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1. Introduction

A. Berner and I. Juhász [BJ84] introduce the following two person infinite game, $G(X)$, played on a separable space X: at the n^{th} play, O picks an open set $U_n \subset X$, then P picks a point $x_n \in U_n$. They say O *wins* if P's points $\{x_n\}_{n\in\omega}$ are dense in X.

Clearly O has a winning strategy in $G(X)$ if X has a countable π-base. (Recall that a π-*base* for X is a collection $\mathcal{B}$ of non-empty open subsets of X such that every non-empty open subset of X contains some member of $\mathcal{B}$, and that the π-*weight*, $\pi w(X)$, of X is the least cardinal of a π-base for X.) It is shown in [BJ84] that $\pi w(X) = \omega$ is equivalent to the existence of a winning strategy for O in $G(X)$.

The focus of this paper is on the question of the existence of a space X in which $G(X)$ is undetermined, i.e., neither player has a winning strategy. It is still an open question whether or not such a space exists in ZFC. In [BJ84] such a space is constructed from the axiom $\diamondsuit$, a consequence of $V = L$, and in [Juh85], Juhász obtains examples from $MA(\omega_1)$ for countable posets. Here we show that such a space exists assuming Martin's Axiom for σ-centered posets (in particular, the continuum hypothesis).

Next we consider *irresolvable spaces*, i.e., spaces which do not have disjoint dense subsets. We show that the existence of an irresolvable space X for which G(X) is undetermined implies the existence of a semi-selective σ-centered filter on ω, the set of natural numbers. (Recall that a filter $\mathcal{F}$ on ω is *semi-selective* if, given $\{F_n\}_{n\in\omega} \subset \mathcal{F}$, there exists $F \in \mathcal{F}$ with $|F \setminus F_n| \leq n$ for all $n \in \omega$, and that $\mathcal{F}$ is *σ-centered* if $\mathcal{F}^+ = \{A \subset \omega : \omega \setminus A \notin \mathcal{F}\}$ can be written as a countable union of subcollections each having the finite intersection property.) K. Kunen [Kun76] showed that if $\aleph_2$ or more random reals are added to a model of CH, then in the resulting model there are no semi-selective ultrafilters. We show that in the same model, there are no semi-selective σ-centered filters, hence no irresolvable spaces X for which G(X) is undetermined. (Note that ultrafilters are σ-centered, so our result is an extension of Kunen's.)

Finally we observe that there are also no such filters in Laver's model for the Borel conjecture ([Lav76]).

All of our spaces are assumed to be regular and T_1.

2. An Undetermined Game

Assuming MA for σ-centered posets, we construct a space X for which G(X) is undetermined. Since we consider irresolvable spaces in Section 3, we construct our X so that it doesn't have disjoint dense subsets. By the next lemma, together with the Berner-Juhász result, it follows that O has no winning strategy in G(X).

Lemma 2.1. (MA_κ for countable posets). *If* X *is a separable space without isolated points and* $\pi w(X) \leq \kappa$, *then* X *has disjoint dense subsets.*

Proof. Let $\mathcal{B}$ be a π-base for X with $|\mathcal{B}| \leq \kappa$. Let Z be a countable dense subset of X. Let the poset P be the set of all pairs $(F_0,F_1) \in ([Z]^{<\omega})^2$ such that $F_0 \cap F_1 = \emptyset$. Define $(F_0',F_1') \leq (F_0,F_1)$ if $F_0' \supset F_0$ and $F_1' \supset F_1$. Clearly this poset P is countable and a generic filter meeting all sets of the form

$$D_B = \{(F_0,F_1) \in P: B \cap F_0 \neq \emptyset \text{ and } B \cap F_1 \neq \emptyset\}$$

for $B \in \mathcal{B}$, defines disjoint dense subsets of Z, hence of X.

The set X will be the set ω of natural numbers. The topology, τ, will be the union of topologies τ_α, $\alpha \leq \underline{c}$, constructed inductively. The next two lemmas will be needed to get us from stage α to stage $\alpha + 1$ in the induction.

Lemma 2.2. (MA_κ for countable posets). *Let* (X,τ) *be a countable regular space of weight at most* κ *with no isolated points, and let* $A \subset X$ *be dense. Then there is a finer regular topology* τ' *of the same weight on* X *such that* (X,τ') *has no isolated points and* A *is open and dense in* (X,τ').

