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SPACES OF CONTINUOUS LINEAR FUNCTIONALS: SOMETHING OLD AND SOMETHING NEW

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Let $C(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space X and $C^*(X)$ be the set of bounded functions in $C(X)$. Let us denote by $C_k(X)$ (respectively by $C_p(X)$) the set $C(X)$ topologized with the compact-open (respectively the point-open) topology. Both $C_k(X)$ and $C_p(X)$ are locally convex spaces. The locally convex compact-open topology on $C(X)$ is generated by the collection of seminorms $\{p_K$: K is a compact subset of X$\}$ where $p_K(f) = \sup\{|f(x)|: x \in K\}$ for $f \in C(X)$. Similarly the locally convex point-open topology on $C(X)$ is generated by the collection of seminorms $\{p_F$: F is a finite subset of X$\}$ where $p_F(f) = \sup\{|f(x)|: x \in F\}$. Let $K(X) = \{K \subseteq X$: K is a compact subset of X$\}$ and $F(X) = \{F \subseteq X$: F is a finite subset of X$\}$.

Basic open sets in $C_k(X)$ (respectively in $C_p(X)$) look like $< f,A,\varepsilon > = \{g \in C(X): |f(x) - g(x)| < \varepsilon$ for all $x \in A\}$ where $f \in C(X)$, $A \in K(X)$ (respectively $A \in F(X)$) and $\varepsilon > 0$.

Let $\Lambda_k(X)$ (respectively $\Lambda_p(X)$) be the set of all continuous linear functionals (real-valued functions) on $C_k^*(X)$ (on $C_p^*(X)$ respectively). Note since $C_k^*(X)$ (respectively $C_p^*(X)$) is a dense linear subspace of the locally convex space $C_k(X)$ (respectively $C_p(X)$), the set of all continuous linear functionals on $C_k^*(X)$ (respectively $C_p^*(X)$) equals the set of all continuous linear functionals on $C_k(X)$ (respectively on $C_p(X)$). In [8], a normed linear space whose underlying set is $\Lambda_k(X)$ has been studied in detail. In [8], the notation $\Lambda(X)$ has been used in place of $\Lambda_k(X)$. A necessary condition for this normed linear space $\Lambda_k(X)$ to be complete is that $C(X) = C^*(X)$, that is, every real-valued continuous function on X must be bounded. In this paper, we want to put the problem of completeness of $\Lambda_k(X)$ in a proper perspective and we show that the problem of completeness of $\Lambda_k(X)$ is essentially a problem of finding a suitable topology on $C^*(X)$. Because of the discussion in this paragraph, from now on, we will be interested only in $C^*(X)$. We want to answer the problem of completeness of $\Lambda_k(X)$ in a more general setting. For this purpose, we first define a new topology on $C^*(X)$ and we will see that the point-open, compact-open and sup-norm topologies on $C^*(X)$ are all special cases of this topology.

1. A New Topology on C* (X)

Let α be a collection of subsets of X which satisfies the following two conditions: (i) each member of α is C^*-embedded and (ii) if $A, B \in \alpha$, then there exists $C \in \alpha$ such that $A \cup B \subseteq C$.

For each $A \in \alpha$, define a seminorm p_A on $C^*(X)$ as follows. For $f \in C^*(X)$, $p_A(f) = \sup \{|f(x)|: x \in A\}$.

Consider the locally convex topology on $C^*(X)$ generated by the collection of seminorms $\{p_A: A \in \alpha\}$. Because of (ii), for each $f \in C^*(X)$, $f + \mathcal{U} = \{f + V: V \in \mathcal{U}\}$ is a neighborhood base at f where $\mathcal{U} = \{V_{p_A,\varepsilon}: A \in \alpha, \varepsilon > 0\}$.

We call this new locally convex topology on $C^*(X)$ α-topology and the corresponding topological space we denote by $C^*_\alpha(X)$. Note when $\alpha = K(X)$ or $F(X)$, we get compact-open or point-open topology on $C^*(X)$ respectively.

The supremum norm on $C^*(X)$ is defined as $\|f\|_\infty = \sup\{|f(x)|: x \in X\}$ for $f \in C^*(X)$. This supermum norm generates a finer topology than the α-topology on $C^*(X)$. We denote this normed linear space by $C^*_\infty(X)$. If α contains X, then $C^*_\alpha(X) = C^*_\infty(X)$; and if, in addition, we assume the members of α to be closed, then $C^*_\alpha(X) = C^*_\infty(X)$ only if α contains X. (see [7], page 7).

