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1. Introduction

There has been renewed interest in mappings on various types of indecomposable continua which arise in dynamical systems. In particular a continuum described by Kuratowski in [4] has been of interest because it is the attracting set for a horseshoe map of Smale which is described in [5]. See also Smale [6] and Barge [1]. Watkins [7] has studied similar continua as inverse limits. The continuum described in [4], which we call M, is often called the Knaster continuum, or the bucket handle. According to Kuratowski, in [4], M was described by Janiszewski in his dissertation in 1911. Kuratowski also acknowledges Knaster's aid in the study of these continua. M is known to be indecomposable and chainable. At the spring topology conference in April of 1988, at The University of Florida, M. Barge asked the following question. For which composants K of M is it true that M can be embedded in the plane in such a way that the points of the image of K are accessible? (For a definition of composant see section 2 below.) Several of the participants at the conference, including Tom Ingram, John Mayer and this author, indicated that they thought this could be done for any composant of M. It is the purpose of this paper to give a particularly simple construction to show that if K is a composant of M, then there is a homeomorphism h of M into E^2 such that each point of h(K) is accessible.

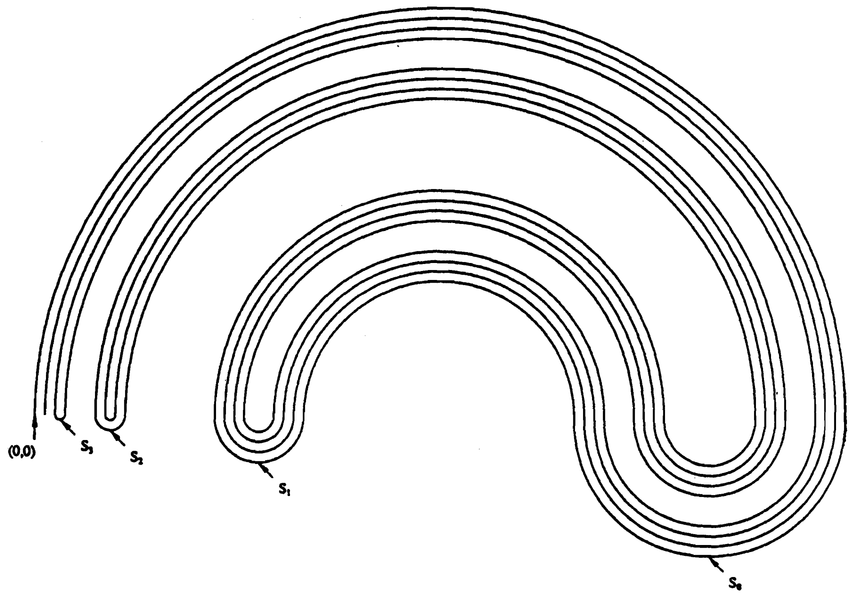
2. Notation

By a continuum is meant a compact, connected metric space. A composant of a continuum H is a subset K of H such that for some point p of H, K is the union of all proper subcontinua of H containing p. In the continua considered here, a composant is a maximal arcwise connected subset. A point p of a continuum H in a Euclidean plane E^2 is said to be accessible if an only if there is an arc $\alpha \subset E^2$ such that $\alpha \cap H = \{p\}$. Janiszewski's example of an indecomposable continuum is simply and elegantly described by Kuratowski in [4]. We include his description here for completeness. Let E denote the Cantor set on the interval [0,1]. Let S denote the set of all semicircles having center (1/2,0), lying, except for their endpoints, above the x-axis and having endpoints in $E \times \{0\}$. The continuum described by Kuratowski, denoted by M, is the union of S and a countable sequence $S_0, S_1, \ldots$ of collections of semicircles described as follows. For each non-negative integer n, S_n is the set of all semicircles having center $(5\cdot 3^{-n}/6, 0)$, lying, except for their endpoints, below the x-axis, and having endpoints in $E \times \{0\}$. For our purposes it will be convenient to represent the points of E using their base 3 representation and to introduce some notation to represent certain subsets of $E \times \{0\}$. We also abuse the notation by identifying E and $E \times \{0\}$ and the reader is warned that he must determine from the context whether a point p in E denotes a number in the Cantor set or a point in $E \times \{0\}$. For each positive integer n, let E_n denote the set of all n-term sequences each term of which is either 0 or 2. We associate with each member $e = p_1, \ldots, p_n$ of E_n the corresponding (left) endpoint $\hat{e}$ of the Cantor set whose base 3 representation is $.p_1 \ldots p_n$. For each element $e \in E_n$ let I_e denote the interval on the x-axis in E^2 from $(\hat{e},0)$ to $(\hat{e} + 3^{-n},0)$. For each positive integer n, let G_n denote the set of all intervals I_e for all $e \in E_n$. Note that S_0 is a collection of semicircles which connect each point of $I_{2,0} \cap E$ to a unique point of $I_{2,2} \cap E$, and S_1 is a collection of semicircles which connect each point of $I_{0,2,0} \cap E$ to a unique point of $I_{0,2,2} \cap E$. In general if $n > 0$, S_n is a collection of semicircles which connect each point of $I_{0,0,\ldots,0,2,0} \cap E$ to a unique point of $I_{0,0,\ldots,0,2,2} \cap E$ where each subscript has n leading 0's.

