open sets is open.

Now, we list some known results which will be helpful in obtaining the main results.

Theorem 2.5 [2]. *If* X *is a regular, Lindelöf, scattered space, then* bX *is Lindelof.*

Theorem 2.6 [8]. *If* X *is a regular, Lindelöf,* P-*space, then* X *is a functionally countable.*

Theorem 2.7 [9]. *If* f *is a barely continuous function from a hereditarily Lindelöf space* X *onto a space* Y, *then* Y *is Lindelöf.*

Theorem 2.8. *If* X *is a* T_2, *Lindelof,* P-*space, then* X *is normal.*

3. ω-Scattered Spaces

A space X is called *ω-scattered* if every non-empty subset A of X has a point x and an open neighborhood U_x of x in X such that $|U_x \cap A| \leq \omega$.

Every scattered space is ω-scattered but the converse is not true, because every countable space is ω-scattered, while the (countable) set of rationals with the usual topology is not scattered.

A space X is C-*scattered* [15], if every non-empty closed subset A of X has a point with a compact neighborhood in A. The following remark shows that ω-scattered spaces and C-scattered space are not comparable.

Remark 3.1. The set of rationals Q with usual topology is ω-scattered. However, it is not C-scattered since no point of Q has a compact neighborhood.

The set of reals $\mathbb{R}$ with usual topology is C-scattered (in fact, it is locally compact) but not ω-scattered.

A point x of a space X is called a *condensation point* of the set $A \subseteq X$ if every neighborhood of the point x contains an uncountable subset of A.

Definition 3.2 [4]. A subset A of a space X is called *ω-closed* if it contains all of its condensation points. The complement of an ω-closed set is called ω-open.

Observe that $A \subseteq X$ is ω-open iff for each x in A there is an open set U in X containing x such that $|U - A| \leq \omega$.

The next theorem characterizes ω-scattered spaces.

Theorem 3.3. *For any space* X *the following are equivalent:*

(i) X *is ω-scattered.*

(ii) *Every nonempty ω-closed subset* A *of* X *contains a point* x *which is not a condensation point.*

(iii) *There exists a well ordering $\leq$ of* X *such that for each* $x \in X$, *the set* $A_x = \{y \in X \mid y \leq x\}$ *has the property that for each* $y \in A_x$ *there exists an open set*

U_y *containing* y *such that* $|U_y \cap (X - A_x)| \leq \omega$, *i.e., for each* $x \in X$, *the set* A_x *is* ω*-open.*

Proof. (i) $\rightarrow$ (ii) is obvious.

(ii) $\rightarrow$ (iii). Let X be a space in which every nonempty closed subset has a point which is not a condensation point. Then X has a point x_1 which is not a condensation point. Now, $X - \{x_1\}$ is ω-closed in X and therefore $X - \{x_1\}$ has a point x_2 which is not a condensation point. Then $X - \{x_1, x_2\}$ is ω-closed. Finally, using transfinite induction one can complete the proof.

(iii) $\rightarrow$ (i). Let A be any nonempty subset of X. Since X is well ordered, A has a first element, say x_0. Now, by the hypothesis $A_{x_0} = \{y \in X | y \leq x_0\}$ is ω-open. Hence, X is ω-scattered.

Definition 3.4 [5]. A function $f: X \rightarrow Y$ is called ω*-continuous* at $x \in X$ if for every open set V containing f(x) there is an ω-open set U containing x such that $f(U) \subseteq V$. If f is ω-continuous at each point of X, then f is ω-continuous on X. A function $f: X \rightarrow Y$ is called *barely* ω*-continuous* if for every non-empty closed subset A of X, $f_{|A}$ has at least one point of ω-continuity.

The following theorem provides a basic tool to obtain some of the main results.

Theorem 3.5. *If* (X,T) *is a topological space and* T_ω *is the topology on* X *having as a base* $\{U - C | U \in T$ *and*

C is finite or countable}, then for any $A \subset X$ *the following holds:*

(i) *A is ω-open if an only if A is open in* (X,T_ω)*, i.e.,* $A \in T_\omega$.

(ii) *A is ω-closed if and only if A is closed in* (X,T_ω)*, i.e.,* $X - A \in T_\omega$.

(iii) $f\colon (X,T) \to Y$ *is ω-continuous if and only if* $f\colon (X,T_\omega) \to Y$ *is continuous.*

(iv) $f\colon (X,T) \to Y$ *is barely ω-continuous if and only if* $f\colon (X,T_\omega) \to Y$ *is barely continuous.*

The proof is straightforward.

Theorem 3.6. *If* (X,T) *is Lindelöf, then* (X,T_ω) *is Lindelöf.*

The proof is straightforward, therefore left for the reader.