Let $\Lambda_\alpha(X)$ be the set of all continuous linear functionals (real-valued) on $C^*_\alpha(X)$ and let $\Lambda_\infty(X)$ be the set of all continuous linear functionals (real-valued) on $C^*_\infty(X)$. Since the sup-norm topology on $C^*(X)$ is finer than the α-topology on it, $\Lambda_\alpha(X) \subset \Lambda_\infty(X)$. Now $\Lambda_\infty(X)$ is a normed linear space with the usual conjugate norm, that is, given $\lambda \in \Lambda_\infty(X)$, we have a norm $\|\lambda\|_* = \sup\{|\lambda(f)|: f \in C^*(X), \|f\|_\infty \leq 1\}$ where $\|\cdot\|_\infty$ is the sup-norm on $C^*(X)$. Consequently we can assign this $\|\cdot\|_*$-norm on $\Lambda_\alpha(X)$ to make it a normed linear space $(\Lambda_\alpha(X), \|\cdot\|_*)$.

Note $\Lambda_\infty(X)$ is actually a particular case of $\Lambda_\alpha(X)$. Here we also mention another particular $\Lambda_\alpha(X)$. Let X be a normal Hausdorff space and $\sigma = \{cl_X A$: A is a σ-compact subset of X$\}$. Note that σ is closed under finite union because $\cup_{n=1}^{k} cl_X A_n = cl_X(\cup_{n=1}^{k} A_n)$. We denote the corresponding $\Lambda_\alpha(X)$ by $\Lambda_\sigma(X)$. While considering $\Lambda_\sigma(X)$, we will always assume X to be a normal Hausdorff space.

2. Basic Properties of $\Lambda_\alpha(X)$

Let $\Lambda_\alpha^+(X) = \{\lambda \in \Lambda_\alpha(X): \lambda \geq 0\}$ where $\lambda \geq 0$ provided that $\lambda(f) \geq 0$ for each $f \in C^*(X)$ such that $f \geq 0$. If $\lambda \in \Lambda_\alpha(X)$ and A is a subset of X, then λ is said to be supported on A provided that whenever $f \in C^*(X)$ with $f|_A = 0$, then $\lambda(f) = 0$. Since λ is linear, this is equivalent to saying that whenever $f,g \in C^*(X)$ with $f|_A = g|_A$, then $\lambda(f) = \lambda(g)$.

The next two lemmas can be proved in manners similar to Lemmas 1.1 and 1.2 in [8].

Lemma 2.1. *For each* $\lambda \in \Lambda_\alpha(X)$, *there exists an element* A *in* α *such that* λ *is supported on* A. *Conversely, if* λ *is a positive linear functional on* $C^*(X)$ *which is supported on an element of* α, *then* $\lambda \in \Lambda_\alpha^+(X)$.

Lemma 2.2. *Let* A *be a closed subset of* X, *let* $F \in \alpha$ *and let* $\lambda \in \Lambda_\alpha(X)$. *If* λ *is supported on each of* A *and* F, *then* λ *is supported on* $A \cap F$.

Now on $\Lambda_\alpha^+(X)$ we give a topology induced by the metric $d_*(\lambda,\mu) = \| \lambda - \mu \|_*$ for $\lambda,\mu \in \Lambda_\alpha^+(X)$.

Theorem 2.3. *$(\Lambda_\alpha^+(X), d_*)$ is a closed subspace of $(\Lambda_\alpha(X), \|\cdot\|_*)$.*

Proof. Let $\lambda \in \Lambda_\alpha(X)\setminus\Lambda_\alpha^+(X)$. Then there exists a $g \in C^*(X)$ such that $g \geq 0$ and $\lambda(g) < 0$. Let r be a positive number such that $\|rg\|_\infty \leq 1$. Define $\varepsilon = -\frac{r}{2}\lambda(g)$. Now suppose $\mu \in \Lambda_\alpha(X)$ is such that $\|\mu - \lambda\|_* < \varepsilon$. Then $|\mu(rg) - \lambda(rg)| < \varepsilon$ so that $\mu(g) - \lambda(g) < \frac{\varepsilon}{r} = -\frac{1}{2}\lambda(g)$. Therefore $\mu(g) < \frac{1}{2}\lambda(g) < 0$ so that $\mu \in \Lambda_\alpha(X)\setminus\Lambda_\alpha^+(X)$.