Proof. Let (X,τ) and $A \subset X$ be as hypothesized. It follows easily from Lemma 2.1 that A can be written as a disjoint union of countably many dense sets $\{A_n\}_{n\in\omega}$. Define $f: X \mapsto \omega + 1$ by $f(x) = n$ if $x \in A_n$ and $f(x) = \omega$ if

$x \notin A$. Let $\hat{X} = \{(x, f(x))\}_{x \in X} \subset X \times (\omega + 1)$. Let $\Pi: \hat{X} \mapsto X$ be the projection. It is easy to check that $\hat{X}$ has no isolated points, and $\Pi^{-1}(A)$ is open and dense in $\hat{X}$. Thus the lemma follows, with τ' being the topology on X induced by Π.

Lemma 2.3. (MA_κ for σ-centered posets). *Let* τ *be a dense-in-itself topology on* ω *of weight at most* κ. *Let* $\mathcal{U}$ *be a collection of at most* κ *many dense open subsets of* (ω, τ). *Suppose also that* $\psi: \omega^{<\omega} \mapsto P(\omega)$ *is such that* $\psi(\sigma)$ *is dense for all* $\sigma \in 2^{<\omega}$. *Then there is some* $\sigma \in \omega^\omega$ *such that*

1. range(σ) *is dense in* (ω, τ):
2. range(σ) $\subset^*$ U *for every* $U \in \mathcal{U}$;
3. $\sigma(n) \in \psi(\sigma \restriction n)$ *for all* $n \in \omega$.

Proof. Let $\mathcal{B}$ be a base for τ of size at most κ, and let the poset P be the set of all pairs (σ, F) satisfying:

1. $\sigma \in \omega^{<\omega}$;
2. $j \in \text{dom}(\sigma) \Rightarrow \sigma(j) \in \psi(\sigma \restriction j)$;
3. $F \in [\mathcal{U}]^{<\omega}$;
4. $B \in F \Rightarrow B \cap \text{range}(\sigma) \neq \emptyset$.

Define $(\sigma', F') \leq (\sigma, F)$ if $\sigma' \supset \sigma$, $F' \supset F$, and for each $i \in \text{dom}(\sigma') \setminus \text{dom}(\sigma)$, we have $\sigma'(i) \in \cap F$.

Since any two members of P having the same first coordinate are compatible, P is σ-centered. For $B \in \mathcal{B}$, let $D_B = \{(\sigma, F) \in P: B \cap \text{range}(\sigma) \neq \emptyset\}$; for $U \in \mathcal{U}$, let $D_U = \{(\sigma, F) \in P: U \in F\}$, and for $n \in \omega$, let $D_n = \{(\sigma, F) \in P: n \in \text{dom}(\sigma)\}$. These are easily seen to be dense in P.

Let G be a filter in P meeting all D_B's, D_U's, and D_n's. Then G defines a function $\sigma: \omega \mapsto \omega$ such that $\sigma(n) \in \psi(\sigma\restriction n)$ for all $n \in \omega$. Because G meets all D_B's, range(σ) is dense in (ω,τ). Pick $U \in \mathcal{U}$, and let $p = (\sigma_p, F_p) \in G$ with $U \in F_p$. Then for each $i > \mathrm{dom}(\sigma_p)$, $\sigma(i) \in U$, whence range$(\sigma) \subset^* U$.

Now we are ready to construct our example.

Theorem 2.4. (MA for σ-centered posets). *There is a countable irresolvable space* X *such that* G(X) *is undetermined.*

Proof. Let $\{E_\alpha\}_{\alpha<\underline{c}}$ index $P(\omega)$, and let $\{\psi_\alpha\}_{\alpha<\underline{c}}$ index all functions $\psi: 2^{<\omega} \mapsto P(\omega)$. We inductively define regular topologies τ_α, $\alpha < \underline{c}$, on ω having weight less than $\underline{c}$ and sets $\{D_\alpha: \alpha < \underline{c}\} \subset P(\omega)$ such that, for all $\beta < \beta'$ $< \alpha$,

1. $(\omega,\tau_{\beta'})$ has no isolated points;
2. $\tau_\beta \subset \tau_{\beta'}$ and $D_{\beta'} \subset^* D_\beta$;
3. $D_{\beta'}$ is dense in $(\omega,\tau_{\beta'})$;
4. D_β is dense open in $(\omega,\tau_{\beta'})$;
5. Either E_β is not dense in $(\omega,\tau_{\beta+1})$, or $D_{\beta+1} \subset E_\beta$;
6. Either $\psi_\beta(\sigma)$ is not dense in $(\omega,\tau_{\beta+1})$ for some $\sigma \in 2^{<\omega}$, or there is some $\sigma \in \omega^\omega$ such that $\sigma(n) \in \psi(\sigma\restriction n)$ for all $n \in \omega$, and $D_{\beta+1} \subset \mathrm{range}(\sigma)$.

