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1. Introduction

A compact connected metric space is called a *continuum*. K. Kuperberg posed a problem whether the pseudo-arc is pseudo-contractible (University of Houston Problem Book, Problem 31). See below for the definition. In connection with this problem, D. Bellamy [1] constructed a map from the Cantor set onto the pseudo-arc which is null pseudo-homotopic. He also asked ([1], Question 1) whether each map from the Cantor set onto the pseudo-arc is null pseudo-homotopic. The purpose of this paper is to answer the above question in the affirmative. More precisely, we show that each map from the Cantor set to the pseudo-arc (not necessarily onto) is null pseudo-homotopic. Moreover, the parameter space can be taken to be the pseudo-arc.

2. Preliminaries

Definition 1. Let X and Y be continua and $f,g\colon X \to Y$ be maps. We say that f and g are *pseudo-homotopic* if there exist a continuum Z, points $a,b \in Z$ and a map $H\colon X \times Z \to Y$ such that $H(x,a) = f(x)$, $H(x,b) = g(x)$ for each $x \in X$. The continuum Z is called the *parameter space* of a *pseudo-homotopy* H.

A map which is pseudo-homotopic to a constant map is said to be *null pseudo-homotopic*. If id_X: X → X is null pseudo-homotopic, then we say that X is *pseudo-contractible*.

Definition 2. 1) Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of sets. The collection $\mathcal{U}$ is called a *chain* provided $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$.

2) A function f: $\{1, \ldots, m\} \to \{1, \ldots, n\}$ is called a *pattern* if $|f(i) - f(i+1)| \leq 1$ for each $i = 1, \ldots, m - 1$.

3) Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ and $\mathcal{V} = \{V_1, \ldots, V_n\}$ be chains, and f: $\{1, \ldots, m\} \to \{1, \ldots, n\}$ be a pattern. We say that $\mathcal{U}$ *follows* f *in* $\mathcal{V}$ if $U_i \subset V_{f(i)}$ for each $i = 1, \ldots, m$. In this case, a function $\bar{f}$: $\mathcal{U} \to \mathcal{V}$ is defined by $\bar{f}(U_i) = V_{f(i)}$. We will identify f and $\bar{f}$.

4) Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a chain cover of a continuum. The links U_1 and U_n are denoted by first $\mathcal{U}$ and last $\mathcal{U}$ respectively. For each k ($1 \leq k \leq n$), $i(U_k)$ is defined by $U_k - \mathrm{cl}\,(\bigcup_{j \neq k} U_j)$.

Definition 3. Let X be a continuum.

1) X is said to be *arc-like* if, for each $\varepsilon > 0$, there exists a chain cover $\mathcal{U}$ of X such that mesh $\mathcal{U} < \varepsilon$.

2) X is said to be *hereditarily indecomposable* if no subcontinuum of X can be represented as the union of two of its proper subcontinua.

3) Hereditarily indecomposable arc-like continuum is topologically unique ([3] and [6]), which is called

the *pseudo-arc*. Throughout this paper, the pseudo-arc is denoted by P.

4) Let p and q be points of X. X is said to be *irreducible between* p *and* q, if X contains no proper subcontinuum which contains both of p and q.

The following theorem is well known and will be used for the proof.

Theorem 4 ([2] and [5]). *Let* $\mathcal{C} = \{C_1, \ldots, C_n\}$ *be a chain cover of* P *and* $x \in i(C_1)$, $y \in i(C_n)$. *Suppose that* P *is irreducible between* x *and* y. *Then for each pattern* $f\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ *with* $f(1) = 1$ *and* $f(m) = n$, *there exists a chain cover* $\mathcal{D} = \{D_1, \ldots, D_m\}$ *which follows* f *in* $\mathcal{C}$, *and* $x \in i(D_1)$, $y \in i(D_m)$.

3. The Main Theorem

Our main theorem is

Theorem 5. *Each map from the Cantor set to the pseudo-arc is null pseudo-homotopic. Furthermore, we can take the parameter space of the pseudo-homotopy as the pseudo-arc.*

In the rest of this paper, C denote the Cantor set. The following theorem is the key step.

Proposition 6. *Suppose that a map* $f\colon C \to P$ *satisfies the following condition:*

there exists a point $a_0 \in P$ *such that* P *is irreducible between* a_0 *and* y, *for each* $y \in f(C)$.

Then f *is pseudo-homotopic to a constant map with the parameter space* P.

Proof. Suppose that P is irreducible between x_0 and y_0. We can take a sequence $(\mathcal{D}_n)_{n \geq 0}$ of open covers of C as follows:

a) Each $\mathcal{D}_n$ is a mutually disjoint clopen cover of C.

b) $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$ for each n.

c) mesh $\mathcal{D}_n \to 0$ as $n \to \infty$.

