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INFINITE PRODUCTS OF COOK CONTINUA

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The relationship between products of continua and homogeneity properties is fascinating and complex. In some cases, products of continua have much nicer homogeneity properties than the factor continua themselves have. Consider, for example, $\Pi_{i=1}^{\infty} X_i$ where for each i, X_i is the unit interval, the triod, or in fact, any AR. Even though the factor continua themselves fail to be even homogeneous, the product continuum is the Hilbert cube Q, which has just about the strongest homogeneity properties possible. (See [A3], [A4], [W].) Stranger still are examples due to J. van Mill, [vM], F. D. Ancel and S. Singh [AS], and F. D. Ancel, P. F. Duvall, and S. Singh [ADS]. A continuum is *rigid* if its only self-homeomorphism is the identity. Van Mill has an example of an infinite-dimensional rigid continuum X such that X^2 is the Hilbert cube; Ancel and Singh, and Ancel, Duvall, and Singh have, for $n \geq 3$, examples of rigid finite-dimensional continua whose squares are $S^n \times S^n$.

On the other hand, in some cases the product operation destroys many homogeneity properties. Consider the situation where M is the Menger universal curve. Except for the fact that M admits no isotopies, M has homogeneity properties as strong as those of the Hilbert cube. (This follows from the work of R. D. Anderson [A1], [A2], and, more recently, Mladen Bestvina [B].) A space is *n-homogeneous* means that for each pair A,B of n-element subsets of the space, there is a self-homeomorphism h that takes A to B. One of M's nice homogeneity properties is the fact that it is n-homogeneous for all n. However, it follows from results of K. Kuperberg, W. Kuperberg, and W. R. R. Transue [KKT], and J. K. Phelps [P1], [P2] that not only does M^2 fail to be even 2-homogeneous, but also if X is *any* continuum, then $M \times X$ is not 2-homogeneous. Further, if n is a positive integer greater than 1, or $n = \infty$, M^n is factorwise rigid, i.e., the only homeomorphisms it admits are product homeomorphisms composed with coordinate switching homeomorphisms. Another product continuum with this factorwise rigidity property is P^n, where P denotes the pseudoarc and n is a positive integer greater than 1 or $+\infty$. (See D. Bellamy and J. Lysko [BL], and D. Bellamy and J. Kennedy [BK].) Here again, if one takes into account composant considerations, this continuum has very strong homogeneity properties. (There has been much work done on the homogeneity properties of this continuum. For some of the most recent and for references to the other, see [K1], [K2] and [L].)

A topological space is *homogeneous* if it is 1-homogeneous. The homogeneity properties of a given space include properties both weaker and stronger than homogeneity itself. By homogeneity properties, we mean those properties of a space that have to do with how subgroups of the full homeomorphism group of the space act on the space. Howard Cook [C] has constructed an example of a continuum whose only self maps are trivial ones. Let us say then that the continuum X is a *Cook continuum* if whenever f: X → X is continuous, then f is the identity or f is a constant map. Cook continua are rigid continua, and they belong at the very end of our spectrum of homogeneity properties. It is the purpose of this paper to investigate the homogeneity properties of X^∞ where X is a Cook continuum, although along the way some more general information is obtained, as well as some additional examples.

Herein, $\mathbb{Z}$ will denote the integers and $\mathbb{N}$ will denote the positive integers.

If X is a space, H(X) will denote the group of all self homeomorphisms of X. A homeomorphism $T \in H(X)$ is said to be *transitive* if there is some $x \in X$ such that the orbit of x under T, $O_T(x) = \{T^n(x) | n \in \mathbb{Z}\}$, is dense in X. If $T \in H(X)$, then the point p in X is a *periodic point* of T if there is $n \in \mathbb{N}$ such that $T^n(p) = p$.

The proofs that follow will make heavy use of the following fact: If X is a zero-dimensional, completely metrizable, separable space that is dense in itself and X contains no nonempty compact open sets, then X is homeomorphic to the irrationals.

Theorem 1. *Suppose* X *is a separable metrizable space. Then* X^{∞} *admits a transitive homeomorphism* T. *Further,* T *admits a dense set of periodic points.*

Proof. For each integer i, let $X_i = X$ and consider $\Pi_{i=-\infty}^{\infty} X_i$. Let D denote a countable dense subset of X and for $i \in \mathbb{N}$, let $D_i = \{(d_1, d_2, \ldots, d_i) | d_j \in D \text{ for } 1 \leq j \leq i\}$. Since D_i is countable, so is $\hat{D} = \cup_{i=1}^{\infty} D_i$. Further, there is a countable set $E = \{\ldots e_{-1}, e_0, e_1, e_2, \ldots\}$ such that if $(d_1, d_2, \ldots, d_i) \in \hat{D}$, then there is some $m \in \mathbb{Z}$ such that $(e_m, e_{m+1}, \ldots, e_{m+i-1}) = (d_1, \ldots, d_i)$. Also, $e = (\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots) \in \Pi_{i=-\infty}^{\infty} X_i$. Suppose that s denotes the shift homeomorphism on $\Pi_{i=-\infty}^{\infty} X_i$, i.e., $s(x) = y$ where $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$, $y = (\ldots, y_{-1}, y_0, y_1, \ldots)$ and $y_i = x_{i+1}$ for $i \in \mathbb{Z}$.