To start, let τ_0 be a metrizable topology on ω with no isolated points, and let $D_0 = \omega$. Suppose we have constructed τ_β and D_β for all $\beta < \alpha$, where $\alpha < \underline{c}$.

Case 1. *α is a limit ordinal.*

Let $\tau_\alpha = \cup_{\beta<\alpha} \tau_\beta$. By Lemma 2.3, there exists $D_\alpha \subset \omega$ such that $D_\alpha \subset^* D_\beta$ for all $\beta < \alpha$ and D_α is dense in (ω,τ_α). It is easy to check that conditions (1) - (6) are satisfied.

Case 2. $\alpha = \gamma + 1$.

By Lemma 2.2, there is a topology $\tau'_\alpha \supset \tau_\gamma$ such that D_γ is dense open in (δ,τ'_α). Define $\psi'_\gamma\colon 2^{<\omega} \mapsto P(\omega)$ by setting $\psi'_\gamma(\sigma) = \psi_\gamma(\sigma)$ if $\psi_\gamma(\sigma)$ is dense in (ω,τ'_α), and $\psi'_\gamma(\sigma) = \omega$ otherwise. Let $\sigma \in \omega^\omega$ satisfy the conclusion of Lemma 2.3 with $\psi = \psi'_\gamma$ and let $\mathcal{D} = \{D_\beta\}_{\beta\le\gamma}$. Let $D'_\alpha =$ range(σ). If $E_\gamma \cap D'_\alpha$ is not dense in (ω,τ'_α), let $D_\alpha = D'_\alpha$. Otherwise let $D_\alpha = E_\gamma \cap D'_\alpha$. Let $\tau_\alpha \supset \tau'_\alpha$ be a regular dense-in-itself topology of the same weight such that D_α is dense open in (ω,τ_α). It is easy to verify that (1) - (6) hold. This completes the inductive construction.

Let $\tau = \cup\{\tau_\alpha\colon \alpha < \underline{c}\}$. We claim that (ω,τ) is a regular irresolvable space in which neither O nor P has a winning strategy. Note that all D_α's are dense open in (ω,τ). By (5), every dense subset of (ω,τ) contains some D_α; thus (ω,τ) is irresolvable. Clearly, (ω,τ) is regular and dense-in-itself because each τ_α, $\alpha < \underline{c}$, is. By Lemma 2.1, $\pi w(\omega,\tau) = \underline{c}$, so by the Berner-Juhász result, O has no winning strategy.

Finally, suppose P plays according to a strategy s. Let $D_\emptyset = \{s(\langle U\rangle)\colon U \in \tau\setminus\{\emptyset\}\}$. For each $n \in D_\emptyset$, pick U(n) such that $s(U(n)) = n$; for $\tau \notin D_{\emptyset}$, let $U(n) = \omega$. Now, for each $n \in \omega$, let

$$D_{\langle n \rangle} = \{s(\langle U(n), n, U\rangle): U \in \tau \setminus \{\emptyset\}\}.$$

For each $m \in D_{\langle n \rangle}$, pick $U(n,m)$ such that $s(\langle U(n), n, U(n,m)\rangle) = m$; if $m \notin D_{\langle n \rangle}$, let $U(n,m) = \omega$. Define

$$D_{\langle n,m \rangle} = \{s(U(n), n, U(n,m), m, U\rangle): U \in \tau \setminus \{\emptyset\}\}.$$

Continuing in this way, we define for each $\sigma \in \omega^{<\omega}$ a dense subset D_σ of (ω, τ) such that if $\sigma \in \omega^\omega$ and $\sigma(n) \in D_{\sigma \restriction n}$ for all n, then O can make P choose range(σ).

The function $\psi: 2^{<\omega} \mapsto P(\omega)$ defined by $\psi(\sigma) = D_\sigma$ is equal to some ψ_α. Then $\psi_\alpha(\sigma)$ is dense in $(\omega, \tau_{\alpha+1})$ for all $\sigma \in 2^{<\omega}$, hence there is some $\sigma \in \omega^\omega$ such that $\sigma(n) \in \psi(\sigma \restriction n)$ for all n and $D_{\alpha+1} \subset$ range(σ). So O can make P choose range(σ), and range(σ) is dense in (ω, τ). Thus s is not a winning strategy. Since s was arbitrary, P has no winning strategy.