3. The Completeness of $\Lambda_\alpha^+(X)$ and $\Lambda_\alpha(X)$

The space $\Lambda_\alpha^+(X)$ is a metric space with the metric d_*. This space is complete provided that if a sequence in $\Lambda_\alpha^+(X)$ is a Cauchy sequence with respect to d_*, then it converges. Likewise the normed linear space $\Lambda_\alpha(X)$ is complete if it is complete with respect to its norm $\|\cdot\|_*$, that is, if it is a Banach space.

We have studied the completeness of $\Lambda_k(X)$ and $\Lambda_k^+(X)$ in [8]. We already know that $\Lambda_\infty(X)$, being the conjugate space of a normed linear space, is always complete.

To establish that the completeness of $\Lambda_\alpha^+(X)$ is equivalent to the completeness of $\Lambda_\alpha(X)$, we need the following theorem which can be proved like Theorem 2.2 in [8].

Theorem 3.1. *Each $\lambda \in \Lambda_\alpha(X)$ can be written as $\lambda = \lambda^+ - \lambda^-$ where λ^+ and λ^- are members of $\Lambda_\alpha^+(X)$. Furthermore, if $\lambda, \mu \in \Lambda_\alpha(X)$, then $\|\lambda^+ - \mu^+\|_* \leq \|\lambda - \mu\|_*$ and $\|\lambda^- - \mu^-\|_* \leq \|\lambda - \mu\|_*$.*

Theorem 3.2. *The metric space* $\Lambda_\alpha^+(X)$ *is complete if and only if the normed linear space* $\Lambda_\alpha(X)$ *is complete.*

Proof. Use Theorems 3.1 and 2.3.

Because of Theorem 3.2, each of the following theorems about $\Lambda_\alpha^+(X)$ is also true for $\Lambda_\alpha(X)$.

Theorem 3.3. *Suppose* X *is infinite and* $F(X) \subseteq \alpha$. *Now if* $\Lambda_\alpha^+(X)$ *is complete, then every countable subset of* X *is contained in some member of* α.

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be any countable subset of X. For each $m \in \mathbb{N}$, define $\lambda_m : C_\alpha^*(X) \to \mathbb{R}$ as follows. For each $f \in C_\alpha^*(X)$, take $\lambda_m(f) = \Sigma_{n=1}^m \frac{1}{2n} f(x_n)$. Each λ_m is a positive linear functional on $C_\alpha^*(X)$ supported on the finite set $\{x_1,\ldots,x_m\}$. Then by Lemma 2.1, λ_m is continuous. Now for each k and m with $k < m$, $d_*(\lambda_k,\lambda_m) = \|\lambda_k - \lambda_m\|_* \leq \Sigma_{n=k+1}^m \frac{1}{2n}$. Therefore (λ_m) is a Cauchy sequence in $\Lambda_\alpha^+(X)$. Since $\Lambda_\alpha^+(X)$ is complete, the (λ_m) converges to some λ in $\Lambda_\alpha^+(X)$. Also $\lambda_m \to \lambda$ implies $\lambda(f) = \lim_{m\to\infty}\lambda_m(f) = \Sigma_{n=1}^\infty \frac{1}{2n} f(x_n)$ for all $f \in C_\alpha^*(X)$.

Now suppose λ has a support Y which belongs to α. We show that $A \subseteq Y$. Suppose not, then there is some m such that $x_m \notin Y$. Since X is completely regular, there is some continuous function f on X with values in the unit interval I such that $f(x_m) = 1$ and $f(Y) = \{0\}$. Since λ is supported on Y, $\lambda(f) = 0$. But $\lambda(f) = \Sigma_{n=1}^\infty \frac{1}{2n} f(x_n) \geq \frac{1}{2m} f(x_m) = \frac{1}{2m} > 0$. With this contradiction, it follows that $A \subseteq Y$.

Corollary 3.4. *If* X *is infinite, then* $\Lambda_p^+(X)$ *and* $\Lambda_p(X)$ *are not complete.*

Theorem 3.5. *If the closure of each countable union of elements of* α *belongs to* α*, then* $\Lambda_\alpha^+(X)$ *is complete.*

Proof. Let (λ_n) be a Cauchy sequence in $\Lambda_\alpha^+(X)$. Consider $\Lambda_\alpha^+(X)$ as a subspace of the complete metric space $\Lambda_\infty^+(X)$. Then (λ_n) is a Cauchy sequence in $\Lambda_\infty^+(X)$ and hence converges to some λ in $\Lambda_\infty^+(X)$. Suppose each λ_n is supported on A_n where $A_n \in \alpha$. We show that λ is supported on $A = cl_X(\bigcup_{n=1}^\infty A_n)$. Let $f \in C^*(X)$ with $f|_A = 0$. Since each λ_n is supported on $A_n \subseteq A$, then each $\lambda_n(f) = 0$ and consequently $\lambda(f) = \lim_{n\to\infty}\lambda_n(f) = 0$. Therefore λ has support A. But by hypothesis $A \in \alpha$. Hence by Lemma 2.1 $\lambda \in \Lambda_\alpha^+(X)$. So $\Lambda_\alpha^+(X)$ is complete.