Theorem 3.7. *If* $f\colon (X,T) \to Y$ *is barely ω-continuous and* (X,T) *is hereditarily Lindelöf, then Y is Lindelöf.*

Proof. It follows from theorem 3.6 that (X,T_ω) is hereditarily Lindelöf. By theorem 3.5, $f\colon (X,T_\omega) \to Y$ is barely continuous. Hence by theorem 2.7, Y is Lindelöf.

Theorem 3.8. (X,T) *is ω-scattered if an only if* (X,T_ω) *is scattered.*

The proof is obvious by the Theorem 3.5.

Definition 3.9 [4]. A space X is called a P*-space if the intersection of countably many open sets is ω-open.

Theorem 3.10. *If* (X,T) *is a* T_2, *Lindelöf* P*-*space, then* (X,T) *is functionally countable.*

Proof. Suppose (X,T) is a Lindelöf P*-space, then by Theorem 3.6, (X,T_ω) is Lindelöf. Now, (X,T_ω) is a T_2, Lindelöf P-space. Thus, by Theorem 2.8, (X,T_ω) is normal. Hence, by Theorem 2.6, (X,T_ω) is functionally countable. Let f: $(X,T_\omega) \to (X,T)$ be the identity function. Then, f is continuous. Since (X,T_ω) is functionally countable, it is easy to see that (X,T) is functionally countable.

Theorem 3.11. (X,T) *is* ω-*scattered if and only if every function* f *on* (X,T) *is barely* ω-*continuous.*

Proof. Suppose (X,T) is ω-scattered. Let f: $(X,T) \to Y$ be a function from (X,T) onto an arbitrary space Y. Let A be any ω-closed subset of X. Then, A contains a point x_0 which is not a condensation point by Theorem 3.3. Now, it is easy to conclude that $f_{|A}$ is ω-continuous at x_0. Hence, f is barely ω-continuous.

For the converse, suppose that any function f from (X,T) onto any space is barely ω-continuous. So, in particular the identity function i_x from (X,T) onto X with discrete topology is barely ω-continuous. Let A be any non-empty ω-closed subset of X. Then, $i_{x|A}$ is ω-continuous at some y in A, i.e., there is an ω-open set U such that $U \cap A = i_x^{-1}(i_x(y)) = \{y\}$. Hence, (X,T_ω) is scattered. Therefore, by Theorem 3.8 (X,T) is ω-scattered.

Notation. Let X be a topological space. Let $X^{(0)} = X$. Let $X^{(1)}$ denote the collection of condensation points of X. With $X^{(\alpha)}$ for an ordinal α, let $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

It is easy to see that X is ω-scattered if and only if $X^{(\alpha)} = \phi$ for some α.

Theorem 3.12. *If* X *is a Lindelöf* ω-*scattered space then* bX *is Lindelöf.*

Proof. Let α be an ordinal such that $X^{(\alpha)} = \phi$. α exists because X is ω-scattered. If $\alpha = 1$, then it is easy to see that X is countable because X is Lindelöf. Hence the result follows. Suppose we have proved the result for all $\beta < \alpha$. That is, if $\beta < \alpha$ and $X^{(\beta)} = \phi$, then bX is Lindelöf.

Case 1. There is $\beta < \alpha$ such that $\beta + 1 = \alpha$ and $X^{(\alpha)} = \phi$. It is easy to see that $X^{(\beta)}$ is a countable closed subset of X. Consider the open cover $\mathcal{U} = \{X - X^{(\beta)}\} \cup \{U_x | \ x \in X^{(\beta)}\}$ where $|U_x \cap X^{(\beta)}| \leq \omega$ for each x and U_x is open in X containing x. Since X is regular, there exists an open cover $\mathcal{H}$ of X such that the closure of members of $\mathcal{H}$ refines $\mathcal{U}$. X is Lindelöf implies $\mathcal{H}$ has a countable subcover $\mathcal{V}$. Now if $V \in \mathcal{V}$ and $\overline{V} \subseteq X - X^{(\beta)}$ then $\overline{V}^{(\beta)} = \phi$, i.e. $b\overline{V}$ is Lindelöf by the inductive assumption. Let $\mathcal{V}' = \{\overline{V} | \ V \in \mathcal{V}$, and $\overline{V} \subseteq (X - X^{(\beta)})\}$. Since $X^{(\beta)}$ is countable we have $bX^{(\beta)}$ is Lindelöf. Now $\mathcal{M} = \{X^{(\beta)}\} \cup \mathcal{V}'$ is countable closed cover of X such that for each $M \in \mathcal{M}$ we have bM is Lindelöf. Hence bX is Lindelöf.

Case 2. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)} = \phi$.