3. Semi-Selective Filters and Irresolvable Spaces

Our task in this section is to show that the existence of an irresolvable space X for which G(X) is undetermined implies the existence of a semi-selective σ-centered filter on ω. In the next section we will discuss models in which no such filters exist.

Let us say that X is *strongly irresolvable* if every open subspace of X is irresolvable.

Lemma 3.1. *If* X *is irresolvable, then* X *contains an open strongly irresolvable subset.*

Proof. Let $\mathcal{U}$ be a maximal disjoint family of open resolvable subsets of X. Then the interior of $X \setminus \cup \mathcal{U}$ is non-empty and strongly irresolvable.

Lemma 3.2. *If* Y *is open in* X, *and* P *has a winning strategy in* G(Y), *then* P *has a winning strategy in* G(X).

Proof. Clear.

Lemma 3.3. *If there is an irresolvable space* X *for which* G(X) *is undetermined, then there is one which is strongly irresolvable.*

Proof. Let X be irresolvable with G(X) undetermined. Let $Y \subset X$ be open and strongly irresolvable. By Lemma 3.2 P has no winning strategy in Y. By Lemma 2.1, $\pi w(Y) > \omega$, so O has no winning strategy either.

Lemma 3.4. *If* X *is regular and* G(X) *is undetermined, then* G(Y) *is undetermined for any countable dense* $Y \subset X$.

Proof. Let X satisfy the hypotheses, and let Y be a countable dense subset of X. By regularity $\pi w(Y) = \pi w(X) > \omega$, so O has no winning strategy in G(Y). And again one easily sees that P does not have a winning strategy in G(Y) because otherwise P would have one in G(X) as well.

Lemma 3.5. *If there is an irresolvable space* X *with* G(X) *undetermined, then there is a countable strongly irresolvable such* X.

Proof. Let X be irresolvable with G(X) undetermined. By 3.3, we may assume that X is strongly irresolvable.

Let Y be a countable dense subset of X. By 3.4, G(Y) is undetermined. Clearly Y is strongly irresolvable, since Y is dense in a strongly irresolvable space.

Lemma 3.6. *Let* X *be a countable strongly irresolvable space with* G(X) *undetermined. Let* $\mathcal{F}$ *be the collection of dense subsets of* X. *Then* $\mathcal{F}$ *is a* σ*-centered semi-selective filter on the set* X.

Proof. If $F \in \mathcal{F}$, then $X\backslash F$ is not dense in any open set, i.e., $X\backslash F$ is nowhere dense. Thus F contains the dense open set $X\backslash(\overline{X\backslash F})$. It follows that $\mathcal{F}$ is a filter.

Let $\mathcal{F}^+ = \{A: X\backslash A \notin \mathcal{F}\}$. Then every $A \in \mathcal{F}^+$ is somewhere dense, hence int $A \neq \emptyset$. For $x \in X$, let $\mathcal{F}_x^+ = \{A \in \mathcal{F}^+: x \in \text{int } A\}$. Then $\mathcal{F}^+ = \cup_{x\in X} \mathcal{F}_x^+$, so is σ-centered.

Finally, to see that $\mathcal{F}$ is semi-selective, suppose $F_n \in \mathcal{F}$ for each $n < \omega$. Consider any strategy for P which picks a point in $\cap_{i\leq n} F_i$ on the n^{th} move. Since the strategy is not winning, there exists $x_n \in \cap_{i\leq n} F_i$ such that $F = \{x_n\}_{n\in\omega}$ is dense. Then $F \in \mathcal{F}$ and $|F\backslash F_n| \leq n$ for all n.

Corollary 3.7. *If there are no semi-selective* σ*-centered filters on* ω*, then* G(X) *is determined for any irresolvable space* X.

4. No Semi-Selective Filters

A filter $\mathcal{F}$ on ω is said to be *rapid* if, given any function $f: \omega \mapsto \omega$, there exist $n(k) > f(k)$ such that $\{n(k): k \in \omega\} \in \mathcal{F}$. It is easy to see that semi-selective filters are rapid. In the model constructed by R. Laver [Lav76] which demonstrated the consistency of the Borel conjecture, it is known [Mil80] that there are no rapid filters. So in this model, G(X) is determined for any irresolvable space X.

K. Kunen [Kun76] showed that if one adds at least $\aleph_2$ random reals to a model of CH, then there are no semi-selective ultrafilters in the resulting model. We show that in fact there are no semi-selective σ-centered filters in this model. Since ultrafilters are trivially σ-centered, this extends Kunen's result.

It will be convenient to use the following characterization of σ-centered filters.