Suppose that o is a basic open set in $\Pi_{i=-\infty}^{\infty} X_i$, i.e., there is a finite subset B of $\mathbb{Z}$ and a collection $\{o(i) | i \in B\}$ of open sets of X such that $o = \{(\ldots, y_{-2}, y_{-1}, y_0, y, \ldots) \in \Pi_{i=-\infty}^{\infty} X_i | \text{ for } i \in B, y_i \in o(i)\}$. Suppose that m_1 is the smallest integer in B and m_2 is the largest. There is $(d_1, d_2, \ldots, d_{m_2-m_1+1}) \in D_{m_2-m_1+1}$ such that $d_{i+1-m_1} \in o(i)$ for $i \in B$. Thus, for some $k \in \mathbb{Z}$, $e_{k+i} = d_i$ for $i \in \{1, \ldots, m_2 - m_1 + 1\}$, and $e_{k+i+1-m_1} = d_{i+1-m_1} \in o(i)$ for $i \in B$. Further, $s^{k+1-m_1}(e) \in o$, for $s^{k+1-m_1}(e) = e'$ where $e' = (\ldots, e'_{-2}, e'_{-1}, e'_0, e'_1, \ldots)$ and $e'_i = e_{i+k+1-m_1} = d_{i+1-m_1} \in o(i)$ for $i \in B$. It follows that $O_s(e)$ is dense in X^∞, and that s is transitive on $\Pi_{i=-\infty}^{\infty} X_i$.

Suppose $\{a_0,\ldots,a_{n-1}\}$ is a finite subset of X. Let $\hat{a} \in \Pi_{i=-\infty}^{\infty} X_i$ be defined as follows: $\hat{a} = (\ldots,\hat{a}_{-1},\hat{a}_0,\hat{a}_1,\ldots)$ and $\hat{a}_i = a_{i \bmod n}$. Then $s^n(\hat{a}) = \hat{a}$, and each finite subset of X gives rise to a periodic point (actually, to exactly n! periodic points) of $\Pi_{i=-\infty}^{\infty} X_i$. Since the collection of all such periodic points is dense in $\Pi_{i=-\infty}^{\infty} X_i$, s admits a dense set of periodic points. Finally, $\Pi_{i=-\infty}^{\infty} X_i$ is homeomorphic to X^∞, and $H(X^\infty)$ contains a homeomorphism T that is transitive and admits a dense set of periodic points.

Thus, no matter what separable metrizable space X is, X^∞ admits at least a dense orbit under the action of the full homeomorphism group $H(X^\infty)$, and in fact, under the action of the subgroup $G_T = \{T^n | n \in \mathbb{Z}\}$. The product operation all by itself induces a sort of weak homogeneity. Let us make the following definition: A space X is *weakly homogeneous* if there is some x in X such that $\{g(x) | g \in H(X)\}$ is dense in X. Also, for $x \in X$ denote by $O(x)$ the orbit of x under $H(X)$, i.e., $O(x) = \{g(x) | g \in H(X)\}$.

It follows then that if X is a Cook continuum, X^∞ admits a transitive homeomorphism, and X^∞ is weakly homogeneous. It is not terribly surprising that for X a Cook continuum, X^∞ admits only the minimum as far as self maps go, but this does require some proof.

Suppose Γ is an indexing set, and $A \subseteq \Gamma$. Let $\pi_A: X^\Gamma \to X^A$ denote the projection map, i.e., if $x \in X^\Gamma$

with γ^{th} coordinate x_γ for $\gamma \in \Gamma$, then $\pi_A(x) \in X^A$ where for $\alpha \in A$, the α^{th} coordinate of $\pi_A(x) = x_\alpha$. If $A = \{\alpha\}$, we will write π_α rather than $\pi_{\{\alpha\}}$.

Theorem 2. *Suppose* Γ *is an indexing set,* X *is a* Cook *continuum, and* $f: X^\Gamma \to X^\Gamma$ *is continuous. There are a collection* $B = \{B(\alpha) | \alpha \in \Gamma\}$ *of mutually disjoint subsets of* Γ *and a subset* $C = \{c_\gamma | \gamma \notin \cup_{\alpha\in\Gamma} B(\alpha)\}$ *of* X *such that* $f(x) = w$ *where for* $\beta \in \Gamma$,

$$\pi_\beta(w) = \begin{cases} \pi_\alpha(x) & \text{if } \beta \in B(\alpha) \\ c_\beta & \text{if } \beta \notin \cup_{\alpha\in\Gamma} B(\alpha). \end{cases}$$

Proof. First, we need to establish some notational conventions which will be used throughout this proof. Let $C(X) = \{g: X \to X | g \text{ is continuous}\}$ and let $\hat{C}(X) = \{g \in C(X) | g \text{ is a constant map}\}$. If 1_X denotes the identity on X, then $C(X) = \hat{C}(X) \cup \{1_X\}$. There is a natural continuous map from $\hat{C}(X)$ onto X, which we will call ψ. (That is, $\psi(g) = c$ where $g \in \hat{C}(X)$ and c is that point of X such that $g(x) = c$ for $x \in X$.)

For $A \subseteq \Gamma$, $x \in X^\Gamma$, define $Z(A,x) = \{y \in X^\Gamma | \pi_\beta(y) = \pi_\beta(x) \text{ for } \beta \notin A\}$, and $\tilde{Z}(A,x) = \{\pi_A(y) | y \in Z(A,x)\}$. Thus, $Z(A,x) \subseteq X^\Gamma$ and $\tilde{Z}(A,x) = X^A$. Again, for $A = \{\alpha\}$ we will write $Z(\alpha,x)$ for $Z(\{\alpha\},x)$ and $\tilde{Z}(\alpha,x)$ for $\tilde{Z}(\{\alpha\},x)$. For $x \in X^\Gamma$, $y \in X$, and $\beta \in \Gamma$, define $x(\beta,y) = w \in X^\Gamma$ where $\pi_p(w) = \pi_p(x)$ for $p \neq \beta$, and $\pi_\beta(w) = y$. Also, let $\psi(A,x)$ denote the natural homeomorphism from $\tilde{Z}(A,x)$ to $Z(A,x)$.