Corollary 3.6. *Suppose* X *is a normal Hausdorff space. Then* $\Lambda_\sigma^+(X)$ *is always complete.*

Proof. Suppose for each n, A_n is a σ-compact subset of X. Then $cl_X(\bigcup_{n=1}^\infty cl_X A_n) = cl_X(\bigcup_{n=1}^\infty A_n) \in \sigma$.

4. Measure Theoretic-Counterparts

In this section, we will talk about the measure-theoretic counterparts of $\Lambda_\alpha(X)$ and $\Lambda_\infty(X)$ with some extra conditions on α and X. So now we introduce some ideas from measure theory.

The algebra generated by the closed sets of X are denoted by $\mathcal{A}_c$ while the σ-algebra they generate is denoted by $\mathcal{B}$, called the *Borel sets*.

For us a *finitely additive measure* (also called signed measure) on $\mathcal{A}_c$ is a real-valued function defined on $\mathcal{A}_c$ satisfying the following two properties (i) $\mu(\emptyset) = 0$; (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{A}_c$ and $A \cap B = \emptyset$.

A finitely additive measure μ is called a countably additive measure or simply a measure provided that (iii) $\mu(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ such that $A_n \in \mathcal{A}_c$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{A}_c$. When a measure μ is defined on $\mathcal{B}$, we call it a Borel measure. A measure μ defined on $\mathcal{B}$ has support A where $A \subseteq X$ and $A \in \mathcal{B}$ if $|\mu|(X \setminus A) = 0$. A finitely additive measure μ defined on $\mathcal{A}_c$ or $\mathcal{B}$ is regular whenever A is in the domain of definition of μ and $\epsilon > 0$, there are closed and open sets C and U such that $C \subseteq A \subseteq U$ and $|\mu|(U \setminus C) < \varepsilon$. Note when μ has compact support, this definition of regularity coincides with the one usually given in the books on measure theory. For more information on measure theory see [4] and [6].

Now we fix some notations.

A (signed) measure μ defined on $\mathcal{B}$ is said to be a finite (signed) measure if $|\mu(A)| < \infty$ holds for each $A \in \mathcal{B}$. It can be shown that a signed measure μ is finite if and only if $|\mu|(X) < \infty$. So a finite signed measure defined on $\mathcal{B}$, has finite total variation. For details on the above, see [1], 26.

Now let $M_b(X)$ be the set of all finite (signed) regular Borel measures on X. Let $M_b^+(X) = \{\mu \in M_b(X) : \mu \geq 0$, that is, μ is a positive measure$\}$. Throughout the remaining part of this paper we will assume the following extra condition on α: the members of α are closed.

Now define $M_{b,\alpha}(X) = \{\mu \in M_b(X): \mu$ has a support $A(\subseteq X)$ such that $A \in \alpha\}$. Let $M^+_{b,\alpha}(X) = \{\mu \in M_{b,\alpha}(X): \mu \geq 0\}$. When $\alpha = K(X)$ or $F(X)$, we write $M_{b,k}(X)$ or $M_{b,p}(X)$ respectively.

The next thing to observe is that given $\mu \in M_b(X)$, $\|\mu\| = |\mu|(X)$ defines a norm on $M_b(X)$. So $(M_b(X), \|\cdot\|)$ is actually a normed linear space. Also $M^+_b(X)$ is a metric space when equipped with the norm p given by $p(\mu_1,\mu_2) = \|\mu_1 - \mu_2\|$ for every $\mu_1,\mu_2 \in M^+_b(X)$. Note $(M_{b,\alpha}(X), \|\cdot\|)$ is a normed linear space while $(M^+_{b,\alpha}(X), p)$ is a metric space.

Before having our first theorem in this section, we need the following two lemmas.