Lemma 4.1. *A filter is σ-centered iff there are ultrafilters F_n, $n \in \omega$, such that $F = \bigcap_{n\in\omega} F_n$.*

Proof. If $F = \bigcap_n F_n$, where each F_n is an ultrafilter, then one easily sees that $F^+ = \bigcup_{n\in\omega} F_n$; so F is σ-centered.

Conversely, if $F^+ = \bigcup_{n\in\omega} F_n$, where each F_n is centered, let F_n' be an ultrafilter containing $F_n \cup F$. It is easy to check that $F = \bigcap_{n\in\omega} F_n'$.

Lemma 4.2. *If there are no semi-selective ultrafilters on ω and if $F = \bigcap_{n\in\omega} F_n$ is a semi-selective filter on ω where each F_n is an ultrafilter, then every element of $\bigcup_{n\in\omega} F_n$ is in infinitely many of the F_n's.*

Proof. Let $A \in F_k$ for some $k \in \omega$, and suppose that $H = \{n\colon A \in F_n\}$ is finite. Since $F\restriction A = \{F \cap A\colon F \in F\}$ is semi-selective, it cannot be an ultrafilter. Thus there is an ultrafilter $F' \notin \{F_n\}_{n\in H}$ extending F with $A \in F'$. There exists $A' \subset A$ with $A' \in F' \setminus \cup_{n\in H} F_n$. Now $A' \in F^+ = \cup_{n\in\omega} F_n$, so A', and hence A, is in some F_n with $n \notin H$. This is a contradiction.

Let $V \models CH$, let $\mathcal{B}_\lambda$ be the product measure algebra on 2^λ, and let G be $\mathcal{B}_{\omega_2}$-generic over V. Suppose that in $V[G]$, F is a semi-selective σ-centered filter on ω. Then $F = \cap_{n\in\omega} F_n$, where each F_n is an ultrafilter. Since $\mathcal{B}_{\omega_2}$ is ccc and CH holds in V, we can reflect these conditions, as well as the conclusion of Lemma 4.2, to $V[G|\mathcal{B}_\lambda]$ for some $\lambda < \omega_2$. (See the proof of Corollary 4.4 for more details).

Proposition 4.3. *In* V, *suppose that* $F = \cap_{n\in\omega} F_n$ *is semi-selective, where each* F_n *is an ultrafilter on* ω, *and that* $A \in F_k$ *for any* $k \in \omega$ *implies* A *is in infinitely many* F_n*'s. Let* G *be* $\mathcal{B}_{\omega_2}$*-generic over* V. *Then in* V[G], *there do not exist ultrafilters* F_n^* *extending* F_n *such that* $\cap_{n\in\omega} F_n^*$ *is semi-selective.*

Proof. Assume the contrary. Then without loss of generality we may assume that there are $\mathcal{B}_{\omega_2}$-names $\dot{F}_n$, $n < \omega$, and $\dot{F}$ such that 1 forces

1. $\dot{F}_n$ is an ultrafilter extending F_n;
2. $\dot{F} = \cap_{n\in\omega} \dot{F}_n$ is semi-selective.

Let μ be the product measure on $\mathcal{B}_{\omega_2}$.

Fact 1. *For each* $\varepsilon > 0$ *and* $n < \omega$, *there is a* $\mathcal{B}_{\omega_2}$-*name* $\dot{X}$ *such that*

$$1 \Vdash \text{"}\dot{X} \in \dot{\mathcal{F}}_n \text{ and } \mu[[m \in \dot{X}]] < \varepsilon \text{ for all } m \in \omega\text{"}.$$

Proof. Let $\varepsilon > 0$ and $n < \omega$. Let $Fn(\omega_2,2)$ denote the set of all functions from a finite subset of ω_2 to 2. If $b \in Fn(\omega_2,2)$, let $[b] \in \mathcal{B}_{\omega_2}$ denote the equivalence class of $\{f \in 2^{\omega_2}: f \supset b\}$. Choose $k \in \omega$ such that $1/2^k < \varepsilon/2$. Choose disjoint subsets $\{A(m)\}_{m<\omega}$ of ω_2 with $|A(m)| = k$, and let $\{b_{m,i}: i < 2^k\}$ index $2^{A(m)}$. For each $i < 2^k$, let $\dot{X}_i$ be defined by $[[m \in \dot{X}_i]] = [b_{m,i}]$, and let $c_i = [[\dot{X}_i \in \mathcal{F}_n]]$. Since $\vee\{[b_{m,i}]: i < 2^k\} = 1$, it is forced by 1 that $\cup\{\dot{X}_i: i < 2^k\} = \omega$. Since $\{[b_{m,i}]: i < 2^k\}$ is an antichain, it is forced by 1 that $\dot{X}_i \cap \dot{X}_j = \emptyset$ if $i \neq j$. Thus, $\vee_{i<2^k} c_i = 1$ and $c_i \wedge c_j = 0$ if $i \neq j$.

