TOPOLOGY PROCEEDINGS

Volume 14, 1989

Pages 249–263

http://topology.auburn.edu/tp/

CONTRACTIBILITY OF CONTINUA ADMITTING ARC-STRUCTURES

by

Akira Koyama

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

CONTRACTIBILITY OF CONTINUA ADMITTING ARC-STRUCTURES

Akira Koyama *

1. Introduction

Throughout this paper a *continuum* means a compact connected metric space. Let X be a continuum. By C(X), we denote the *hyperspace* of subcontinua of X with the Hausdorff metric. For a given sequence $\{Z_n\}$ of subsets of X, we denote the *limit inferior*, the *limit superior*, and the *limit* of $\{Z_n\}$, by $\mathrm{Li}_n Z_n$, $\mathrm{Ls}_n Z_n$ and $\mathrm{Lim}_n Z_n$, respectively (see [8] for the definitions).

An *arc-structure* A on a continuum X is a function $A: X \times X \to C(X)$ such that for $x \neq y$ in X, the set A(x,y) is an *arc from* x *to* y in X and such that the following conditions are satisfied for all x, y and z in X:

(a) $A(x,x) = \{x\}$,

(b) $A(x,y) = A(y,x)$, and

(c) $A(x,z) \subset A(x,y) \cup A(y,z)$ with the equality prevailing whenever y belongs to A(x,z).

Throughout this paper a pair (X,A) means a *continuum* X *with a given arc-structure* A on X.

A pair (X,A) is *arc-smooth at a point* p *in* X if the induced function $A_p: X \to C(X)$ defined by $A_p(x) = A(p,x)$ is continuous. A pair (X,A) is *weakly arc-smooth at a point* p *in* X if for a given convergent sequence $\{x_n\}$ in X,

* Dedicated to Professor Y. Kodama on his 60th birthday.

$\mathrm{Li}_n\, A(p,x_n) = A(p,x)$ for some $x \in X$ (not necessarily $x = \lim_n x_n$). A pair (X,A) is *(weakly) arc-smooth* if there is a point in X at which (X,A) is (weakly) arc-smooth (see [4], [5], [6] and [7]).

In [4] and [5], Fugate, Gordh and Lum defined and investigated continua admitting arc-structures and arc-smooth continua as higher dimensional analogs of dendroids and smooth dendroids. In [6] and [7], weakly smooth dendroids were generalized to weakly arc-smooth continua. The following characterizations of weakly arc-smooth continua, analogous to well-known characterizations of weakly smooth dendroids [10], [11], were obtained:

Theorem 1. *The following statements are equivalent.*

(1) (X,A) *is weakly arc-smooth at a point* p *in* X.

(2) $A_p(X) = \{A_p(x) \mid x \in X\} \subset C(X)$ *is compact.*

(3) (X,A) *is hereditarily* T-*convex and* $\Gamma_p \cup \Gamma_p^{-1}$ *is closed in* $X \times X$, *where* $\Gamma_p = \{(x,y) \mid y \in A(p,x)\}$ *and* $\Gamma_p^{-1} = \{(x,y) \mid (y,x) \in \Gamma_p\}$.

(4) (X,A) *is hereditarily* T-*convex and* $T_A(x) \subset L_p(X) \cup M_p(x)$ *for every* $x \in X$, *where* $L_p(x) = \{y \mid y \in A(p,x)\}$ *and* $M_p(x) = \{y \mid x \in A(p,y)\}$.

Here a subset Z of (X,A) is *convex (with respect to* A*)* if $A(x,y) \subset Z$ for every pair (x,y) of points in Z. The set function T_A for (X,A) is defined by the formula:

$$T_A(x) = \left\{ y \in X \;\middle|\; \begin{array}{l}\text{each convex subcontinua of } (X,A) \\ \text{with } y \text{ in its interior contains } x\end{array} \right\}$$

(X,A) is T-*convex* if $T_A(x)$ is convex for every $x \in X$. Moreover, (X,A) is said to be *hereditarily* T-*convex* if for every convex subcontinuum Z of (X,A), the pair $(Z, A|Z \times Z)$ is T-convex (see [6] and [7]).

The purpose of this paper is to investigate contractibility of continua admitting arc-structures. First, we will characterize arc-smoothness on a class of weakly arc-smooth continua by convex-hereditary contractibility. Next, we will introduce convex-contractibility of continua admitting arc-structures and characterize this property in the term of the set function T_A. Those results are some generalizations of ones in [1] and [3].