Step 1. For each $x \in C$, there exists a chain cover $\mathcal{V}_x$ of P such that

1-1) $f(x) \in i(\text{first } \mathcal{V}_x)$ and $a_0 \in i(\text{last } \mathcal{V}_x)$.

1-2) mesh $\mathcal{V}_x < 1/4$ ([2], [4]).

By c) and the continuity of f, we can take an integer $n(x) > 0$ such that

1-3) $f(\mathcal{D}_{n(x)}(x)) \subset i(\text{first } \mathcal{V}_x)$,

where, $\mathcal{D}_{n(x)}(x)$ denotes the unique member of $\mathcal{D}_{n(x)}$ which contains x.

The collection $\{\mathcal{D}_{n(x)}(x) \mid x \in C\}$ forms an open cover of C, so we can take finitely many points $x_1, \dots, x_r \in C$ such that $C = \bigcup_{i=1}^{r} \mathcal{D}_{n(x_i)}(x_i)$. Define n_1 as

1-4) $n_1 = \max\{n(x_i) \mid 1 \leq i \leq r\}$.

Then noticing b), we have

1-5) for each $D \in \mathcal{D}_{n_1}$, there exists a chain cover $\mathcal{V}_D^1$ such that $f(D) \subset i(\text{first } \mathcal{V}_D^1)$ and $a_0 \in i(\text{last } \mathcal{V}_D^1)$

For each member D of $\mathcal{D}_{n_1}$, we define a chain cover $\mathcal{U}^1_D$ of P as follows.

1-6) (The number of links of $\mathcal{U}^1_D$) = (The number of links of $\mathcal{V}^1_D$)

1-7) $x_0 \in i(\text{first } \mathcal{U}^1_D)$ and $y_0 \in i(\text{last } \mathcal{U}^1_D)$.

Now we have an open cover $D \times \mathcal{U}^1_D$ of $D \times P$, for each $D \in \mathcal{D}_{n_1}$.

Step 2. Fix a member D_1 of $\mathcal{D}_{n_1}$. For each $x \in D_1$, we can take a chain cover $\mathcal{V}^2_x$ of P such that

2-1) $f(x) \in i(\text{first } \mathcal{V}^2_x)$ and $a_0 \in i(\text{last } \mathcal{V}^2_x)$.

2-2) mesh $\mathcal{V}^2_x < 1/8$ and $\mathcal{V}^2_x$ is a closure refinement of $\mathcal{V}^1_{D_1}$ (that is, for each $V \in \mathcal{V}^2_x$, there exists $U \in \mathcal{V}^1_{D_1}$ such that $cl(V) \subset U$).

Again by c), there exists an integer $m(x) > 0$ such that

2-3) $f(\mathcal{D}_{m(x)}(x)) \subset i(\text{first } \mathcal{V}^2_x)$.

The collection $\{\mathcal{D}_{m(x)}(x) \mid x \in D_1\}$ forms an open cover of D_1, so there exist finitely many points $y_1,\ldots,y_s \in D_1$ such that $D_1 = \bigcup_{j=1}^{s} \mathcal{D}_{m(y_j)}(y_j)$.

Repeating these processes for all members of $\mathcal{D}_{n_1}$, we obtain finitely many points $y_1,\ldots,y_t$ and chain covers $\mathcal{V}^2_{y_1},\ldots,\mathcal{V}^2_{y_t}$. Define n_2 as

2-4) $n_2 = \max\{m(y_j) \mid 1 \le j \le t\}$.

Then we have

2-5) for each $D_2 \in \mathcal{D}_{n_2}$, there exists a chain cover $\mathcal{V}^2_{D_2}$ such that $f(D_2) \subset i(\text{first } \mathcal{V}^2_{D_2})$ and $a_0 \in i(\text{last } \mathcal{V}^2_{D_2})$.

Next, we define a pattern as follows. For each $D_2 \in \mathcal{D}_{n_2}$, take the unique $D_1 \in \mathcal{D}_{n_1}$ which contains D_2. Then by the choice of $\mathcal{V}^2_{D_2}$ (2-2),5)), $\mathcal{V}^2_{D_2}$ is a closure refinement of $\mathcal{V}^1_{D_1}$. So we can find a pattern

$f_{D_2D_1}: \mathcal{V}_{D_2} \to \mathcal{V}_{D_1}$ such that

$f_{D_2D_1}(\text{first } \mathcal{V}^2_{D_2}) = \text{first } \mathcal{V}^1_{D_1}$ and

$f_{D_2D_1}(\text{last } \mathcal{V}^2_{D_2}) = \text{last } \mathcal{V}^1_{D_1}$. (Recall the remark in Definition 2).