Let $\varepsilon_i = \mu(c_i)$. We claim that there are $M_i < \omega$ such that

$$m > M_i \Rightarrow \mu(c_i \wedge [b_{m,i}]) < \varepsilon_i/2^{k-1}.$$

To see this choose a finite subset S of $Fn(\omega_2,2)$ such that if $c = \cup\{[s]: s \in S\}$, then $\mu(c\Delta c_i) < \frac{\varepsilon_i}{2^{k+1}}$. Choose M_i such that if $m > M_i$, then $dom(b_{m,i}) \cap dom(s) = \emptyset$ for all $s \in S$. Then $\mu([b_{m,i}] \wedge c) = 1/2^k \cdot \mu(c)$. Since $[b_{m,i}] \wedge c_i < ([b_{m,i}] \wedge c) \vee (c_i \wedge c)$, it is easy to check that $\mu([b_{m,i}] \wedge c_i) < \varepsilon_i/2^{k-1}$.

Now define $\dot{X}$ so that for each $i < 2^k$ and $m < M_i$, $c_i \wedge [[m \in \dot{X}]] = 0$, and for $m > M_i$, $c_i \wedge [[m \in \dot{X}]] =$

$a \wedge [[b_{m,i}]] = c_i \wedge [[m \in \dot{X}_i]]$. Then $1 \Vdash$ "$\dot{X} \in F_n$" since $c_i \Vdash$ "$\dot{X} = \dot{X}_i \setminus M_i$". And $m > \max\{M_i: i < 2^k\}$ implies

$$[[m \in \dot{X}]] = \bigvee \{c_i \wedge b_{m,i}: i < 2^k\} < \sum_{i=0}^{2^k-1} \frac{\varepsilon_i}{2^{k-1}} =$$

$$\frac{1}{2^{k-1}} \sum_{i=0}^{2^k-1} \varepsilon_i = \frac{1}{2^{k-1}} < \varepsilon.$$

Thus Fact 1 follows.

Fact 2. *For each* $\varepsilon > 0$, *there exists a* B_{ω_2}*-name* $\dot{X}$ *such that for each* $m < \omega$, $\mu[[m \in \dot{X}]] < \varepsilon$ *and* $1 \Vdash$ "$\dot{X} \in \dot{F}$".

Proof. Let $\varepsilon > 0$, and for each $n < \omega$, by Fact 2 choose $\dot{X}_n$ such that $1 \Vdash$ "$\dot{X}_n \in \dot{F}_n$" and for each $m \in \omega$, $\mu[[m \in \dot{X}_n]] < \varepsilon/2^{n+1}$. Let $\dot{X}$ be such that $1 \Vdash$ "$\dot{X} = \bigcup_{n\in\omega} \dot{X}_n$". Then $1 \Vdash$ "$\dot{X} \in \bigcap_{n\in\omega} \dot{F}_n = \dot{F}$", and $[[m \in \dot{X}]] = \bigvee_{n\in\omega}[[m \in \dot{X}_n]]$, so $\mu[[m \in \dot{X}]] < \Sigma_{n\in\omega} \varepsilon/2^{n+1} = \varepsilon$. That completes the proof of Fact 2.

Recall that if F is a filter on ω and $\{z_n\}_{n<\omega}$ is a sequence of numbers, the "F-$\lim z_n = \ell$" means that $\{n: |z_n - \ell| < \varepsilon\} \in F$ for each $\varepsilon > 0$.

Fact 3. *For each* $n \in \omega$, *there is a* B_{ω_2}*-name* $\dot{X}_n$ *such that*

$1 \Vdash$ "$\dot{X}_n \in \dot{F}$" *and* F_n-$\lim\{\mu[[m \in \dot{X}_n]]\}_{m\in\omega} = 0$.

Proof. Let us assume without loss of generality that $n = 0$. Choose $I_1 \in F_1 \setminus F_0$. Let $n_2 > 1$ be the least such that $I_1 \notin F_{n_2}$. Such an n_2 exists because $\omega \setminus I_1$ must be in F_m for infinitely many $m \in \omega$. Choose $I_2 \in F_{n_2} \setminus F_0$ with $I_2 \cap I_1 = \emptyset$. Let $n_3 > n_2$ be least such that $(I_1 \cup I_2) \notin F_{n_3}$, and pick $I_3 \in F_{n_3} \setminus F_0$ with $I_3 \cap (I_1 \cup I_2) = \emptyset$. Continuing in this manner, we can choose disjoint subsets $I_1, I_2, \ldots$ of ω such that for each $m > 0$, $i_k \in F_m$ for some k. Clearly we can also ensure that $\omega = \cup\{I_k: 1 \le k < \omega\}$.