Definitions of undefined terms may be found in [5] and [6].

2. Results

A continuum is *hereditarily contractible* if each of its subcontinua is contractible. Charatonik and Grabowski [3] have shown the following characterization of smooth fans:

Theorem 2. ([3], Corollary 17). *A fan is hereditarily contractible if and only if it is smooth.*

In order to extend Theorem 2 to continua admitting arc-structures, we introduce a new class of such continua. A pair (X,A) is an *arc-fan (shortly, a-fan) with the top* p provided that for $x \neq y$ in X, $A(p,x) \cap A(p,y) \neq \{p\}$ implies that $A(p,x) \subset A(p,y)$ or $A(p,y) \subset A(p,x)$. A pair (X,A) is an *a-fan* if there is a point in X with which (X,A) is an a-fan.

If a continuum X is a cone over a compactum or a star-like continuum in ℓ^2, there is an arc-structure A on X such that the pair (X,A) is an a-fan. If X is a dendroid, an a-fan (X,A) with top p is the fan with the top p. Hence, since a hereditarily contractible continuum is a dendroid, by Theorem 2, we have

Corollary 3. *An a-fan* (X,A) *is hereditarily contractible if and only if* X *is a smooth fan.*

Therefore we introduce a kind of hereditary contractibility on a pair (X,A), which characterizes arc-smoothness of weakly arc-smooth a-fans. (X,A) is *convex-hereditarily contractible* (with respect to A) if every convex subcontinuum of X is contractible. By [5], Lemma I-2-B and Theorem I-6-A, an arc-smooth continuum is convex-hereditarily contractible. For an a-fan, we have the following theorem, which is an extension of Theorem 2.

Theorem 4. *If an a-fan* (X,A) *is weakly arc-smooth and convex-hereditarily contractible, then it is arc-smooth.*

For the proof, we need the following lemmas. We define the *end-set* $E(X,A)$ of a pair to be

$\{e \in X \mid \text{if } e \in A(x,y), \text{ then } e = x \text{ or } e = y\}$ (see [5], I-9).

Because of the equivalence of (1) and (3) in Theorem 1, the first Lemma can be proved with the argument similar to of [5], Lemma I-9-A.

Lemma 5. *If a pair* (X,A) *is weakly arc-smooth at* p, *then each arc* A(p,x) *is contained in an arc* A(p,e) *with* e *in the end-set* E(X,A).

Proof. Suppose that some point x of X fails to lie in an arc of the required form. Then there exists a sequence $\{y_n\}$ in X satisfying $x \le_p y_1 \le_p y_2 \le_p \cdots \le_p y_n \le_p y_{n+1} \le_p \cdots$ and such that no point e satisfies $y_n \le_p e$ for all n. Passing to a subsequence, if necessary, assume that the sequence $\{y_n\}$ converges. Then for each n,

$$y_n \in \mathrm{cl}(\cup A(p,y_n)) = \mathrm{Li}_n A(p,y_n) = A(p,z)$$
$$\text{for some } z \in X.$$

The former set-equality holds since the sequence $\{A(p,y_n)\}$ is nested and the latter since X is weakly arc-smooth at p. But we have a contradiction to the assumption.

Lemma 6. *If an a-fan* (X,A) *with the top* p *is arc-smooth, then it is arc-smooth at* p.

Proof. Suppose that (X,A) is arc-smooth at $q \ne p$. Let $\{x_n\}$ be any convergent sequence in X with $\lim_n x_n = x$. Then

$$(+) \quad \mathrm{Lim}_n A(q,x_n) = A(q,x).$$

We consider two cases.