Suppose that $\alpha \in \Gamma$, $x \in X^{\Gamma}$. Consider $Z(\alpha,x)$ and let $B(\alpha,x) = \{\beta \in \Gamma | \pi_{\beta} f | Z(\alpha,x)$ is a homeomorphism$\}$. If for some $x \neq y \in X^{\Gamma}$ and $\beta' \in \Gamma$, $\beta' \in B(\alpha,x) - B(\alpha,y)$, then if $H = \{z \in X^{\Gamma} | \pi_{\beta'} f | Z(\alpha,z)$ is a homeomorphism$\}$ and $K = \{z \in X^{\Gamma} | \pi_{\beta'} f | Z(\alpha,z)$ is a constant map$\}$, H and K are disjoint nonempty closed sets whose union is X^{Γ}, which is not possible. Then for $x,y \in X^{\Gamma}$, $B(\alpha,x) = B(\alpha,y)$ and we can define $B(\alpha) = B(\alpha,x)$. (Note that it is possible that $B(\alpha) = \emptyset$.) Also, for $\alpha,\beta \in \Gamma$ and $\alpha \neq \beta$, $B(\alpha) \cap B(\beta) = \emptyset$, for otherwise we have a contradiction to the fact that if $x \in X^{\Gamma}$, $\pi_{\gamma} f | Z(\{\alpha,\beta\},x)\colon Z(\{\alpha,\beta\},x) \to X$ is a function.

Suppose $\gamma \notin \cup_{\alpha\in\Gamma} B(\alpha)$. Then for each $\alpha \in \Gamma$, $x \in X^{\Gamma}$, there is a point $c_{\gamma}(\alpha,x) \in X$ such that $(\pi_{\gamma} f | Z(\alpha,x))(z) = c_{\gamma}(\alpha,x)$ for each $z \in X$. Fix $x_0 \in X^{\Gamma}$. For each $\beta' \neq \alpha$ define $\Theta_{\beta'}\colon \tilde{Z}(\beta',x_0) = X \to \hat{C}(X)$ by $\Theta_{\beta'}(y) = \pi_{\gamma}(f | Z(\alpha,x_0(\beta',y))) \circ \psi(\alpha,x_0(\beta',y))$ for $y \in X$. Now $\Theta_{\beta'}$ is a continuous function, as is $\psi\colon \hat{C}(X) \to X$. It follows that $\psi\Theta_{\beta'}\colon X \to X$ is a constant map for each β'. (Otherwise $\psi\Theta_{\beta'} = 1_X$, $\psi\Theta_{\beta'} = \pi_{\gamma} \circ (f | Z(\beta',x_0)) \circ \psi(\beta',x_0)$, and $\gamma \in \cup_{\alpha\in\Gamma} B(\alpha)$.) Since $x_0 \in Z(\beta',x_0)$, $\pi_{\beta'}(x_0) \in \tilde{Z}(\beta',x_0)$ and $\psi\Theta_{\beta'}(\pi_{\beta'}(x_0)) = c_{\gamma}(\alpha,x_0) = \psi\Theta_{\beta'}(y)$ for each $y \in X$. Further, if $\hat{\beta},\beta'$ are 2 elements of $\Gamma - \{\alpha\}$, then $x_0 \in Z(\beta',x_0) \cap Z(\hat{\beta},x_0)$ so that for $z \in X^{\Gamma}$ which has the same coordinates as x_0 except for perhaps the α^{th}, β'^{th} and $\hat{\beta}^{th}$-coordinates, $\pi_{\gamma} \circ (f | Z(\alpha,z)) \circ \psi(\alpha,z)(y) = c_{\gamma}(\alpha,x_0)$ for $y \in X$. By induction, if B is any finite subset of $\Gamma - \{\alpha\}$, $z \in X^{\Gamma}$ with $\pi_p(z) = \pi_p(x_0)$ for $p \notin B \cap \{\alpha\}$, $\pi_{\gamma} \circ (f | Z(\alpha,z)) \circ$

$\psi(\alpha,z)(y) = c_\gamma(\alpha,x_0)$ for $y \in X$. Then Θ: $X^\Gamma \to \hat{C}(X)$ defined by $\Theta(z) = \pi_\gamma f|Z(\alpha,z)$ is a continuous map and for a dense subset of X^Γ (namely all points of X^Γ whose coordinates are the same as those of x_0 except for a finite subset of Γ), $\Theta(z)$ is the constant map from X to itself that takes each point of X to $c_\gamma(\alpha,x_0)$. Thus, for any $z \in X^\Gamma$, $\Theta(z)$ is that same constant map. Likewise, $c_\gamma(\alpha,x_0) = c_\gamma(\alpha',x_0)$ for any $\alpha' \in \Gamma$, and it makes sense then to speak of $c_\gamma = c_\gamma(\alpha,x_0) = c_\gamma(\alpha',x_0)$ for $\alpha' \in \Gamma$.

Thus, we can classify completely the continuous self maps of X^Γ. For f: $X^\Gamma \to X^\Gamma$ continuous, $z \in X^\Gamma$, $f(z) = w$ where for $\gamma \in \Gamma$,

$$\pi_\gamma(w) = \begin{cases} \pi_\alpha(z) & \text{if } \gamma \in B(\alpha) \\ c_\gamma & \text{if } \gamma \notin \cup_{\alpha\in\Gamma} B(\alpha). \end{cases}$$

Corollary 3. *Suppose* Γ *is an indexing set,* X *is a* Cook *continuum, and* $f \in H(X^\Gamma)$. *There is a one-to-one surjective mapping* σ_f: $\Gamma \to \Gamma$ *such that for* $x \in X^\Gamma$, $f(x) = w$, *where for* $\gamma \in \Gamma$, $\pi_\gamma(w) = \pi_\alpha(x)$ *and* $\sigma_f(\alpha) = \gamma$. *In other words,* $H(X^\Gamma)$ *consists of precisely the coordinate switching homeomorphisms* (plus the identity).