By Fact 2, for $k \ge 1$, we can choose $\dot{X}_k$ such that $1 \Vdash$ "$\dot{X}_k \in \dot{F}$" and $\mu[[m \in \dot{X}_k]] < \frac{1}{k}$ for each $m < \omega$. Now define $\dot{X}_0$ so that $[[m \in \dot{X}_0]] = [[m \in \dot{X}_k]]$ if $m \in I_k$.

Clearly $F_0\text{-}\lim\{\mu[[m \in \dot{X}_0]]\}_{m\in\omega} = 0$, so we need only check that $1 \Vdash$ "$\dot{X}_0 \in \dot{F}$".

Let $0 < i < \omega$, and let $k \ge 1$ be such that $I_k \in F_i$. By definition of $\dot{X}_0$, $1 \Vdash$ "$\dot{X}_0 \cap I_k = X_k \cap I_k$". But $1 \Vdash$ "$\dot{X}_k \in \dot{F}_i$", so $1 \Vdash$ "$\dot{X}_0 \cap I_k \in \dot{F}_i$". Hence $1 \Vdash$ "$\dot{X}_0 \in \cap_{i>0} F_i$". If $b \Vdash$ "$\dot{X}_0 \notin \dot{F}_0$", then $b \Vdash$ "$\omega \setminus \dot{X}_0 \in \dot{F}_0 \setminus \cup_{i>0} \dot{F}_i$", which is a contradiction to Lemma 4.2. Thus $1 \Vdash$ "$\dot{X}_0 \in \dot{F}$", and the proof of Fact 3 is complete.

Now fix a family $\{\dot{X}_n: n \in \omega\}$ as in Fact 3. Since $1 \Vdash$ "$\dot{F}$ is semi-selective", there is a name $\dot{X}$ such that $1 \Vdash$ "$\dot{X} \in \dot{F}$ and $|\dot{X} \setminus \dot{X}_n| < n$ for all $n \in \omega$".

Fact 4. *For each* $n \in \omega$, $F_n\text{-}\lim\{\mu[[m \in \dot{X}]]\}_{m\in\omega} = 0$.

Proof. Suppose on the contrary that $A \in F_n$ and $\mu[[m \in \dot{X}]] \ge \varepsilon$ for all $m \in A$. We may assume $\mu[[m \in \dot{X}_n]] < \varepsilon/2$ for each $m \in A$, and hence $\mu[[m \in \dot{X} \setminus \dot{X}_n]] > \varepsilon/2$ for each

$m \in A$. But then $b = \wedge_{m \in H}[[m \in \dot{X} \setminus \dot{X}_n]] \neq 0$ for some subset H of ω of size n, whence $b \Vdash$ "$|\dot{X} \setminus \dot{X}_n| > n$", a contradiction. This completes the proof of Fact 4.

Now we aim for a contradiction that will complete the proof of the Proposition. We know by Fact 4, that for each $k \in \omega$, the set $F_k = \{m \in \omega: \mu[[m \in \dot{X}]] < 1/2^k\}$ is in $\mathcal{F}$. Since $\mathcal{F}$ is semi-selective, there is an $F \in \mathcal{F}$ such that $|F \setminus F_k| < k$ for all $k \in \omega$. This implies that the series $\Sigma_{m \in F}\, \mu[[m \in \dot{X}]]$ converges. Our contradiction will be that $1 \Vdash$ "$|\dot{X} \cap F| < \omega$". Let $b_k = [[\dot{X} \cap (F \setminus k) \neq \emptyset]]$. Note that $\mu(b_k)$ decreases to 0 as $k \to \infty$. Let G be an arbitrary $\mathcal{B}_{\omega_2}$-generic filter. Since

$$\{b \in \mathcal{B}_{\omega_2}: \exists n\ (b \wedge b_n = 0)\}$$

is dense, $b_k \in G$ for some k, whence $V[G] \models \dot{X}_G \cap (F \setminus k) = \emptyset$.