Case 1. $p \in A(q,x)$: If there exists an endpoint $e \in E(X,A)$ such that $A(p,x_n) \subset A(p,e)$ for almost all n, it is clear that $\mathrm{Lim}_n\ A(p,x_n) = A(p,x)$. Hence, by Lemma 5 and the fan structure of A, we may assume that for each $n \geq 1$,

$$(*)\quad A(q,x_n) = A(q,p) \cup A(p,x_n) \text{ and}$$
$$A(q,p) \cap A(p,x_n) = \{p\}.$$

If there exists $y \in \mathrm{Ls}_n\ A(p,x_n)\setminus A(p,x)$, there exists a sequence $n(1) < n(2) < \ldots$ and points $y_i \in A(p,x_{n(i)})$ such that $y = \lim_i y_i$. Then by (+),

$$y \in \mathrm{Ls}_n\ A(p,x_n) \subset \mathrm{Ls}_n\ A(q,x_n) = A(q,x).$$

Hence $y \in A(q,x)\setminus A(p,x) = A(q,p)\setminus\{p\}$. By (*), $p \in \mathrm{Ls}_i\ A(q,y_i)$. Therefore, by [6], Lemma 3.4(2), we have that

$$\mathrm{Lim}_i\ A(q,y_i) \subset A(q,p) \neq A(q,y).$$

even if the limit exists. It contradicts to the arc-smoothness of (X,A) at q. Thus, $\mathrm{Ls}_n\ A(p,x_n) \subset A(p,x)$.

On the other hand, by [6], Lemma 3.4(2), $\mathrm{Li}_n\ A(p,x_n)$ is convex and contains both the points p and x. Hence $A(p,x) \subset \mathrm{Li}_n\ A(p,x_n)$. It follows that $\mathrm{Lim}_n\ A(p,x_n) = A(p,x)$.

Case 2. $p \notin A(q,x)$: If $p \in A(q,x_n)$ for infinitely many n, $A(q,x_n) = A(q,p) \cup A(p,x_n)$ for infinitely many n, and by (+),

$$p \in A(q,p) \subset \mathrm{Ls}_n\ A(q,x_n) = \mathrm{Lim}_n\ A(q,x_n) = A(q,x).$$

This is a contradiction. Hence there exists $e \in E(X,A)$ such that

$$x_n \in A(p,e) \text{ for almost all } n.$$

The existence of such an endpoint e is guaranteed by Lemma 5. Then $A(p,x) = \mathrm{Lim}_n A(p,x_n)$.

In both cases, we have that $\mathrm{Lim}_n A(p,x_n) = A(p,x)$. Therefore (X,A) is arc-smooth at p.

Proof of Theorem 4. Suppose that (X,A) is an a-fan with the top p and is weakly arc-smooth at a point q. By Lemma 6, it suffices to consider whether (X,A) is arc-smooth at p. We assume that (X,A) is not arc-smooth at p. Thus, there exists a convergent sequence $\{x_n\}$ in X such that the sequence $\{A(p,x_n)\}$ is convergent in $C(X)$, but putting $x = \mathrm{Lim}_n x_n$, we have $\mathrm{Lim}_n A(p,x_n) \neq A(p,x)$. We note that by [6], Lemma 3.4(2), $\mathrm{Lim}_n A(p,x_n)$ is convex, and therefore $A(p,x) \subsetneq \mathrm{Lim}_n A(p,x_n)$.

By Lemma 5, there is a point $e \in E(X,A)$ such that $A(p,x) \subset A(p,e)$. If $x_n \in A(p,e)$ for infinitely many $n \geq 1$, then $\mathrm{Lim}_n A(p,x_n) = A(p,x) \subset A(p,e)$. This is a contradiction to the choice of the sequence $\{x_n\}$. Hence, passing to subsequences if necessary, we may assume that

(1) $x_n \notin A(p,e) \cup A(p,q)$ for every $n \geq 1$,

(2) $A(p,x_n) \cap A(p,x_m) = \{p\}$ if $n \neq m$.

Since (X,A) is an a-fan with the top p, by (1),

(3) $A(q,x_n) = A(p,q) \cup A(p,x_n)$ for every $n \geq 1$.

Since (X,A) is weakly arc-smooth at q,

(4) $\mathrm{Li}_n A(q,x_n) = A(q,y)$ for some $y \in X$.

Then, by (3) and (4),

$$\begin{aligned} A(q,y) = \mathrm{Li}_n A(q,x_n) &= A(q,p) \cup \mathrm{Li}_n A(p,x_n) \\ &= A(q,p) \cup \mathrm{Lim}_n A(p,x_n) \\ &= \mathrm{Lim}_n A(q,x_n). \end{aligned}$$

Hence we have that

(5) $A(p,x) \subsetneqq \mathrm{Lim}_n\ A(p,x_n) \subset A(q,y) = \mathrm{Lim}_n\ A(q,x_n)$.