Proof. Since f is one-to-one and onto, $B(\alpha)$ (defined in the proof of Theorem 2) must not be empty, but must be degenerate for each α. Define then $\alpha_f(\alpha) = \gamma$ where $\{\gamma\} = B(\alpha)$. Further, $\cup_{\alpha\in\Gamma} B(\alpha) = \Gamma$, for otherwise f is not onto. The rest follows.

Theorem 4. *Suppose* X *is a nondegenerate continuum. If* $S = \{f \in H(X^{\infty}) \mid$ *for some* $\sigma \in H(\mathbb{N})$, $\pi_{\sigma(i)}f(x) = \pi_i(x)$ *for* $i \in \mathbb{N}$ *and* $x \in X^{\infty}\}$, *then* S *is a closed subgroup of* $H(X)^{\infty}$ *and* S *is homeomorphic to the irrationals.*

Proof. It is clear that S is a subgroup of $H(X^{\infty})$. Since the irrationals are homeomorphic to $\mathbb{N}^{\infty}$, we can think of an irrational as being simply a function from $\mathbb{N}$ into $\mathbb{N}$. Further, if $\mathcal{B}$ denotes the collection of all subsets of $\mathbb{N}$ containing two integers, then $\mathcal{B}$ is countable. For $B \in \mathcal{B}$, $i \in \mathbb{N}$, let $F_{Bi} = \{(x_1,x_2,\ldots) \in \mathbb{N}^{\infty} \mid$ for each $j \in B$, $x_j = i\}$. Since each F_{Bi} is closed in $\mathbb{N}^{\infty}$, $\cup_{Bi}F_{Bi}$ is an F_{σ}-subset of $\mathbb{N}^{\infty}$, and $D = \mathbb{N}^{\infty} - \cup_{Bi}F_{Bi}$ is a G_{δ}-subset of $\mathbb{N}^{\infty}$. Note that $D \neq \emptyset$ and each point of D is a limit point of D. By a classical result of Alexandroff, D is a completely metrizable space. Since D also has the property that it has no isolated points and it is zero-dimensional, D is homeomorphic to the irrationals.

For each n, $i \in \mathbb{N}$, let $G_{ni} = \{(x_1,x_2,\ldots) \in \mathbb{N}^{\infty} \mid x_i = n\}$. Then each G_{ni} is open in $\mathbb{N}^{\infty}$, and $G_n = \cup_i G_{ni}$ is open in $\mathbb{N}^{\infty}$. Further, $G = \cap_{n\in\mathbb{N}} G_n$ is a nonempty G_{δ}-set in $\mathbb{N}^{\infty}$. It has no isolated points, and is also homeomorphic to the irrationals; and this is also the case with $D \cap G = E$, which is the set we really want to discuss, for any member of E represents a one-to-one functions from $\mathbb{N}$ onto $\mathbb{N}$, and E is therefore in one-to-one correspondence with $H(\mathbb{N})$.

Now $H(\mathbb{N})$ is in one-to-one correspondence with S, but more than this E is homeomorphic to S. Then S is homeomorphic to the irrationals. Further, S is completely metrizable, so by a classical result of Mazurkiewicz, S is a G_δ-subset of $H(X^\infty)$. Since S is also a subgroup of $H(X^\infty)$, and G_δ-subgroups of completely metrizable groups are always closed subgroups, it follows that S is closed in $H(X^\infty)$.

Corollary 5. *If* X *is a* Cook *continuum,* $H(X^\infty) = S = \{f \in H(X^\infty) \mid$ *for some* $\sigma \in H(\mathbb{N})$, *if* $x \in X^\infty$, $i \in \mathbb{N}$, $\pi_{\sigma(i)} f(x) = \pi_i x\}$. *Thus,* S *is homeomorphic to the irrationals.*

Proof. This follows from Corollary 3 and Theorem 4.

Remark 6. It follows from Theorem 1 and Corollary 5 that if X is a Cook continuum, then X^∞ is weakly homogeneous and $H(X^\infty)$ is zero-dimensional and homeomorphic to the irrationals. A space X is *nearly homogeneous* if for each x, $Gx = \{h(x) \mid h \in H(X)\}$ is dense in X. Thus, a nearly homogeneous space is weakly homogeneous, but not every weakly homogeneous space is nearly homogeneous, for X^∞ with X a Cook continuum is weakly homogeneous without being nearly homogeneous. (Consider $x \in X$ and $\hat{x} \in X^\infty$ defined by $\pi_i \hat{x} = x$ for $i \in \mathbb{N}$. Then $G\hat{x} = \{\hat{x}\}$.)

There are also nearly homogeneous continua with full homeomorphism groups topologically equivalent to the irrationals. One such example is Example 2 of [Kl] which essentially is a simple closed curve which has been "pinched" at a countable dense collection of places to yield a regular curve which admits exactly 2 orbits under its full homeomorphism group: One is a countable dense set and the other is a dense G_δ-set. Beverly Brechner [B] has proven that any regular curve which contains no free arc (i.e., each arc in the space is nowhere dense in the space) has a zero dimensional homeomorphism group. Suppose Y is a compact metric space, H(Y) is a completely metrizable separable topological group, which is, in addition, zero-dimensional. Now H(Y) must also be dense in itself, and can't contain any compact open sets. (If it did, the dense G_δ orbit would contain compact open sets in its relative topology.) Thus, H(Y) is homeomorphic to the irrationals.