Lemma 4.1. *Suppose* Y *is a* Borel *subset of a completely regular Hausdorff space* X. *Let* $\mathcal{B}(X)$ *and* $\mathcal{B}(Y)$ *be the* σ-*algebras of* Borel *subsets of* X *and* Y *respectively. Then* $\mathcal{B}(X) \cap Y = \mathcal{B}(Y)$ *where* $\mathcal{B}(X) \cap Y = \{B \cap Y: B \in \mathcal{B}(X)\}$.

Proof. Define $\mathcal{D} = \{A \in \mathcal{P}(X): A = E \cup (B\setminus Y); E \in \mathcal{B}(Y)$ and $B \in \mathcal{B}(X)\}$ where $\mathcal{P}(X)$ is the power set of X. Note $X\setminus(E \cup (B\setminus Y)) = (Y\setminus E) \cup ((X\setminus(B\setminus Y))\setminus Y)$. Now it can be easily shown that $\mathcal{D}$ is a σ-algebra on X containing all the closed subsets of X. Hence $\mathcal{B}(X) \subseteq \mathcal{D}$. So $\mathcal{B}(X) \cap Y \subseteq \mathcal{D} \cap Y$. But $\mathcal{D} \cap Y = \mathcal{B}(Y)$. So $\mathcal{B}(X) \cap Y \subseteq \mathcal{B}(Y)$. Note $\mathcal{B}(X) \cap Y$ is a σ-algebra on Y and if C is a closed subset of Y, then $C = C' \cap Y$ for some closed subset C' of X which means $C \in \mathcal{B}(X) \cap Y$. Hence $\mathcal{B}(Y) \subseteq \mathcal{B}(X) \cap Y$. Therefore $\mathcal{B}(X) \cap Y = \mathcal{B}(Y)$.

Lemma 4.2. *If* A *is a compact subset of a completely regular Hausdorff space* X, *then for every closed set* $B \subseteq X \setminus A$, *there exists a continuous function* $f: X \to I$ *such that* $f(x) = 0$ *for* $x \in A$ *and* $f(x) = 1$ *for* $x \in B$.

Proof. See [5], page 168.

Theorem 4.3. *Suppose* $\alpha \subseteq K(X)$, *that is, the members of* α *are compact. Then* $(M_{b,\alpha}(X), \|\cdot\|)$ *is isometrically isomorphic to* $(\Lambda_\alpha(X), \|\cdot\|_*)$ *while* $M^+_{b,\alpha}(X)$ *is identified with* $\Lambda^+_\alpha(X)$ *under this isometric isomorphism.*

Proof. Define $F: M_{b,\alpha}(X) \to \Lambda_\alpha(X)$ by $F(\mu)(f) = \int f\, d\mu$ for each $\mu \in M_{b,\alpha}(X)$ and $f \in C^*_\alpha(X)$. Let K be a compact support of μ belonging to α, that is, $|\mu|(X\setminus K) = 0$ and $K \in \alpha$. Then for each $f \in C^*_\alpha(X)$, $|F(\mu)(f)| = |\int f\, d\mu| = |\int_K f\, d\mu| \le \int_K |f| d|\mu| \le |\mu|(K) \cdot p_K(f)$ and so $F(\mu)$ is continuous. Clearly $F(\mu)$ is linear. Hence $F(\mu) \in \Lambda_\alpha(X)$.

Also $\|F(\mu)\|_* \le \sup\{|\mu|(K)p_K(f): f \in C^*(X), \|f\|_\infty \le 1\}$

$= |\mu|(K) = \|\mu\|$.

Now we prove the reverse inequality, that is, $\|\mu\| \le \|F(\mu)\|_*$.

Note $|\mu|(K) = \sup\{\Sigma|\mu(A_i)|: \{A_i\}$ is a finite disjoint collection of $\mathcal{B}$ with $\cup A_i \subseteq K\}$. So given $\varepsilon > 0$, there exist $A_1, \ldots, A_n \in \mathcal{B}$ such that A_i's are pairwise disjoint and $\Sigma_{i=1}^n |\mu|(A_i)| > |\mu|(K) - \varepsilon$. Since μ is regular there exist compact sets C_i and open sets U_i such that $C_i \subseteq A_i \subseteq U_i$ and $|\mu|(U_i \setminus C_i) < \varepsilon/n$ for $1 \le i \le n$. Since the compact subsets C_i's are pairwise disjoint, pairwise disjoint open sets V_i exist such that $C_i \subseteq V_i$. Now let