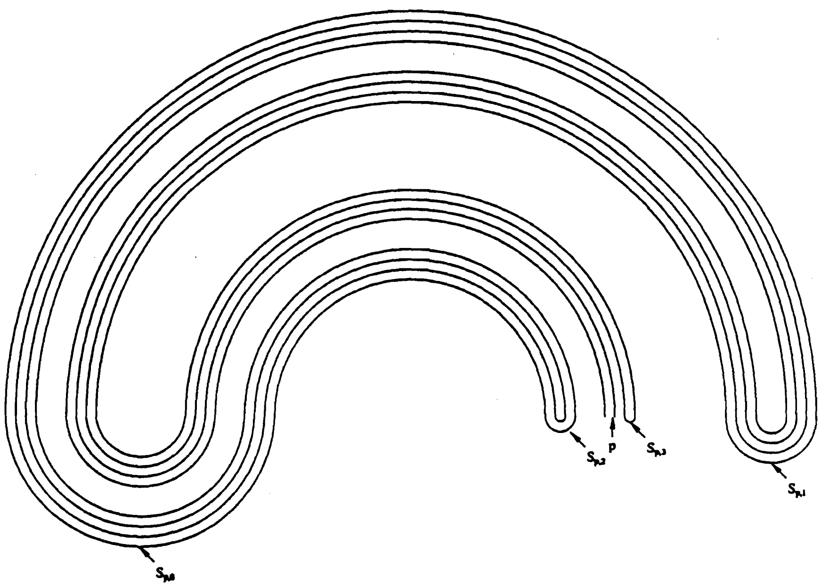
3. Examples

We shall modify slightly the construction given in section 2 to describe for each point p of E a continuum M_p containing $(p,0)$ which is homeomorphic to M under a homeomorphism which takes $(0,0)$ onto $(p,0)$. In case $p = 0$, then $M_p = M$ and the homeomorphism will be the identity homeomorphism. Let p denote a point of E. Let

M



M_p for $p = .202020\cdots$



M_p denote the continuum which is the union of S (described in section 2) and the countable collection of semicircles $S_{p,0}, S_{p,1}, \ldots$ described as follows. $S_{p,0}$ is the same as S_0 if $p \in I_0$. If $p \in I_2$, then let $S_{p,0}$ be a similar set of semicircles below I_0. That is, $S_{p,0}$ is the set of all semicircles having center the midpoint of I_0, lying, except for their endpoints, below the x-axis and containing exactly two points of $I_0 \cap E$. We next define for each positive integer n, a similar collection of semicircles having endpoints in a member of G_{n+1} which does not contain p. Let n denote a positive integer. There are exactly two members of G_{n+1} which lie in the member of G_n which contains p. Let I_e denote the one of these two members of G_{n+1} which does not contain p. Let $S_{p,n}$ denote the collection of all semicircles having center the midpoint of I_e, lying, except for their endpoints, below the x-axis, and containing exactly two points of $I_e \cap E$. This completes the description of M_p. Note that if $p = (0,0)$, then M_p is Janiszewski's example.

4. Homeomorphisms

To see that for each point p in E, M_p is homeomorphic to M one may use methods entirely analogous to those of Bing in [2]. Thus we give only an outline of a proof here. It is well known that M is chainable. For definitions and an introduction to chainable (or snakelike) continua, see Bing [3]. It is easy to see that there is a sequence $\{C(i)\}_{i=1}^{\infty}$ of chains such that $\bigcap_{i=1}^{\infty} C(i)^* = M$ with the following properties. For each positive integer n, (1) C(n) has mesh less than 2^{-n}, (2) each member of C(n+1) has a closure which lies in a member of C(n), (3) each member of C(n) contains the closure of a member of C(n+1), (4) each member of G_n intersects only one member of C(n), and is a subset of that member, and (5) the first link of C(n) contains the origin. From Bing [2] we say that the chain $\{d_1, d_2, \ldots, d_n\}$ follows the pattern $(1,a_1), (2,a_2), \ldots, (n,a_n)$ in the chain F if for $1 \leq i \leq n$, d_i lies in the a_i-th link of F. It is also easy to see that, for each point p in E, there is a similar sequence of chains $\{C_p(i)\}_{i=1}^{\infty}$ covering M_p having the same properties except that p is in the first link of each of the chains. Moreover, these chains can be chosen such that for each positive integer n, $C_p(n)$ has the same number of links as C(n), and $C_p(n+1)$ follows the same pattern in $C_p(n)$ that C_{n+1} follows in C_n. It follows from Theorem 11 of [2] that there is a homeomorphism h from M onto M_p. We shall use the fact that if x is a point of M and y is a point of M_p, and $m_1, m_2, \ldots$ is a sequence of positive integers such that for each positive integer n, x is in the m_nth link of C(n) and y is in the m_nth link of $C_p(n)$, then h(x) = y. This gives, for example, that h(0,0) = p, since for each positive integer n, (0,0) is in the first link of C(n) and p is in the first link of $C_p(n)$.