Now we define a continuum

$$K = A(q,y) \cup [\bigcup_{n\geq 1} A(p,x_n)].$$

Since $\mathrm{Lim}_n\ A(p,x_n) \subset A(q,y)$, K is a convex subcontinuum, and $(K,A|K \times K)$ is not arc-smooth at p. Note that, by the condition (c) of the definition of weak arc-smoothness,

$$K = A(q,y) \cup A(p,y)\ [\bigcup_{n\geq 1} A(p,x_n)]$$

and p is an only one ramification point of K. Now we show that any subcontinuum L of K has one of the following properties:

(6) If $p \notin L$, then $L \subset A(q,y)$ or $L \subset A(p,x_n)\setminus\{p\}$ for some n.

(7) If $p \in L$, then $L = A(q',y') \cup [\bigcup_{n\geq 1} A(p,x_n')]$ for some $p \in A(q',y') \subset A(q,y)$ and $x_n' \in A(p,x_n)$ for each $n \geq 1$.

If $p \notin L$, by (1) and (2),

$L \cap A(p,x_n) \cap A(q,y) = \emptyset = L \cap A(p,x_n) \cap A(p,x_m)$ if $n \neq m$.

Hence, by the *Sierpinski's Theorem* (see [9], Theorem V.3.6),

$L \subset A(q,y)$ or $L \subset A(p,x_n)\setminus\{p\}$ for some n.

Suppose that $p \in L$. For each $n \geq 1$, let x_n' be the point of L such that

$L \cap A(p,x_n) \subset A(p,x_n')$.

If there exists an n_0 and a point w such that $p \neq x'_{n_0}$ and $w \in A(p,x'_{n_0})\setminus L$, then, by (1) and (2), $A(q,y) \cup [\bigcup_{n\neq n_0} A(p,x_n)] \cup A(p,w)$ and $A(w,x'_{n_0})$ give a separation of L in K. This is a contradiction of connectedness of L. Thus, $A(p,x'_n) = L \cap A(p,x_n)$ for all n. Similarly, take the points q' and y' of L such that

$$L \cap A(q,y) \subset A(q',y').$$

Then, by (5),

$$p \in \mathrm{Li}_n\, A(p,x'_n) \subset \mathrm{Ls}_n\, A(p,x'_n) \subset L \cap \mathrm{Ls}_n\, A(p,x'_n) = L \cap \mathrm{Lim}_n\, A(p,x_n) \subset L \cap A(q,y) \subset A(q',y').$$

Hence $\mathrm{Ls}_n\, A(p,x'_n)$ is a subcontinuum of $A(q',y')$ containing p. Since $[\bigcup_{n\geq 1} A(p,x'_n)] \cup \mathrm{Ls}_n\, A(p,x'_n) \subset L \subset [\bigcup_{n\geq 1} A(p,x'_n)] \cup A(q',y')$ and L is a continuum, $L = [\bigcup_{n\geq 1} A(p,x'_n)] \cup A(q',y')$.

Because of the properties (6) and (7), K is hereditarily arcwise connected and hereditarily unicoherent. Namely, K is a fan with the top p. Moreover, by our assumption, K is not smooth. Hence, by Theorem 2, K contains a non-contractible subfan, which is convex with respect to A. But this contradicts the convex-hereditary contractibility of (X,A). It follows that (X,A) is arc-smooth at p.

Corollary 7. For an a-fan (X,A) *with the top p, the following statements are equivalent:*

(1) (X,A) *is weakly arc-smooth and convex-hereditarily contractible.*

(2) (X,A) *is arc-smooth.*

Since there is a non-smooth dendroid with two ramification points, which is weakly smooth and hereditarily contractible (see [3], p. 237), the assumption "a-fan" in Theorem 4 is essential. In the case that X is a dendroid, the assumption that X is weakly arc-smooth is not needed. But, in general, there is an a-fan which is convex-hereditarily contractible but is not weakly arc-smooth. Namely, we have

Example 8. Each point in the Euclidean plane is represented by the polar coordinate system (r,θ). Let

$$D = \{(r,\theta) \mid 0 \leq r \leq 1\} \text{ and}$$

$$E_n = \{(r,\theta) \mid r = 1 + \theta/n,\ 0 \leq \theta \leq \pi/2\},$$

$$n = 1, 2, 3, \ldots$$

We define a continuum

$$X = D \cup (\bigcup_{n\geq 1} E_n).$$

Let $p = (1,0) \in D \cap (\bigcup_{n\geq 1} E_n)$, and for each $x \in X$, we define an arc $A(p,x)$ as follows:

$$A(p,x) = \begin{cases} \text{the unique arc from } p \text{ to } x \text{ in } E_n, \text{ if } x \in E_n, \\ \text{the straight line segment from } p \text{ to } x \text{ in } D, \text{ if } x \in D. \end{cases}$$

Then the correspondence A induces an arc-structure on X, which is also denoted by A.