$W_i = U_i \cap V_i$. Then $C_i \cap (X \setminus W_i) = \emptyset$. Hence by Lemma 4.2, there exists a continuous function $f_i : X \to I$ such that $f_i(C_i) = \{1\}$ and $f_i(X \setminus W_i) = 0$. Let $a_i = \frac{|\mu(A_i)|}{\mu(A_i)}$ if $\mu(A_i) \neq 0$ and if $|\mu(A_i)| = 0$, let $a_i = 0$. Let $f = \Sigma_{i=1}^n a_i f_i$. Since W_i's are pairwise disjoint, $\|f\|_\infty \le 1$.
Now $|\int f\, d\mu - \Sigma_{i=1}^n |\mu(A_i)||$

$$= |\Sigma_{i=1}^n a_i \int f_i d\mu - \Sigma_{i=1}^n |\mu(A_i)||$$

$$= |\Sigma_{i=1}^n a_i \int_{W_i} f_i d\mu - \Sigma_{i=1}^n |\mu(A_i)||$$

$$= |\Sigma_{i=1}^n [a_i \int_{C_i} f_i d\mu - |\mu(A_i)|] + \Sigma_{i=1}^n a_i \int_{W_i \setminus C_i} f_i\, d\mu|$$

$$= |\Sigma_{i=1}^n [a_i \mu(C_i) - a_i \mu(A_i)] + \Sigma_{i=1}^n a_i \int_{W_i \setminus C_i} f_i\, d\mu|$$

$$\le \Sigma_{i=1}^n |a_i||\mu(C_i) - \mu(A_i)| + \Sigma_{i=1}^n |a_i| \int_{W_i \setminus C_i} |f_i| d|\mu|$$

$$\le \Sigma_{i=1}^n |\mu(A_i \setminus C_i)| + \Sigma_{i=1}^n |\mu|(W_i \setminus C_i)$$

$$\le \Sigma_{i=1}^n |\mu|(A_i \setminus C_i) + \Sigma_{i=1}^n |\mu|(W_i \setminus C_i)$$

$$< n \cdot \frac{\varepsilon}{n} + n \cdot \frac{\varepsilon}{n} = 2\varepsilon.$$

So $\|F(\mu)\|_* \ge |\int f\, d\mu| > \Sigma_{i=1}^n |\mu(A_i)| - 2\varepsilon > |\mu|(K) - 3\varepsilon = \|\mu\| - 3\varepsilon$. Therefore $\|\mu\| - 3\varepsilon < \|F(\mu)\|_* \le \|\mu\|$. Hence $\|F(\mu)\|_* = \|\mu\|$, that is, F is an isometry.

Now we need to show that F is onto. Suppose $\lambda \in \Lambda_\alpha(X)$. Then λ can be written as $\lambda = \lambda^+ - \lambda^-$ where

$\lambda^+, \lambda^- \in \Lambda_\alpha^+(X)$. Now if λ has a compact support K belonging to α, then both λ^+ and λ^- have compact support K. To show F is onto, we try to get $\mu_1, \mu_2 \in M_{b,\alpha}^+(X)$ such that $\lambda^+ = F(\mu_1)$ and $\lambda^- = F(\mu_2)$. So $\lambda = \lambda^+ - \lambda^- = F(\mu_1) - F(\mu_2) = F(\mu_1 - \mu_2) = F(\mu)$ where $\mu = \mu_1 - \mu_2 \in M_{b,\alpha}(X)$. So we just need to consider λ^+. Define $\lambda_K^+ \colon C_\infty^*(K) \to \mathbb{R}$ as follows. For each $f \in C_\infty^*(K)$, choose an $f_K \in C_\alpha^*(X)$ such that $f_{K|K} = f$. Then define $\lambda_K^+(f) = \lambda^+(f_K)$. Since λ^+ is supported on K, λ_K^+ is well-defined. Also since λ^+ is linear, so is λ_K^+. Finally λ_K^+ is continuous since $\sup\{|\lambda_K^+(f)| : f \in C^*(K), \|f\|_\infty \leq 1\} = \sup\{|\lambda^+(f)| : f \in C^*(X), \|f\|_\infty \leq 1\} = \|\lambda^+\|_* < \infty$. By the Riesz Representation Theorem (see [1]), there exists a $\mu_K \in M_b^+(K)$ such that $\lambda_K^+(f) = \int_K f\, d\mu_K$ for all $f \in C^*(K)$.