Corollary 4.4. *If* $\aleph_2$ *random reals are added to a model of* CH, *then there are no semi-selective* σ*-centered filters in the resulting model.*

Proof. Suppose CH holds in V, G is $\mathcal{B}_{\omega_2}$-generic over V, and in $V[G]$, $\mathcal{F} = \cap_{n<\omega} \mathcal{F}_n$ is semi-selective, where each $\mathcal{F}_n$ is an ultrafilter. Let $\dot{\mathcal{F}}$ and $\dot{\mathcal{F}}_n$, $n \in \omega$, be $\mathcal{B}_{\omega_2}$-names for $\mathcal{F}$ and the $\mathcal{F}_n$'s.

Each $p \in \mathcal{B}_{\omega_2}$ is the equivalence class modulo sets of measure 0 of a subset of 2^{ω_2} of the form

$$\{x \in 2^{\omega_2}: x \restriction A \in Y\}$$

for some countable $A \subset \omega_2$ and $Y \subset 2^A$. For each p, pick some such A with sup A minimal and call it the support of p, denoted supp(p). For a nice name π of a subset of ω, let

$$\mathrm{supp}(\pi) = \bigcup\{\mathrm{supp}(p): (\exists n)((\check{n},p) \in \pi)\}.$$

Then $|\mathrm{supp}(\pi)| \leq \omega$.

We may assume that each name $\dot{N}$ for a subset of $\mathcal{P}(\omega)$ is a set of pairs of the form (π,p) where π is a nice name for a subset of ω and $p \in \mathcal{B}_{\omega_2}$. For $\alpha < \omega_2$, let

$$G \upharpoonright \alpha = \{p \in G: \mathrm{supp}(p) \subset \alpha\}$$

and

$$\dot{N} \upharpoonright \alpha = \{(\pi,p) \in \dot{N}: \mathrm{supp}(\pi) \cup \mathrm{supp}(p) \subset \alpha\}.$$

Note that $G \upharpoonright \alpha$ is $\mathcal{B}_\alpha$-generic over V, and $\dot{N} \upharpoonright \alpha$ is a $\mathcal{B}_\alpha$-name for a subset of $\mathcal{P}(\omega)$. (By abuse of notation, we use p and π to denote members of $\mathcal{B}_{\omega_2}$ and $\mathcal{B}_{\omega_2}$-names, as well as the corresponding members of $\mathcal{B}_\alpha$ and $\mathcal{B}_\alpha$-names, as long as their supports are contained in α.)

Using the fact that $\mathcal{B}_{\omega_2}$ is ccc and that $V[G \upharpoonright \alpha] \models$ CH for each $\alpha < \omega_2$, by a standard closing up argument we can find $\lambda < \omega_2$ with cofinality ω_1 such that:

1. for each $n \in \omega$ and $A \in \mathcal{P}(\omega) \cap V[G \upharpoonright \lambda]$ (i.e. A has a $\mathcal{B}_\lambda$-name, say π), there is a $\mathcal{B}_\lambda$-name π' such that

$$1 \Vdash \text{"}\pi' \in F_n \text{ and either } \pi' \subset \pi \text{ or } \pi' \cap \pi = \emptyset;\text{"}$$

2. for each countable subcollection $\{F_i\}_{i<\omega}$ of $\mathcal{F} \cap V[G \upharpoonright \lambda]$, there exists $F \in V[G \upharpoonright \lambda]$ such that $|F_i \setminus F| \leq i$ for each i;

3. if $k \in \omega$ and $A \in \mathcal{F}_k \cap V[G \upharpoonright \lambda]$, then $A \in \mathcal{F}_n \cap V[G \upharpoonright \lambda]$ for infinitely many n;

4. the sequence of names $\{\dot{\mathcal{F}}_n\}_{n<\omega}$ is in $V[G \upharpoonright \lambda]$.

From (1) it follows that $\mathcal{F}_n \cap V[G \upharpoonright \lambda] = (\dot{\mathcal{F}}_n \upharpoonright \lambda)_G \in V[G \upharpoonright \lambda]$ and that $\mathcal{F}_n \cap V[G \upharpoonright \lambda]$ is an ultrafilter. From (2), (3) and (4), it follows that $\mathcal{F} \cap V[G \upharpoonright \lambda] = \cap_{n\in\omega} \mathcal{F}_n \cap V[G \upharpoonright \lambda]$ is a semi-selective σ-centered filter in $V[G \upharpoonright \lambda]$ such that each $A \in \mathcal{F} \cap V[G \upharpoonright \lambda]$ is in infinitely many $\mathcal{F}_n \cap V[G \upharpoonright \lambda]$. Thus the conditions of Proposition 4.3 are satisfied with $V = V[G \upharpoonright \lambda]$, and we have a contradiction.

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