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In [RZ] Reed and Zenor showed that a connected, locally connected, locally compact normal Moore space is metrizable. This result re-opened interest in the general question of metrization of manifolds, pending the solution of Wilder's Problem ([RZ], [R]).

Recall that a manifold is a connected regular T_1-space for which there is a natural number n such that each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$. Hence manifolds are locally compact and locally connected, but not necessarily metrizable or, equivalently, paracompact. The Reed-Zenor theorem has as a corollary that normal Moore manifolds are metrizable.

For an excellent source of information on non-metrizable manifolds see Peter Nyikos' article in [Ny1].

A natural generalization of a developable space is a quasi-developable space. Recall that a space X is developable (quasi-developable) if there exists a sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of open covers of X (collections of open subsets of X) such that for each $x \in X$, if U is open in X and $x \in U$ then there is a natural number n such that $st(x, \mathcal{G}_n) \neq \emptyset$ and $st(x, \mathcal{G}_n) \subset U$. If a quasi-developable space is perfect (= closed sets are G_δ sets) then it is developable [B]. A regular T_1 space that is developable is a Moore space. It is shown in [BL] that if $\langle \mathcal{G}_n : n \in \omega \rangle$

is a quasi-development for X and if $x \in U$ where U is open in X then there exists n such that $\emptyset \neq st(x, G_n) \subset U$ and x is an element of only one member of G_n.

In this note an example of a quasi-developable 2-manifold that is not developable is given. A different example was independently obtained by Peter Nyikos [Ny2]. Also partial results are proved concerning the metrizability of quasi-developable manifolds.

Let all spaces in this paper be T_1-spaces. The following lemma (proved in [RuZ]) is needed to develop techniques used in constructing the example.

Lemma 1 [RuZ]. *Let* $\{U_n: n \in \omega\}$ *be a nested sequence of open connected subsets of* $D' = (-1,1) \quad (0,1)$ *such that* $\cap\{cl(U_n, D'): n \in \omega\} = 0$ *where* $cl(U_n, D')$ *denotes the closure of* U_n *in* D' *with the relative topology from* $\mathbf{R}^2$. *Furthermore let* $p_n \in U_n$ *for each* $n \in \omega$. *Then there is a homeomorphism* g *of* D' *into* D' *such that:*

(i) $D' - g(D')$ *is homeomorphic to* $J = [0,1)$,

(ii) $D' - g(D') \subset cl(\{g(p_n): n \in \omega\}, D')$ *and*

(iii) $D' - g(D') \subset Int(cl(g(U_n), D'), D')$ *for each* $n \in \omega$.

where Int(A,B) *denotes the interior of* A *in* B.

This lemma is a tool in the following definition.

Definition 1. Let M be a 2-manifold, D a subspace of M homeomorphic to D', $\{U_n: n \in \omega\}$ a nested sequence of open connected subsets of D with $\cap\{cl(U_n, M): n \in \omega\} = \emptyset$

and $p_n \in U_n$ for each $n \in \omega$. A Rudin-Zenor extension of M with respect to D, $\{U_n: n \in \omega\}$ and $\{p_n: n \in \omega\}$ is a topological space M' described as follows:

Let g be a homeomorphism of D into D as in Lemma 1. Let g' be a homeomorphism of J onto D - g(D) where J is a copy of [0,1) disjoint from M. Let g* be the union of g and g' (thus g* maps $D \cup J$ onto D). Then M' is the unique topological space satisfying:

(i) the underlying set of M' is $M \cup J$,

(ii) M and $J \cup D$ are open in M',

(iii) M keeps its original topology as a subspace of M', and

(iv) the subspace topology on $D \cup J$ is such that g* is a homeomorphism.

Notice the Rudin-Zenor extension of M adds one copy of J to M.

Definition 2. Let M be a 2-manifold and A an index set. Let $\mathcal{D} = \{D_\alpha: \alpha \in A\}$ where each D_α is a subspace of M homeomorphic to D'. For each $\alpha \in A$ let $U_\alpha = \{U(\alpha,n): n \in \omega\}$ be a decreasing sequence of connected open subsets of D_α such that $\cap\{cl(U(\alpha,n),M): n \in \omega\} = \emptyset$ and let $\mathcal{U}x = \{U_\alpha: \alpha \in A\}$. For each $\alpha \in A$ and $n \in \omega$, let $p(\alpha,n) \in U(\alpha,n)$ and let $P_\alpha = \{p(\alpha,n): n \in \omega\}$. Let $\mathcal{P} = \{P_\alpha: \alpha \in A\}$. Let $\mathcal{J} = \{J_\alpha: \alpha \in A\}$ where each J_α is a copy of [0,1), $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$ and each J_α is disjoint from M. The free Rudin-Zenor extension of M relative to $(\mathcal{D},\mathcal{U},\mathcal{P},\mathcal{J})$, denoted by FRZ(M), is the unique topological space such that

(i) the underlying set of FRZ(M) is $\cup\{J_\alpha: \alpha \in A\} \cup M$,

(ii) for each $\alpha \in A$, $M \cup J_\alpha$ is an open subspace of FRZ(M), and

(iii) for each $\alpha \in A$ the subspace topology of $M \cup J_\alpha$ is a Rudin-Zenor extension of M.