It only remains to show that an element $\mu_1 \in M_b^+(X)$ can be found such that $\mu_1(B) = \mu_K(B \cap K)$ for all $B \in \mathcal{B}$. Then μ_1 would be supported on K so that μ_1 would be in $M_{b,\alpha}^+(X)$ and thus for each $f \in C^*(X), \lambda^+(f) = \lambda_K^+(f|_K) = \int_K f|_K\, d\mu_K = \int f\, d\mu_1 = F(\mu_1)(f)$ which shows that $\lambda^+ = F(\mu_1)$.

First observe that because of Lemma 4.1, μ_1 is well-defined on $\mathcal{B}$. So we only need to show that μ_1 is regular. Let $B \in \mathcal{B}$ and let $\varepsilon > 0$. Since μ_K is regular, there exists a compact subset C of K and an open subset U of K such that $C \subseteq B \cap K \subseteq U$ and $\mu_K(U \setminus C) < \varepsilon$. Let $V = U \cup (X \setminus K)$ which is open in X. Then $C \subseteq B \subseteq V$ and $\mu_1(V \setminus C) = \mu_K((V \setminus C) \cap K) = \mu_K(U \setminus C) < \varepsilon$. Therefore μ_1 is regular and is thus an element of $M_{b,\alpha}^+(X)$.

Note when $\alpha = \mathcal{F}(X)$ or $\mathcal{K}(X)$, the above theorem tells us what is exactly the measure-theoretic counterpart of $\Lambda_p(X)$ or of $\Lambda_k(X)$ respectively. Note that when $\alpha = \mathcal{F}(X)$, $M_{b,\alpha}(X)$ is actually the linear space over $\mathbb{R}$ generated by the set of Dirac's measures on X. This fact explains why $\Lambda_p(X)$ and $\Lambda_p^+(x)$ cannot be complete because a limit of a Cauchy sequence in $M_{b,p}(X)$ or in $M_{b,p}^+(X)$ may converge to a regular Borel measure on X with infinite support.

Now what is the measure-theoretic counterpart of $\Lambda_\infty(X)$? To answer this question, we introduce a new measure space. Let $M_c(X)$ be the set of all bounded finitely additive regular measures defined on $\mathcal{A}_c$. Again $M_c(X)$ is a normed linear space with the total variation norm. Let $M_c^+(X) = \{\mu \in M_c(X): \mu \geq 0\}$.

Theorem 4.4. *If* X *is a normal and Hausdorff, then* $(M_c(X), \|\cdot\|)$ *is isometrically isomorphic to* $(\Lambda_\infty(X), \|\cdot\|_*)$ *while* $M_c^+(X)$ *is identified with* $\Lambda_\infty^+(X)$ *under this isometric isomorphism.*

Proof. See [3], pages 78-83.

But what about the countable additiveness of elements of $M_c(X)$? When X is countably compact, we have the following answer.

Theorem 4.4. *If* X *is countably compact and if* μ *is a bounded regular finitely additive measure defined on* $\mathcal{A}_c$, *then* μ *is countably additive on* $\mathcal{A}_c$, *that is,* $\mu(\cup_{n=1}^\infty A_n) = \Sigma_{n=1}^\infty \mu(A_n)$ *whenever* (A_n) *is a countable family of pairwise* *disjoint sets from* A_c *with union in* A_c. *Moreover* μ *has a regular countably additive extension to the* σ*-algebra* $\mathcal{B}$ *of Borel subsets of* X.

Proof. See Theorem 3.11 in [7]. Also see [3].

Now the last theorem can be used to improve Theorem 4.4 to the following version.

Theorem 4.6. *If* X *is countably compact, normal and Hausdorff, then* $M_b(X)$ *is isometrically isomorphic to* $\Lambda_\infty(X)$ *while* $M_b^+(X)$ *is identified with* $\Lambda_\infty^+(X)$ *under this isometric isomorphism.*

Since $\Lambda_\alpha(X) \subseteq \Lambda_\infty(X)$, for a countably compact, normal Hausdorff space, we have the following measure-theoretic counterpart of $\Lambda_\alpha(X)$.

Theorem 4.7. *If* X *is countably compact, normal and Hausdorff, then* $M_{b,\alpha}(X)$ *is isometrically isomorphic to* $\Lambda_\alpha(X)$ *while* $M_{b,\alpha}^+(X)$ *is identified with* $\Lambda_\alpha^+(X)$ *under this isometric isomorphism.*

5. Density

The density d(X) of a space X is the smallest infinite cardinal number m such that X has a dense subset which has cardinality less than or equal to m. Now a space X is separable if and only if $d(X) = \aleph_0$. If X is a subspace of a metrizable space Y, then $d(X) \leq d(Y)$.