Consider the cover $\mathcal{U} = \{X - X^{(\beta)} \mid \beta < \alpha\}$ of X. Since X is regular, there exists an open cover $\mathcal{H}$ of X such that the closures of members of $\mathcal{H}$ refines $\mathcal{U}$. X is Lindelöf implies $\mathcal{H}$ has a countable subcover $\mathcal{V}$. Then for each $V \in \mathcal{V}$, $\overline{V}$ is in some $X - X^{(\beta)}$ for $\beta < \alpha$. Hence, for each $V \in \mathcal{V}$, $\overline{V}^{(\beta)} = \phi$. By the inductive assumption, for each $V \in \mathcal{V}$, $b\overline{V}$ is Lindelöf. Therefore, bX is Lindelöf.

Theorem 3.13. (i) *If* (X,T) *is a regular, Lindelöf, ω-scattered space, then* (X,T) *is functionally countable.*

(ii) *If* X *is a regular, Lindelöf, ω-scattered space such that each point of* X *is a* G_δ*-set, then* $|X| \leq \omega$.

Proof. (i) It follows from Theorem 3.12 that b(X,T) is Lindelöf. Also b(X,T) is a T_2 P-space. Hence by Theorem 2.6 and 2.8, b(X,T) is functionally countable. Let f: b(X,T) → (X,T) be the identity function. Then, f is continuous. Since b(X,T) is functionally countable, (X,T) is functionally countable.

The proof of (ii) follows easily from the Theorem 3.12.

Theorem 3.14. *If* (X,T) *is hereditarily Lindelöf ω-scattered space, then* (X,T) *is countable.*

Proof. Suppose (X,T) is hereditarily Lindelöf ω-scattered space. Let i_X be the identity function from

(X,T) into X with discrete topology. Then i_X is barely ω-continuous. Hence, by Theorem 3.7, $i_X(X)$ is Lindelöf. Therefore, X is countable.

In [8], the following theorem is attributed to Rudin [13] and Pelcyzynski and Semadeni [12].

Theorem 3.15. *For a compact Hausdorff space the following are equivalent:*

(i) X *is scattered.*

(iii) X *is functionally countable.*

It is natural to ask whether Theorem 3.15 remains true if we replace scattered by ω-scattered. The following theorem gives an affirmative answer to this question.

Theorem 3.16. *For a compact Hausdorff space* X *the following are equivalent:*

(i) X *is scattered.*

(ii) X *is* ω*-scattered.*

(iii) X *is functionally countable.*

Proof. (i) $\to$ (ii) is obvious

(ii) $\to$ (iii). It follows from Theorem 3.13.

(iii) $\to$ (i) follows from Theorem 3.15.

4. Product of Lindelöf ω-Scattered Spaces

Theorem 4.1. *If* bX *and* Y *are Lindelöf spaces, then* $X \times Y$ *is Lindelöf.*

The proof that $bX \times Y$ is Lindelöf follows an argument similar to the one used in ([6], Vol. II, page 16) to prove that the product of two compact spaces is compact. Since $X \times Y$'s topology is weaker than $bX \times Y$'s, $X \times Y$ is Lindelöf.

Theorem 4.2. *If* X *is a regular, Lindelöf, ω-scattered space and* Y *is any Lindelöf space, then* $X \times Y$ *is Lindelöf.*

Proof. It follows from Theorem 3.12 that bX is Lindelöf. Hence, by Theorem 4.1, $X \times Y$ is Lindelöf.

Corollary 4.3. *A finite product of Lindelöf ω-scattered spaces is Lindelöf.*

In [10], it was shown that a countable product of Lindelöf P-spaces is Lindelöf. Using this result we can obtain the following theorem.

Theorem 4.4. *A countable product of regular, Lindelöf, ω-scattered spaces is Lindelöf.*

Proof. Let $\{X_n \mid n \leq \omega\}$ be a family of Lindelöf ω-scattered spaces. Then, by Theorem 3.12, each bX_n is a Lindelöf. Hence $\prod_{n\leq\omega} bX_n$ is Lindelöf. Since $\prod_{n\leq\omega} bX_n$ maps continuously onto $\prod_{n\leq\omega} X_n$, we obtain that $\prod_{n\leq\omega} X_n$ is Lindelöf.

In [7], Kunen proved that if each X_n is a Hausdorff compact scattered space, then the box product $\square_{n\leq\omega} X_n$ is c-Lindelöf.

In view of Theorem 3.16, we can state Kunen's result as follows:

Theorem 4.5. *If each* X_n *is a Hausdorff compact, ω-scattered space, then the box product* $\square_{n \leq \omega} X_n$ *is c-Lindelöf.*

5. Metrizability of ω-Scattered Spaces

In [18], it was shown that every regular, first countable, paracompact, scattered space is metrizable. In this section, we obtain a generalization of this result using ω-scattered spaces.

Definition 5.1 [11]. A space X is called *σ-discrete* if it is a union of countably many closed discrete subspaces.

