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Research Announcement:

FIXED POINTS OF ORIENTATION REVERSING HOMEOMORPHISMS OF THE PLANE

by

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Le h be a homeomorphism of the plane R^2 onto itself, and let X be a plane continuum invariant under h.

In 1951, M. L. Cartwright and J. C. Littlewood (see [5]) proved that if X does not separate the plane, then h has a fixed point in X. Simpler proofs of the Cartwright-Littlewood theorem were later provided by O. H. Hamilton in [6] and by Morton Brown in [4]. In [2], H. Bell proved the Cartwright-Littlewood theorem for an arbitrary homeomorphism of the plane (see also [1] and [3]).

In this note, we do not assume that the continuum X does not separate the plane. We assume that h is an orientation reversing homeomorphism of the plane onto itself.

Marcy Barge asked whether h has always a fixed point in X, and in some cases, for instance if X has exactly two complementary domains, whether h has two fixed points in X. The following results, whose proofs are based on Bell's theorem, answer Barge's question:

Theorem 1. *If* X *has exactly two complementary domains, then* h *has at least two fixed points in* X.

Denote by:

(1) $[X,h]$ the union of X and the bounded complementary domains of X which contain no fixed points of h,

(2) $P(X,h)$ the set of fixed points of h in the bounded complementary domains of X,

(3) $LP(X,h)$ the set of the limit points of $P(X,h)$ in X,

(4) $Q(X,h)$ the set of the fixed points of h in X.

Lemma 1. *If* $LP(X,h) = \emptyset$, *then* $[X,h]$ *has finitely many complementary domains.*

Lemma 2. *If* $[X,h]$ *does not separate* R^2, *then* h *has a fixed point in* X.

Lemma 3. *If* $LP(X,h)$ *consists of exactly one point, then there is an orientation reversing homeomorphism* f *of* R^2 *onto itself, and there exists a continuum* Y *invariant under* f *such that* 1) Y *has finitely many complementary domains, and* 2) *if* $Q(X,h)$ *contains* n (finitely many) *points, then* $Q(Y,f)$ *contains at most* $n - 1$ *points.*

Lemma 4. *Let* $k > 2$ *be an integer. If* $[X,h]$ *has* k *complementary domains, then there exist an orientation reversing homeomorphism* f *of* R^2 *onto itself, a continuum* Y *invariant under* f, *and an integer* j, $2 \leq j \leq k - 1$, *such that* 1) Y *has* j *invariant complementary domains, and* 2) *the cardinality of* $Q(Y,f)$ *does not exceed the cardinality of* $Q(X,h)$.

Theorem 2. *If at least one of the bounded complementary domains of* X *is invariant under* h, *then* h *has at least two fixed points in* X. *Otherwise* h *has at least one fixed point in* X.

Proof. By Lemma 2, if there are no invariant bounded complementary domains, then h has a fixed point in X.

If LP(X,h) contains more than one point, then clearly h has at least two fixed points in X.

By Lemma 3, if LP(X,h) contains exactly one point which is the only point of Q(X,h), then there exists an orientation reversing homeomorphism f of the plane and a continuum Y invariant under f, such that f has no fixed points in Y, and Y has finitely many complementary domains. If [Y,f] does not separate the plane, then f has a fixed point in Y. If [Y,f] separates the plane, then by Lemma 4 applied inductively, f has a fixed point in Y. Hence, Q(X,h) contains at least two points.

Assume now that X has at least one invariant bounded complementary domain U. If LP(X,h) = ∅, then [X,h] has finitely many complementary domains. The continuum [X,h] ∪ U separates the plane. Either [X,h] ∪ U has exactly two complementary domains, or by Lemma 4, applied inductively, we obtain an orientation reversing homeomorphism g of the plane having an invariant continuum Z with exactly two complementary domains. By Theorem 1, g has at least two fixed points in Z. Therefore, h has at least two fixed points in X.

The proofs of the above results are given in [7]. The following generalizations of Theorem 2 are contained in [8]:

Theorem 3. *If* X *has at least* 2^k, $k \geq 0$, *bounded complementary domains which are invariant under* h, *then* h *has at least* k + 2 *fixed points in* X.

Theorem 4. *If* X *has infinitely many complementary domains which are invariant under* h, *then* h *has infinitely many fixed points in* X.

References

1. Harold Bell, *A fixed point theorem for planar homeomorphisms*, Bull. Amer. Math. Soc. 82 (1976), 778-780.
2. Harold Bell, *A fixed point theorem for plane homeomorphisms*, Fund. Math. 100 (1978), 119-128.
3. Beverly Brechner, *Prime ends, indecomposable continua, and the fixed point property*, Top. Proc. (1) 4 (1979), 227-234.
4. Morton Brown, *A short short proof of the Cartwright-Littlewood fixed point theorem*, Proc. Amer. Math. Soc. 65 (1977), 372.
5. M. L. Cartwright and J. C. Littlewood, *Some fixed point theorems*, Ann. of Math. 54 (1951), 1-37.
6. O. H. Hamilton, *A short proof of the Cartwright-Littlewood fixed point theorem*, Canad. J. Math. 6 (1954), 522-524.
7. K. Kuperberg, *Fixed points of orientation reversing homeomorphisms of the plane*, Proc. Amer. Math. Soc., to appear.

8. K. Kuperberg, *A lower bound for the number of fixed points of orientation reversing homeomorphisms*, Proceedings of the Workshop on the Geometry of Hamiltonian Systems, MSRI, to appear.

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