Theorem 5.1. *For each space* X, $d(\Lambda_\alpha^+(X)) = d(\Lambda_\alpha(X))$.

Proof. Use Theorem 3.1.

Corollary 5.2. *$\Lambda_\alpha^+(X)$ is separable if an only if $\Lambda_\alpha(X)$ is separable.*

For each $x \in X$, define the evaluation function at x, ϕ_x: $C_\alpha^*(X) \to \mathbb{R}$ by taking $\phi_x(f) = f(x)$ for each $f \in C_\alpha^*(X)$. Now ϕ_x is a positive linear functional on $C_\alpha^*(X)$ which is supported on $\{x\}$. Now if $\{x\} \in \alpha$, then by Lemma 2.1 $\phi_x \in \Lambda_\alpha^+(X)$.

For the remainder of this section, the notation $|X|$ stands for the cardinality of X.

Theorem 5.3. *Suppose $F(X) \subseteq \alpha$. Then $|X| \leq d(\Lambda_\alpha^+(X))$.*

Proof. Since $F(X) \subseteq \alpha$, $\phi_x \in \Lambda_\alpha^+(X)$ for all $x \in X$. Define the evaluation function Φ: $X \to \Lambda_\alpha^+(X)$ by taking $\Phi(x) = \phi_x$. Since $C^*(X)$ separate points, then Φ is one-to-one. Therefore $|\Phi(X)| = |X|$. Now let x and y be distinct points of X. Then $d_*(\phi_x,\phi_y) = \|\phi_x - \phi_y\|_* = \sup\{|\phi_x(f) - \phi_y(f)|: f \in C^*(X), \|f\|_\infty \leq 1\} = \sup\{|f(x) - f(y)|: f \in C^*(X), \|f\|_\infty \leq 1\} \geq 1$. So $\Phi(X)$ is a discrete subset of $\Lambda_\alpha^+(X)$ and hence $|\Phi(X)| \leq d(\Phi(X))$. Therefore $|X| = |\Phi(X)| \leq d(\Phi(X)) \leq d(\Lambda_\alpha^+(X))$.

Corollary 5.4. *Suppose $F(X) \subseteq \alpha$ and $\Lambda_\alpha^+(X)$ is separable. Then X is countable.*

In order to establish a more general theorem on separability of $\Lambda_\alpha^+(X)$, we need to discuss the separability of $M_{b,\alpha}^+(X)$. Note that the proof of Theorem 3.3 in [8] actually shows that if X is countable, then $M_b(X)$ is separable. So when X is countable, $M_{b,\alpha}(X)$ and $M^+_{b,\alpha}(X)$ are also separable.

Theorem 5.5. *Suppose* $F(X) \subseteq \alpha \subseteq K(X)$. *Then* $\Lambda^+_\alpha(X)$ *is separable if and only if* X *is countable.*

Proof. Suppose $\Lambda^+_\alpha(X)$ is separable. Then by Corollary 5.4, X is countable. Conversely, let X be countable. Now since $\alpha \subseteq K(X)$, by Theorem 4.3, $\Lambda^+_\alpha(X)$ is isomorphic to $M^+_{b,\alpha}(X)$. So $d(\Lambda^+_\alpha(X)) = d(M^+_{b,\alpha}(X))$. Hence $\Lambda^+_\alpha(X)$ is separable.

Lastly, we talk about the separability of $\Lambda_\infty(X)$. Note that Theorem 5.1 gives us $d(\Lambda^+_\infty(X)) = d(\Lambda_\infty(X))$. This means that $\Lambda_\infty(X)$ is separable if and only if $\Lambda^+_\alpha(X)$ is separable.

Theorem 5.6. $\Lambda_\infty(X)$ *is separable if and only if* X *is compact and countable.*

Proof. If $\Lambda_\infty(X)$ is separable, then $\Lambda_k(X)$ is separable and so X is countable. Again, since $\Lambda_\infty(X)$ is the conjugate space of the normed linear space $C^*_\infty(X)$, $C^*_\infty(X)$ is separable. But this implies that X is compact (see [9], page 54). Conversely, let X be compact and countable. Since X is compact, $C^*_k(X) = C^*_\infty(X)$ and consequently $\Lambda_\infty(X) = \Lambda_k(X)$. But X is countable and so $\Lambda_k(X)$ is separable. Hence $\Lambda_\infty(X)$ is separable.

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