It has been conjectured that if Z is a homogeneous continuum then H(Z) is not zero-dimensional. In fact, many topologists suspect that H(Z) must be infinite-dimensional, although it may well be totally disconnected (as is the case with Menger universal curve). (Recall that Z is homogeneous if $Gz = Z$ for $z \in Z$.)

Howard Cook also has an example of a continuum N with H(N) homeomorphic to the Cantor set [C]. No such continuum can be weakly homogeneous, as we now prove.

Theorem 7. *If* Z *is a weakly homogeneous nondegenerate continuum, then* H(Z) *is not compact. Further, if* H(Z) *is locally compact, then if* $z \in Z$ *with* Gz *a nondegenerate connected subset of* Z, Gz *is a first category,* F_σ*-set in* Z.

Proof. If Z is weakly homogeneous, there is some $z_0 \in Z$ such that Gz_0 is dense in Z. If H(Z) is compact, so is Gz_0, since $E_{z_0} : H(Z) \to Gz_0$ defined by $E_{z_0}(g) = g(z_0)$ is a continuous surjective map. Then $Gz_0 = Z$ and Z is a homogeneous continuum. This is impossible for if Z is homogeneous, H(Z) cannot even be locally compact. (See P3.)

Suppose H(Z) is locally compact and Z is weakly homogeneous and nondegenerate. Suppose $z \in Z$ and Gz is second category in Z, with Gz connected and nondegenerate. From a theorem of James Keesling [K], it follows that H(Z) is zero-dimensional. It follows from Effros' Theorem that if Gz is second category in Z, Gz is a G_δ-set in Z and is, considered as space, completely metrizable. Then there is some open and compact set u in H(Z) such that $1_z \in u$ and $uz \neq Gz$. Further, since Gz is not countable, every point of Gz is a limit point of Gz and uz is open in the relative topology on Gz and compact in the whole space. But this can't be, for then Gz is the union of two closed disjoint nonempty sets, and is not connected.

Hence, Gz is not second category in Z, so it must be first category in Z. Since H(Z) has a basis of compact open sets, Gz must also be an F_σ-set in Z.

Theorem 8. *If* X *is a* Cook *continuum and* $h \in H(X^\infty)$, *then* h *admits a dense set of periodic points.*

Proof. If $h \in H(X^\infty)$, there is some $s \in H(\mathbb{N})$ such that for $x \in X^\infty$, $h(x) = w$ where for $i \in \mathbb{N}$, $\pi_i(w) = \pi_j(x)$

and $s(j) = i$. Suppose o is a basic open set in X^∞. Then there are some $n \in \mathbb{N}$ and a collection $\{o_1, \ldots o_n\}$ of open sets in X such that $o = \{y \in X^\infty | \pi_i y \in o_i$ for each $i \leq n\}$.

Consider s. If $i \in \mathbb{N}$, $Or(i) = \{k \in \mathbb{N} |$ for some integer j, $s^j(i) = k\} \subseteq \mathbb{N}$ and $\{Or(i) | i \in \mathbb{N}\}$ partitions $\mathbb{N}$. Denote $Or(i)$ by $\{\ldots.p(i, -2), p(i, -1), p(i, 0), p(i, 1), \ldots\}$, where $s(p(i,m)) = p(i,m + 1)$ for $m \in \mathbb{Z}$. Suppose $\{Or(i_1), \ldots Or(i_\alpha)\}$ denotes $\{Or(i) | Or(i) \cap \{1, \ldots n\} \neq \emptyset\}$, where i_j is the first representative of $Or(i_j)$ contained in $\{1, \ldots n\}$. There is $M \in \mathbb{N}$ such that $\{1, \ldots n\} \subseteq \cup_{j=1}^{\alpha} \{p(i_j, 0), \ldots . p(i_j, M - 1)\}$, and if $|Or(i_j)| < \infty$, then $|Or(i_j)|$ divides M. Choose $y \in X$ and, for $p(i_j,k) \in \cup_{j=1}^{\alpha}\{p(i_j,0,p(i_j,1), \ldots (p(i_j,M - 1)\}$, choose $y_{p(i_j,k)} \in \pi_{p(i_j,k)} o$. Then define $z \in X^\infty$ as follows:

(1) For $\ell \notin \cup_{j=1}^{\alpha} Or(i_j)$, $\pi_\ell(z) = y$

(2) For $\ell \in Or(i_j)$, there is k such that $\ell = p(i_j,k)$, and there is an integer ℓ' such that $k \in \{\ell'M, \ell'M + 1, \ldots, (\ell' + 1)M-1\}$. Let $\pi_\ell(z) = \pi_{p(i_j,k)}(z) = y_{p(i_j,t)}$ where $k = \ell'M + t$ and $t \in \{0, \ldots M - 1\}$.