5. Main Results

Our principal observation is that if K_p is the composant of M which contains p, then, under the homeomorphism h described above, the image of K_p is a composant of M_p each point of which is accessible. To see this we first note that each point of the composant of M_p which contains the origin, 0, is accessible. To show that the composant of M containing p maps onto the composant of M_p containing 0, it suffices to show that h(p) = 0. To see this we shall show that for each positive integer n, if the k-th link of C(n) contains p, then the k-th link of $C_p(n)$ contains 0. Let n denote a positive integer. The chain C(n) contains exactly 2^n links which intersect E and each of them contains exactly one member of G_n. We shall denote the indices of these links by $m_1, m_2, \ldots m_{2n}$, where we have $m_1 = 1 < m_2 < m_3 < \ldots < m_{2n}$. The composant K_0 of M which contains 0 contains an arc α from 0 to the point $(3^{-n+1}, 0)$ and α contains exactly one point of each member of G_n. We shall enumerate the points of E on this arc in the order in which they occur starting with 0. For $1 \leq j \leq 2^n$, m_j is the index of the link of C(n) which contains the jth point of E on α. In order to describe our algorithm for determining these points we introduce some notation. If $x = .x_1x_2 \ldots$ (base 3) is a point in E then define N(x) to be the point in E obtained by changing all the 0's to 2's and 2's to 0's in the base 3 representation of x. We refer to this a complementing the digits by analogy with base 2. Note that N(x) is the point in E symmetric to x about the point (1/2,0), and thus is the point of E joined to x in M by a semi-circle in the closed upper half plane. If $x \neq 0$ define $F(x)$ to be the point determined as follows. Let j denote the first integer such that x_j is not 0, and complement each digit x_u of x where $u > j$. Note that $F(x)$ is the point of $I_{x_1,x_2,\ldots x_j} \cap E$ which is joined to x in M by a semi-circle in the closed lower half plane. Let $p_1 = 0$. Let $p_2 = N(p_1)$. As indicated above, K_0 contains an arc in the closed upper half plane whose endpoints are p_1 and p_2, and which intersects E only in p_1 and p_2. Next define $p_3 = F(p_2)$ and note that K_0 contains an arc from p_2 to p_3 which intersects E only in p_2 and p_3. We continue by defining p_{j+1} to be $N(p_j)$ or $F(p_j)$ according as j is odd or even, for $j < 2^n$. For each j, $1 \leq j < 2^n$, there is an arc in K_0 from p_j to p_{j+1} intersecting E only in p_j and p_{j+1}. One of these points, say p_k, lies in a member of G_n which contains p. Thus p lies in the m_k-th link of $C(n)$. We next show that 0 lies in the m_k-th link of $C_p(n)$. To see this we describe an arc in the composant K_p of M_p having p as one endpoint, containing exactly one point from each member of G_n, and with these points ordered on the arc in the order from p to the other endpoint. The algorithm is much the same as the one described for K_0. If $x = .x_1x_2 \ldots$ is a point of $E - p$, then by $G(x)$ is meant the point determined as follows. Let j denote the first integer such that x_j is different from the j-th digit of p (base 3) and complement each digit x_u of x where $u > j$. Using G as F was used before, define $p_1 = p$ and p_{j+1} to be $N(p_j)$

or $G(p_j)$ for $j < 2^n$ according as j is odd or even. Once again p_j and p_{j+1} are the endpoints of an arc lying in the closed upper or lower half plane according as n is odd or even and intersecting E only in p_j and p_{j+1}. To see that 0 is in the m_k-th link of $C_p(n)$ we make a different but equivalent definition of our algorithm. Note that if x is a point of E - 1, then F(N(x)) is the point obtained by complementing each digit in x up to, and including, the first one that is 0. But this is exactly the algorithm for counting in binary. Starting with 0, if we successively apply the composite map F(N), we obtain the points .200..., .0200..., .2200..., .00200..., ... If we reverse the digits in these sequences and change the 2's to 1's, we obtain the integers 1,2,3... in binary. Let t denote a positive integer. To find the sequence $.x_1x_2...$ obtained from .00... by applying the composite map F(N) t-times, one may write down the integer t in binary as $t_it_{i-1}...t_1$ and for each integer j, let x_j be 2 if and only if t_j is 1. In the case of N and G the result is similar. After t applications of G(N) to the sequence $.x_1x_2...$, the resultant sequence $.y_1y_2...$ will have the property that y_j is different from x_j if and only if t_j is 1. Assume now that the integer k (defined above) is even and recall that p lies in the m_k-th link of C(n). Also the point p_k which results from the application of F(N) k/2 times, lies in the member of G_n containing p. Thus p_k and p agree in the first k digits of their base 3 representation. It should be clear that applying F(N) k/2 times to

.00... will result in changing some of the first k digits from 0's to 2's. The application of G(N) k/2 times to the resulting sequence p_k, or to p since p and p_k agree in the first k digits, will change these 2's back to 0's. This then completes the proof in the case where F and N, and thus also G and N, are applied the same number of times. The case where k is odd and there is an extra application of N should be clear.

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