Notice that FRZ(M) adds $|A|$ many copies of J to M and that FRZ(M) is a T_1-space.

Theorem 1. *Every free Rudin-Zenor extension is locally* $\mathbf{R}^2$. *It is Hausdorff (and thus a* 2*-manifold) if the following property* (*) *holds:*

(*) *for each* $\alpha, \beta \in A$, $\alpha \neq \beta$, *there exists* $n \in \omega$ *such that*

$$\mathrm{cl}(U(\alpha,n),M) \cup \mathrm{cl}(U(\beta,n),M) = 0.$$

Proof. FRZ(M) is locally $\mathbf{R}^2$ since, for each $\alpha \in A$, $M \cup J_\alpha$ is a Rudin-Zenor extension of M. The only difficult case for Hausdorffness of FRZ(M) is when $x \in J_\alpha$, $y \in J_\beta$ and $\alpha \neq \beta$. Property (*) covers this case.

In order to construct the desired example two topological spaces must be reviewed.

Example 1. (Example 2.17 of Gary Gruenhage's article in [G]). Let B be a Bernstein subset of $\mathbf{R}$ and let $\{B_\alpha: \alpha < 2^\omega\}$ be an enumeration of all countable subsets of B such that $\mathrm{cl}(B_\alpha,\mathbf{R})$ is uncountable. For each $\alpha < 2^\omega$ choose

$$x_\alpha \in cl(B_\alpha, \mathbf{R}) \setminus (B \cup \{x_\beta : \beta < \alpha\})$$

and choose points $x_\alpha(m) \in B_\alpha$ such that the sequence $\langle x_\alpha(m) : m \in \omega \rangle$ converges to x_α in $\mathbf{R}$. Let $H = \{x_\alpha : \alpha < 2^\omega\}$ and $X = B \cup H$. Topologize X by letting points of B be isolated and, if $N(x_\alpha, k) = \{x_\alpha\} \cup \{x_\alpha(m) : n \geq k\}$ for each $k \in \omega$, by letting $\{N(x_\alpha, k) : k \in \omega\}$ be a local base at x_α. Then X is a locally compact quasi-developable space such that H is not a G_δ-subset of X (the details of these results are in [G]).

Example 2. This example is the Prüfer Manifold $P(\mathbf{R})$ ([Ra]) (see example 2.7 of Peter Nyikos' article in [Ny1]). To construct this example collared copies of the real line (i.e. $[0,1) \times \mathbf{R}$) are attached at each point of the x-axis to the open upper half plane. Thus the Prüfer manifold as a point set can be visualized as a subset of $\mathbf{R}^3$. In fact

$$P(\mathbf{R}) = \{(x,y,z) : x \in \mathbf{R},\ y > 0,\ z = 0\} \cup (\cup\{\{x\} \times [0,-1) \times \mathbf{R} : x \in \mathbf{R}\}).$$

Let M(x) denotes the collared real line that is attached at the point x on the x-axis. A Prüfer manifold can be obtained from each subset S of $\mathbf{R}$ by attaching an M(x) to the open upper half plane at each point x of S. The resulting Prüfer manifold P(S) is a developable 2-manifold that inherits its topology from $P(\mathbf{R})$. Notice that if S is a countable discrete in itself (i.e. S contains no limit points) subset of $\mathbf{R}$ then P(S) is homeomorphic to $\mathbf{R}^2$

(which is homeomorphic to $D' = (-1,1) \times (0,1)$). Also notice that $P(S)$ as a point set is contained in $\mathbf{R}^3$.

Using these two examples the desired example can be constructed.

Example 3. There exists a quasi-developable 2-manifold Z that is not developable.

Consider the set $X = B \cup H$ of Example 1 as a subset of the x-axis and let $P(B)$ be the Prüfer 2-manifold constructed over the Bernstein set B. Recall that $H = \{x_\alpha: \alpha < 2^\omega\}$.