Note that z will be well-defined, even if for a given $\ell \in Or(i_j)$ there are many k such that $\ell = p(i_j,k)$. Further, $z \in o$ and for $t \in \{0, \ldots M - 1\}$, $\ell' \in \mathbb{Z}$, $\pi_{p(i_j,t)}(z) = \pi_{p(i_j,\ell'M+t)}(z)$, and $s^{\ell'M}(p(i_j,t) = p(i_j,\ell'M + t)$. Suppose $\beta \in \mathbb{N}$. There is i such that

$\beta \in Or(i)$. If $Or(i) \notin \{Or(i_1),\ldots Or(i_\alpha)\}$, $\pi_\beta(z) = y$ and $\pi_{\beta'}(z) = y$ for each $\beta' \in Or(i)$, so $\pi_{s^M(\beta)}(h^M(z)) =$

$\pi_{s^M(\beta)}(z) = \pi_\beta(z) = y$. If $Or(i) = Or(i_j)$ for some $j, \beta =$

$p(i_j,k)$ for some k and there is some $\tilde{\ell}$ such that $k \in \{\tilde{\ell}M, \tilde{\ell}M + 1, \ldots, (\tilde{\ell} + 1)M - 1\}$. Then there is $t \in \{0,\ldots M - 1\}$ such that $k = \tilde{\ell}M + t$, so that $s^M(\beta) = s^M(p(i_j,k)) =$ $p(i_j,k + M) = p(i_j,\tilde{\ell}M + t + M) = p(i_j,(\tilde{\ell} + 1)M + t)$ and $\pi_{s^M(\beta)}(h^M(z)) = \pi_{p(i_j,(\tilde{\ell}+1)M+t)}(h^M(z)) = \pi_{p(i_j,t)}(z) =$

$\pi_{p(i_j,(\tilde{\ell}+1)M+\beta)}(z) = \pi_\beta(z) = \pi_{p(i_j,k)}(z) = y_{p(i_j,t)}$.

Thus, $h^M(z) = z$. It follows that X^∞ admits a dense set of periodic points under the action of h.

When James T. Rogers, Jr., and I were writing our paper "Orbits of the pseudocircle" [KR], we were able to determine that no orbit of the pseudocircle (under the action of the full homeomorphism group) was G_δ in the space, but other than knowing that any orbit must at least be a Borel set in the space, were not able to say anything further about the Borel class of any orbit. It occurred to us then that in all cases in which we knew the least Borel class to which a given orbit in a given space belonged, the orbits were either G_δ-sets in the space or F_σ-sets in the space, and we wondered if there were further possibilities, i.e., orbits that were neither G_δ nor F_σ in the space. The next result shows that there are; specifically, for X a Cook continuum, X^∞ admits under the action of $H(X^\infty)$ orbits which are neither G_δ nor F_σ. (For a discussion of Borel sets and classes of Borel sets, see [Kvl], p. 344-351.)

Notation. If $x \in X^\infty$, let $x^* = \{\pi_i x \mid i \in \mathbb{N}\}$. If $y \in x^*$, let $B_x(y) = |\{i \mid \pi_i x = y\}|$. Note that $B_x(y) \in \mathbb{N} \cup \{\infty\}$.

Theorem 9. *If* $x \in X^\infty$, *then* $Gx = \{h(x) \mid h \in H(X^\infty)\}$ *is an* $F_{\sigma\delta}$*-set in* X^∞. *Further, the following statements may be made about the orbits of* X^∞ *under* $H(X^\infty)$:

(1) *There are closed orbits. Further,* Gx *is a closed orbit if and only if for some* $x_0 \in X$, $\pi_i x = x_0$ *for* $i \in \mathbb{N}$, *and* $Gx = \{x\}$. (Each closed orbit is degenerate and each finite orbit is degenerate.)

(2) *There are countably infinite orbits. Each countably infinite orbit is both a* G_δ*-set in* X^∞ *and an* F_σ*-set in* X^∞. *Further, if* Gx *is countable,* x^* *is finite and there is unique* $y_0 \in x^*$ *such that* $B_x(y_0) = \infty$.

(3) *The orbit* Gx *is homeomorphic to the irrationals if and only if* x^* *is discrete in itself and either* (a) x^* *is infinite, or* (b) x^* *is finite, but there are at least two points* $y_1, y_2 \in x^*$ *such that* $B_x(y_1) = B_x(y_2) = \infty$. *Those orbits* Gx *homeomorphic to the irrationals are precisely the orbits of* X^∞ *that are* G_δ *but not* F_σ.

(4) *Every* F_σ*-orbit is countable, and the orbit* Gx *is* F_σ *or* G_δ *iff it is discrete in itself.*

Proof. (0) Suppose $x \in X^\infty$ and $x^* = \{q_1, q_2, \ldots\}$. Now $Gx \subseteq x^{*\infty}$ and since x^* is an F_σ-set, $x^{*\infty}$ is $F_{\sigma\delta}$. Suppose $\mathcal{N}$ denotes the collection of all finite subsets of $\mathbb{N}$. Then $\mathcal{N} \times x^*$ is a countable set, and for $q_i \in x^*$, $N \in \mathcal{N}$ such that $|N| < B_x(q_i)$, let $A_{iN} = \{z \in X^\infty | \pi_j(z) = q_i$ for $j \in N$, $\pi_j(z) \neq q_i$ for $j \notin N\}$. Since A_{iN} is G_δ in X^∞, $X^\infty - A_{iN}$ is F_σ in X^∞. Further, $Gx \subseteq X^\infty - A_{iN}$ for each q_i, allowable N. (Note that $N = \phi \in \mathcal{N}$.) Further, for each q_i such that $B_x(q_i) < \infty$, $|N| > |B_x(q_i)|$, let $E_{iN} = \{z \in X^\infty | \pi_j(z) = q_i$ for $j \in N$, $\pi_j(z) \neq q_i$ for $j \notin N\}$. Then E_{iN} is G_δ in X^∞, and $X^\infty - E_{iN}$ is F_σ in X^∞. Thus, $Gx = x^{*\infty} \cap (\cap\{X^\infty - A_{iN} | q_i \in x^*, |N| < B_x(q_i)\}) \cap (\cap\{X^\infty - E_{iN} | q_i \in x^*, \infty > |N| > B_x(q_i)\})$, which means that Gx is a countable intersection of F_σ-sets and Gx is an $F_{\sigma\delta}$-set.