Definition 5.2. A space X is called F_σ*-screenable* if every open cover of X has a σ-discrete closed refinement.

Definition 5.3. A subset Y of a space X is called *locally countable* if for each $y \in Y$ there is an open neighborhood U_y in X containing y such that $|U_y \cap Y| \leq \omega$.

Lemma 5.4. *If* X *is* F_σ*-screenable (or metalindelöf) and locally countable, then* X *is σ-discrete.*

Proof. We prove the lemma when X is F_σ-screenable and locally countable. The other case follows similarly. By the assumptions, X has an open cover $\mathcal{U} = \{U_\beta \mid \beta \in \Gamma\}$ such that $|U_\beta| \leq \omega$ for each $\beta \in \Gamma$. X is F_σ-screenable implies there exists a σ-discrete closed refinement $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ where $\mathcal{F}_i = \{F_{i\alpha} \mid \alpha \in \Lambda_i\}$ for $i \in N$. Since each U_β is countable and $\mathcal{F}$ refines $\mathcal{U}$, we see that $|F_{i\alpha}| \leq \omega$ for each i and α. Hence $F_{i\alpha}$ is σ-discrete for each i and α. Let $F_{i\alpha} = \{x_{ij\alpha} | j \in N\}$ and $G_{ij} = \{x_{ij\alpha} | \alpha \in \Lambda_i\}$. Then it is obvious that G_{ij} is discrete, closed and $X = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} G_{ij}$. Therefore, X is σ-discrete.

Lemma 5.5. *If* X *is* F_σ*-screenable and* ω*-scattered, then* X *is* σ*-discrete.*

Proof. Let α be an ordinal such that $X^{(\alpha)} = \phi$. α exists because X is ω-scattered. If $\alpha = 1$, then it is easy to see that X is locally countable and by Lemma 5.4 the result follows. Suppose we have proved the result for all $\beta < \alpha$ and $X^{(\beta)} = \phi$, then X is σ-discrete.

Case 1. There is $\beta < \alpha$ such that $\alpha = \beta + 1$ and $X^{(\alpha)} = \phi$. It is easy to see that $X^{(\beta)}$ is a closed locally countable subset of X. Consider the open cover $\mathcal{U} = \{X - X^{(\beta)}\} \cup \{U_x | x \in X^{(\beta)}\}$ where $|U_x \cap X^{(\beta)}| \leq \omega$ for each x and U_x is open in X containing x. X is F_σ-screenable implies $\mathcal{U}$ has a σ-discrete closed refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where $\mathcal{V}_n = \{V_{n\lambda} | \lambda \in \Lambda_n\}$. Note that each $V_{n\lambda}$ is F_σ-screenable and ω-scattered. Also if $V_{n\lambda} \subseteq X - X^{(\beta)}$ then $V_{n\lambda}^{(\beta)} = \phi$, i.e. $V_{n\lambda}$ is σ-discrete by the inductive assumption. Let $\mathcal{V}' = \{V | V \in \mathcal{V}, \text{ and } V \subseteq X - X^{(\beta)}\}$, then $\mathcal{V}'$ covers $X - X^{(\beta)}$. Since $X^{(\beta)}$ is a closed subset of X, it follows by Lemma 5.4 that $X^{(\beta)}$ is σ-discrete. Now $\mathcal{M} = \{X^{(\beta)}\} \cup \mathcal{V}'$ is a σ-discrete closed cover of X with each member is σ-discrete. Hence, it is easy to conclude that X is σ-discrete.

Case 2. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)} = \varphi$.

Consider the cover $\mathcal{U} = \{X - X^{(\beta)} \mid \beta < \alpha\}$ of X. Let $\mathcal{V}$ be a σ-discrete closed refinement of $\mathcal{U}$. Then, each $V \in \mathcal{V}$ is in some $X - X^{(\beta)}$ for $\beta < \alpha$. Hence $V^{(\beta)} = \phi$ for each $V \in \mathcal{V}$. Therefore, for each $V \in \mathcal{V}$, V is σ-discrete by the inductive assumption. Hence, it is easy to conclude that X is σ-discrete.

Theorem 5.6. *If* X *is a regular, first countable, paracompact,* ω*-scattered space, then* X *is metrizable.*

Proof. It follows from Lemma 5.5 that X is σ-discrete. Now, it is well known that a σ-discrete first countable space is developable. Thus X is developable. Therefore by Bing's metrization theorem (see [1], p. 408), X is metrizable.

Corollary 5.7 [18]. *If* X *is a regular, first countable, scattered, paracompact space, then* X *is metrizable.*

Finally, we suggest, the following questions.

Question 5.8. Which spaces (X,T) have (X,T_ω) paracompact?

Question 5.9. When are regular Lindelöf, ω-scattered spaces, scattered?

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