For each $\alpha < 2^\omega$, let

$$D_\alpha = \{(x,y,z): x \in \mathbf{R},\ y > 0,\ z = 0\} \cup (\cup\{M(x_\alpha(n)): n \in \omega\}).$$

Since $\{x_\alpha(n): n \in \omega\}$ is discrete in itself as a subset of $\mathbf{R}$, D_α is an open subset of $P(B)$ that is homeomorphic to D'. Let $\mathcal{D} = \{D_\alpha: \alpha < 2^\omega\}$.

For each $\alpha < 2^\omega$, let $U(\alpha,n) = A(\alpha,n) \cup B(\alpha,n)$ where

$$A(\alpha,n) = \{(x,y,z) \in \mathbf{R}^3: |(x_\alpha,0,0) - (x,y,0)| < 1/n, y > 0\}$$

and

$$B(\alpha,n) = \cup\{M(x_\alpha(m)): |x_\alpha - x_\alpha(m)| < 1/n\}.$$

It follows that $U(\alpha,n)$ is an open connected subset of and that $D_\alpha, U(\alpha,n) \supset U(\alpha,n+1)$ for each $n \in \omega$, and

$$\cap\{cl(U(\alpha,n),P(B)): n \in \omega\} = \emptyset.$$

Let $U_\alpha = \{U(\alpha,n): n \in \omega\}$ and $\mathcal{U} = \{U_\alpha: \alpha < 2\omega\}$. Let $p(\alpha,n) = (x_\alpha(n),0,0)$ for each $\alpha < 2^\omega$ and $n \in \omega$. Notice that $p(\alpha,n) \in U(\alpha,n)$. Let $P_\alpha = \{p(\alpha,n): n \in \omega\}$ and $\mathcal{P} = \{P_\alpha: \alpha < 2^\omega\}$.

Let $\mathcal{J} = \{J_\alpha: \alpha < 2^\omega\}$ where each J_α is a copy of $[0,1)$ disjoint from $P(B)$ and if $\alpha \neq \beta$, then $J_\alpha \cap J_\beta = 0$.

Let Z be FRZ(P(B)) with respect to $(\mathcal{D},\mathcal{U},\mathcal{P},\mathcal{J})$. Notice that P(B) satisfies property (*). Thus FRZ(P(B)) is a 2-manifold.

To see that FRZ(P(B)) is not perfect consider the subspace

$$Y = \cup\{J_\alpha: \alpha < 2^\omega\} \cup \{(x,0,0): x \in B\}.$$

Notice that $B' = \{(x,0,0) \in \text{FRZ}(P(B)): x \in B\}$ is an open subset of Y. Hence if FRZ(P(B)) was perfect, then B' would be an F_σ-set in Y. Assume $B' = \cup\{F'_n: n \in \omega\}$ where F'_n is closed in Y. There exists $n \in \omega$ such that $|F'_n| > \omega$. Let $F_n = \{x \in B: (x,0,0) \in F'_n\}$. Then F_n as a closed subset in the space X of Example 1 contains a B_α. In this space x_α is a limit of B_α and hence of F_n. Thus, in Y, J_α is contained in $\text{cl}(F'_n,Y)$. But $J_\alpha \cap B' = \emptyset$. Thus B' is not an F_σ and it follows that FRZ(P(B)) is not perfect.

The following theorem is used to show that FRZ(P(B)) is quasi-developable.

Theorem 1. *Let* X *be a regular, locally quasi-developable,* T_1*-space. The following are equivalent:*

(i) X *is quasi-developable,*

(ii) X *is weakly submetacompact, and*

(iii) *X has a σ-relatively discrete cover by quasi-developable sets.*

Proof. (i) → (ii) see [BL]. For (ii) → (iii) let $O(x)$ be an open quasi-developable subset of X containing x for each $x \in X$. Then $\{O(x): x \in X\}$ has a σ-relatively discrete refinement (that is also a cover) by quasi-developable subsets. For (iii) → (i) let $X = \cup\{\cup F(n): n \in \omega\}$ where $F(n) = \{F(n,\alpha): \alpha \in I_n\}$ is a relatively discrete collection of quasi-developable (hence weakly submetacompact) subsets of X. For each $F(n,\alpha) \in F_n$ there exists an open set $U(n,\alpha)$ such that

$$U(n,\alpha) \cap (\cup F_n) = F(n,\alpha).$$

Fix n and α and for each $x \in F(n,\alpha)$ let $O(x)$ be an open quasi-developable set that contains x such that $O(x) \subset U(n,\alpha)$. Since $\{O(x) \cap F(n,\alpha): x \in F(n,\alpha)\}$ is an open cover of $F(n,\alpha)$ it has a σ-relatively discrete refinement $R(n,\alpha) = \langle R(n,\alpha,k): k \in \omega\rangle$ that covers $F(n,\alpha)$. Fix k. For each $R \in R(n,\alpha,k)$ let $V(R)$ be an open set in X such that

$$\{V(R) \cap F(n,\alpha): R \in R(n,\alpha,k)\}$$

witnesses that $R(n,\alpha,k)$ is a relatively discrete collection. If $R \in R(n,\alpha,k)$ let $x(R) \in F(n,\alpha)$ such that R refines $O(x(R))$. Let $\langle G(n,\alpha,k,R,m): m \in \omega\rangle$ be a quasi-development for $O(x(R)) \cap V(R) \cap U(n,\alpha)$. Let

$$H(n,k,m) = \{G \in G(n,\alpha,k,R,m): F(n,\alpha) \in F_n,\ R \in R(n,\alpha,k)\}$$