(1) Suppose $x_0 \in X$. Then if $x \in X^\infty$ such that $\pi_i(x) = x_0$ for each $i \in \mathbb{N}$, $Gx = \{x\}$. Suppose x^* contains more than one point. Further suppose $q_1 \in x^*$ and $B_x(q_1) < \infty$. There is some $q_2 \neq q_1 \in x^*$. If $\mathcal{A} = \{N \subseteq \mathbb{N} | |N| = B_x(q_1)\}$, $\mathcal{A}$ is countable, and for each $A \in \mathcal{A}$, there is some $x(A) \in Gx$ such that for $j \in A$, $\pi_j(x(A)) = q_1$. Since for $A \neq A' \in \mathcal{A}$, $x(A) \neq x(A')$, Gx if infinite. Further, Gx is not closed, for there is some $\hat{x} \in \overline{Gx}$ such that for each $j \in \mathbb{N}$, $\pi_j(\hat{x}) \neq q_1$.

On the other hand, if $q_1 \in x^*$, $B_x(q_1) = \infty$, $q_2 \neq q_1 \in x^*$, then for each $n \in \mathbb{N}$, there is some $y(n) \in Gx$ such that $\pi_i(y(n)) = q_1$ for each $i \leq n$. It follows that the point q, each coordinate of which is q_1, is in $\overline{Gx}$ but is not in Gx. Then Gx is neither closed nor finite.

(2) Suppose $x^* = \{q_1, q_2, \ldots q_n\}$ (i.e., x^* is finite), and $B_x(q_1) = \infty$, but $B_x(q_i) < \infty$ for $i \in \{2, \ldots n\}$. If $\mathcal{A} = \{A \subseteq \mathbb{N} \mid |A| = \Sigma_{i=2}^n B_x(q_i)\}$, $\mathcal{A}$ is a countable set, and for each $A \in \mathcal{A}$, there are only finitely many points of Gx such that $\pi_j(y) \neq q_1$ for each $j \in A$. It follows that Gx is countable, infinite, and an F_σ-set. Further, Gx is discrete in itself, and is therefore a completely metrizable space. Then Gx is a G_δ-set in X^∞. (See p. 430, [Kv1].)

Suppose Gx is countable. If x^* is not finite, then for $y \in Gx$, there are infinitely many choices for $\pi_1 y$. Given $\pi_1 y$ there remain infinitely many choices for $\pi_2 y$ etc. Continuing this reasoning, it is not difficult to see that there is a one-to-one function from Gx onto $\mathbb{N}^\infty$, and since $\mathbb{N}^\infty$ is uncountable, so is Gx. Then x^* is finite.

Suppose Gx is countable and there are $y_0, y_1 \in x^*$ such that $y_0 \neq y_1$, $B_x(y_0) = B_x(y_1) = \infty$. If $\mathbb{N}'$ denotes the odd integers, and we consider $\pi_{\mathbb{N}'}(X^\infty)$, then the collection of all points in $\pi_{\mathbb{N}'}(X^\infty)$ each of whose coordinates is y_0 or y_1 is a Cantor set in $\pi_{\mathbb{N}'}(X^\infty)$ and is uncountable. However, each point in this Cantor set is the projection of many points in Gx, which means Gx is uncountable, too, a contradiction. Then there is unique y_0 in x^* such that $B_x(y_0) = \infty$.

(3) Suppose first that x^* is discrete in itself and infinite, and $x^* = \{q_1, q_2, \ldots\}$. Then $x^{*\infty}$ is homeomorphic to the irrationals and is G_δ in X^∞. Further, suppose $\mathcal{N}$ denotes the collection of all finite subsets of $\mathbb{N}$, and for $i \in \mathbb{N}$, $N \in \mathcal{N}$ such that either (a) $|N| < B_x(q_i)$, or (b) $B_x(q_i) < |N| < \infty$, define $A_{iN} = \{y \in x^{*\infty} | \pi_j y = q_i$ for $j \in N$, and $\pi_j y \neq q_i$ for $j \notin N\}$. Since x^* is discrete in itself, A_{iN} is closed in $x^{*\infty}$ and $x^{*\infty} - A_{iN}$ is open in $x^{*\infty}$. Then $\cap\{x^{*\infty} - A_{iN} | i \in \mathbb{N}$ and $N \in \mathcal{N}$ such that $|N| < B_x(q_i)$ or $|N| > B_x(q_i)\} = Gx$ is G_δ in $x^{*\infty}$. Since Gx is dense in itself, Gx is homeomorphic to the irrationals.

If x^* is finite, but there are two different points q_0 and q_1 in x^* such that $B_x(q_0) = \infty = B_x(q_1)$, let $x^* = \{q_0, q_1, \ldots q_k\}$. Then $x^{*\infty}$ is a Cantor set, and if A_{iN} is defined as it was in the previous paragraph for the previous case, but with $i \in \{0, 1, \ldots k\}$, $Gx = \cap\{x^{*\infty} - A_{iN} | i \in N$ and $n \in \mathcal{N}$ such that $|N| < B_x(q_i)$ or $|N| > B_x(q_i)$ is a G_δ-set in $x^{*\infty}$. Further, Gx is dense in itself and no open nonempty subset of Gx is compact. Since Gx considered as space is completely metrizable and zero-dimensional, Gx is homeomorphic to the irrationals.