Applying Theorem 4, there exists a chain cover $\mathcal{U}^2_{D_2}$ of P such that

2-6) $\mathcal{U}^2_{D_2}$ follows $f_{D_2D_1}$ in $\mathcal{U}^1_{D_1}$.

2-7) $x_0 \in i(\text{first } \mathcal{U}^2_{D_2})$ and $y_0 \in i(\text{last } \mathcal{U}^2_{D_2})$.

Now, we have a covering $D_2 \times \mathcal{U}^2_{D_2}$ of $D_2 \times P$, for each $D_2 \in \mathcal{D}_{n_2}$.

Step 3. Continuing these processes, we obtain a subsequence $(n_k)_{k \geq 1}$ satisfying the following conditions.

3-1) $x_0 \in i(\text{first } \mathcal{U}^k_{D_k})$ and $y_0 \in i(\text{last } \mathcal{U}^k_{D_k})$.

3-2) $f(D_k) \subset i(\text{first } \mathcal{V}^k_{D_k})$ and $a_0 \in i(\text{last } \mathcal{V}^k_{D_k})$.

3-3) For each $D_k \supset D_{k+1}$ $(D_\alpha \in \mathcal{D}_{n_\alpha}, \alpha = k,k+1)$, there exists a pattern $f_{D_{k+1}D_k}$ such that $\mathcal{U}^{k+1}_{D_{k+1}}$ ($\mathcal{V}^{k+1}_{D_{k+1}}$ resp.) follows $f_{D_{k+1}D_k}$ in $\mathcal{U}^k_{D_k}$ ($\mathcal{V}^k_{D_k}$ resp.).

3-4) $f_{D_{k+1}D_k}$ (first $\mathcal{U}^{k+1}_{D_{k+1}}$) = first $\mathcal{U}^k_{D_k}$, and $f_{D_{k+1}D_k}$ (last $\mathcal{U}^{k+1}_{D_{k+1}}$) = last $\mathcal{U}^k_{D_k}$.

The same conditions hold for $\mathcal{V}^{k+1}_{D_{k+1}}$ and $\mathcal{V}^k_{D_k}$.

3-5) mesh $\mathcal{V}^k_{D_k} < 1/2^{k+1}$ for each $k \geq 1$.

There are then more and more chains, both $\mathcal{V}$'s and $\mathcal{U}$'s at each stage than there were before. Each chain at the k-level has several different refining chains at (k+1)-level.

Finally, we define $H: C \times P \to P$ as follows. For each $x \in C$, there exists the unique sequence $D_1(x) \supset D_2(x) \supset \ldots$ with $D_k(x) \in \mathcal{D}_{n_k}$ such that $\{x\} = \bigcap_{k \geq 1} D_k(x)$.

Then we have two sequences $\{\mathcal{U}^k_{D_k(x)}\}_{k \geq 1}$ and $\{\mathcal{V}^k_{D_k(x)}\}_{k \geq 1}$ of chain covers of P. By the standard method of constructing a map between the pseudo-arcs, we have a map $H|x \times P: x \times P \to P$ such that

3-6) $H(x \times \mathcal{U}^k_{D_k}(i)) \subset st(\mathcal{V}^k_{D_k(x)}(i), \mathcal{V}^k_{D_k(x)})$ for each $\mathcal{U}^k_{D_k(x)}(i) \in \mathcal{U}^k_{D_k(x)}$.

Notice the following.

3-7) If $x,y \in D_k \in \mathcal{D}_{n_k}$, then $\mathcal{U}^i_{D_i(x)} = \mathcal{U}^i_{D_i(y)}$ and $\mathcal{V}^i_{D_i(x)} = \mathcal{V}^i_{D_i(y)}$ for each $i = 1,\ldots,k$.

Using this fact, it is easy to see that the map H defined as above is continuous and $H(x,x_0) = f(x)$, $H(x,y_0) = a_0$ for each $x \in C$. This completes the proof.

Proof of Theorem 5.

Let $f: C \to P$ be a map. Take a nondegenerate proper subcontinuum Q of P. By [3], Q is a retract of P. Fix a retraction $r: P \to Q$ and a homeomorphism $h: P \to Q$. Fix a point a_0 of P which lies in a different composant from Q. Applying Proposition 6 to $h \circ f: C \to Q$ and a_0, we have a map $H: C \times P \to P$ and points x_0 and $y_0 \in P$ such that $H|C \times x_0 = h \circ f$ and $H|C \times y_0 = a_0$. Define $F: C \times P \to P$ as $F = h^{-1} \circ r \circ H$. Then $F|C \times x_0 = f$ and $F|C \times y_0 = h^{-1}(a_0)$. This completes the proof of Theorem 5.

Corollary 7. *Any Cantor set in the pseudo-arc* P *is pseudo-contractible in* P.

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