Then $H = \langle H(n,k,m): n \in \omega,\ k \in \omega,\ m \in \omega\rangle$ is a quasi-development for X. To see this let $x \in U$ where U is open in X. There exists n and α such that $x \in F(n,\alpha)$ and there exists $k \in \omega$ and $R \in \mathcal{R}(n,\alpha,k)$ such that $x \in R$. Then there exists m such that

$$st(x,\mathcal{G}(n,\alpha,k,R,m)) \subset U \cap O(x(R)) \cap V(R) \cap U(n,\alpha).$$

Hence $st(x,\mathcal{H}(n,k,m) \subset U$.

Notice that the underlying set in FRZ(P(B)) is $P(B) \cup (\cup\{J_\alpha: \alpha < 2^\omega)$. Since P(B) as a subspace is developable it has a σ-relatively discrete cover and since $\{J_\alpha: \alpha < 2^\omega\}$ is a pairwise disjoint collection it is σ-relatively discrete. Since FRZ(P(B)) is a manifold it is locally quasi-developable. Hence, by the preceding theorem, FRZ(P(B)) is quasi-developable.

The same argument as Peter Nyikos gives in [Ny1] shows that FRZ(P(B)) is not normal.

The following question remains open:

Question 1. Is every hereditarily normal quasi-developable manifold paracompact?

A partial affirmative answer is given if $2^{\omega_1} > 2^\omega$.

Theorem 2. *Assume* $2^{\omega_1} > 2^\omega$. *Every hereditarily normal quasi-developable manifold is paracompact.*

Note that an actually stronger result was announced without proof by one of the authors (see the remark after Theorem 2.5 together with Lemma 2.1 in [Ba]).

According to that result "quasi-developable manifold" can be weakened to "connected, locally c.c.c., hereditarily weakly submeta-Lindelöf space" in Theorem 2 (weakly submeta-Lindelöf = weakly $\delta\theta$-refinable). Since the proof of the more general result has not appeared in print we feel justified in giving a proof of Theorem 2 here.

Proof of Theorem 2. First recall a result of Taylor [Ta] showing each first-countable hereditarily normal space has the following property under $2^{\omega_1} > 2^{\omega}$:

(*) if C is a cub subset of ω_1 and $\{x_\alpha: \alpha \in C\}$ is a weakly σ-discrete set of distinct points then there is a stationary subset $S \subset C$ such that $\{x_\alpha: \alpha \in S\}$ has an expansion by pairwise disjoint open sets.

Now suppose indirectly that there is a non-paracompact, hereditarily normal, quasi-developable manifold X. Then X has a connected open submanifold Y of weight ω_1. Let $\{U_\alpha: \alpha \in \omega_1\}$ be an open cover of Y by separable open subsets. Since Y is connected we can choose, for each $\alpha \in \omega_1$, a point

$$y_\alpha \in \mathrm{cl}(\cup\{U_\beta: \beta < \alpha\}) \setminus \cup\{U_\beta: \beta < \alpha\}.$$

Let C be a cub subset of ω_1 such that $L = \{y_\alpha: \alpha \in C\}$ consists of distinct points. Note that L is locally countable and, thus, a σ-scattered space which is hereditarily weak submetacompact and, hence, weakly σ-discrete ([Ny2], Corollary 3.5). By (*) there is a stationary set $S \subset \omega_1$ such that $\{y_\alpha: \alpha \in S\}$ has a pairwise disjoint expansion $\{B_\alpha: \alpha \in S\}$ by open sets. Since

$$y_\alpha \in \mathrm{cl}(\cup\{U_\beta\colon \beta < \alpha\}) \setminus \cup\{U_\beta\colon \beta < \alpha\}.$$

for each $\alpha \in S$ there is an $f(\alpha) < \alpha$ such that $B_\alpha \cap U_{f(\alpha)} \neq \emptyset$. By the pressing down lemma there is a $\beta \in \omega_1$ such that $f(\alpha) = \beta$ for uncountably many $\alpha \in S$. Therefore uncountably many of the B_α's intersect U_β violating the separability of U_β.

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