Suppose $x^{*\infty}$ is not discrete in itself, but Gx is homeomorphic to the irrationals. Then there is $q_0 \in x^*$ such that $q_0 \in \overline{x^* - \{q_0\}}$. For $i \in \mathbb{N}$, let $F_i = \{z \in X^\infty | \pi_i(z) = q_0\}$. Each F_i is closed and nowhere dense in X^∞. Further, $Gx = \cup_{i\in\mathbb{N}}(F_i \cap Gx)$ and each $F_i \cap Gx$ is closed and nowhere dense in the relative topology on Gx. But then $Gx - F_i$ is dense and open in Gx, which is complete, as space, so $\cap_{i\in\mathbb{N}}(Gx - F_i)$ should be a dense G_δ subset of Gx. But, of course, it is empty, so we have a contradiction, and x* is discrete in itself.

Suppose Gx is homeomorphic to the irrationals. If x* is finite and there is only one $q_0 \in x^*$ such that $B_x(q_0) = \infty$, then Gx is only countably infinite. Hence, if x* is finite, there must be at least 2 points q_0 and $q_1 \in x^*$ such that $B_x(q_0) = \infty = B_x(q_1)$.

Suppose that Gx is an orbit of X^∞ that is G_δ but not F_σ. Then Gx is uncountable and either x* is infinite or there are 2 points $q_0, q_1 \in x^*$ such that $B_x(q_1) = \infty = B_x(q_0)$. If some open subset of Gx is compact, then each point of Gx is in a compact open set so Gx is a countable union of closed sets. Thus, no open subset of Gx is compact. Also, Gx is dense in itself and zero-dimensional. Then Gx is homeomorphic to the irrationals.

(4) Suppose Gx is not discrete in itself and Gx if F_σ. Suppose that x_i denotes the ith coordinate of x, $q_0 \in x^*$ such that $q_0 \in \overline{x^* - \{q_0\}}$, and $Gx = \cup_{i=1}^{\infty} A_i$ where A_i is closed in X^∞. Denote x* by $\{q_0, q_1, \ldots\}$.

There is a least integer m_1 such that $B_1 = \{z \in X^\infty | \pi_i z = q_0 \text{ for } i \leq m_1\}$ does not intersect A_1. Then A_1 and B_1 are disjoint closed sets, and there is a basic open set u_1 such that $B_1 \subseteq u_1$ and $\overline{u_1} \cap A_1 = \phi$. Since q_0 is a limit point of x* and all permutations of x are in Gx, there is a point $z_1 \in u_1 \cap Gx$. Now if $k > m_1$, $\pi_k(u_1) = X$. There is least $k_1 \leq m_1 + 1$ such that no permutation of the finite sequence $(x_1, \ldots x_{k_1})$ is contained as a subsequence of $(\pi_1 z_1, \ldots \pi_{m_1} z_1)$. Then choose $\hat{z}_1$ to be that point of $Gx \cap u_1$ whose ith coordinate for $i \leq m_1$ is $\pi_i(z_1)$ and whose $m_1 + 1$-coordinate is a member of $(x_1, \ldots x_{k_1})$ chosen so that some permutation of $(x_1, \ldots x_{k_1})$ is contained in $(\pi_1 \hat{z}_1 \ldots \pi_{m_1} \hat{z}_1, \pi_{m_1+1} \hat{z}_1)$. There is a least integer $m_2 > m_1 + 1$ such that $B_2 = \{z \in X^\infty | \pi_i z = \pi_i \hat{z}_1$ for $j \leq m_1 + 1$, $\pi_i z = q_0$ for $m_1 + 1 < i \leq m_2\}$ does not intersect A_2. Then A_2 and B_2 are disjoint closed sets, and there is a basic open set u_2 such that $B_2 \subseteq u_2$ and $\overline{u_2} \cap A_2 = \phi$. Again there is a point $z_2 \in u_2 \cap Gx$, and we may actually choose z_2 so that $\pi_i z_2 = \pi_i \hat{z}_1$ for $i \leq m_1 + 1$. Further, there is a least $k_2 \leq m_2 + 1$, $k_2 > k_1$, such that no permutation of the finite sequence $(x_1,\dots x_{k_2})$ is contained as a subsequence of $(\pi_1 z_2,\dots \pi_{m_2} z_2)$. Then choose z_2 to be that point of $Gx \cap u_2$ whose ith coordinate for $j \le m_2$ is $\pi_i(z_2)$ and whose $m_2 + 1$-coordinate is a member of $(x_1 \dots x_{k_2})$ chosen so that some permutation of $(x_1,\dots x_{k_1})$ is contained in $(\pi_1 \hat{z}_2,\dots \pi_{m_2+1} \hat{z}_2)$. Continue this process, obtaining a sequence $\hat{z}_1, \hat{z}_2, \dots$ of points of Gx. Note that $\hat{z}_1, \hat{z}_2, \dots$ converges to the point $\hat{z}$ such that (i) $\pi_i \hat{z} = \pi_i \hat{z}_1$ for $i \le m_1 + 1$ (ii), $\pi_i \hat{z} = \pi_i \hat{z}_j$ for $j > 1$, $m_{j-1} + 1 < i \le m_j + 1$, and that $\hat{z} \notin \cup_{i=1}^{\infty} A_i$, but $\hat{z} \in Gx$. Thus we have a contradiction. If Gx is not discrete in itself then it is neither F_σ nor G_δ (part 3).

If Gx is F_σ, then it is discrete in itself, and part 3 tells us that Gx must be countable, for otherwise it is homeomorphic to the irrationals and can't be an F_σ-set. Finally, what we have just proved combined with part 3 gives us that every orbit that is either G_δ or F_σ is discrete in itself.

Corollary 10. *If* X *is a* Cook *continuum,* X^∞ *admits orbits under the action of its homeomorphism group that are neither* F_σ *nor* G_δ *in the space.*

Proof. Choose $x \in X^\infty$ so that $x^* = \{q_0, q_1, \dots\}$ where the sequence $q_1, q_2, \dots$ converges to q_0.

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