It is clear that (X,A) is an a-fan with the top p which is not weakly arc-smooth. On the other hand, since each convex subset of (X,A) is either contained in a straight line segment in D or some arc E_n, or a star-like subset (with respect to A) containing p, (X,A) is convex-hereditarily contractible. Hence (X,A) is an a-fan with the desired property.

We remark that this example also shows that weak arc-smoothness in [6], Lemma 3.4, is essential.

Fugate, Gordh and Lum [5], I.6, introduced the notion of a $\leq_p$-contraction for a pair (X,A). Namely, a $\leq_p$-*contraction* is a homotopy H: $X \times I \to X$ satisfying the following conditions for each $x \in X$:

(a) $H(x,0) = x$,

(b) $H(x,1) = p$, and

(c) $H(x,t) \in A(p,x)$ for each $t \in I$.

Moreover, they showed that (X,A) *is arc-smooth at p if and only if* (X,A) *admits a* $\leq_p$-*contraction*. Here we introduce a weaker notion of $\leq_p$-contractions, and characterize it by the set function T_A. A *convex-contraction* for (X,A) is a homotopy F: $X \times I \to X$ satisfying the following conditions:

(d) $H(x,0) = x$ for each $x \in X$,

(e) $H(\{x\} \times [s,t])$ is convex with respect to A for each $x \in X$ and $s, t \in I$, and

(f) $H(X \times \{1\})$ is a singleton.

(X,A) is *convex-contractible* if it admits a convex-contraction for (X,A).

Obviously, each $\leq_p$-contraction is a convex-contraction. A criterion of convex-contractibility by the set function T_A is obtained as follows. The proof is a slight modification of one given in [1], Theorem 2. Hence we omit the proof here.

Theorem 9. *If a pair* (X,A) *contains closed subsets* K *and* L *such that* $K \cap T_A(L) = \emptyset = L \cap T_A(K)$ *and* $T_A(K) \cap T_A(L) \neq \emptyset$, *then it is not convex-contractible.*

Finally, we show that there is a pair (X,A) and points a,b in X such that $b \notin T_A(a)$, $a \notin T_A(b)$, $T_A(a) \cap T_A(b) \neq \emptyset$ and X is contractible. It follows that there is a gap between contractibility and convex-contractibility and that Theorem 9 holds only for convex-contractibility.

Example 10. We use the same notation as in Example 8. Let

$$F_n = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} r = (2n+3)\theta/2n(n+1) + 1 - \pi/4n(n+1), \\ \pi/4 \leq \theta \leq \pi/2 \end{array} \right\}$$

for each $n = 1,2,\ldots,$ and define a continuum

$$Y = X \cup [\bigcup_{n \geq 1} F_n].$$

For each $y \in Y$, we define an arc $B(p,y)$ as follows:

$$B(p,y) = \begin{cases} \text{the unique arc from } p \text{ to } y \text{ in } E_n \cup F_n, \\ \text{if } y \in E_n \cup F_n, \\ \text{the straight line segment from } p \text{ to } y, \\ \text{if } y \in D. \end{cases}$$

Then the correspondence B induces an arc-structure on Y, which is also denoted by B.

Clearly the pair (Y,B) is an a-fan with the top p and Y is contractible. In order to show that (Y,B) is not convex-contractible, by Theorem 9, it suffices to find points $a,b \in Y$ such that $a \notin T_B(a)$, $b \notin T_B(a)$ and $T_B(a) \cap T_B(b) \neq \emptyset$. Let a and b be the midpoints of $B(p,(1,\pi/2))$ and $B(p,(1,\pi/4))$, respectively. Then it is clear that $a \notin T_B(b)$ and $b \notin T_B(a)$.

Next, we show that $T_B(a) \cap T_B(b) \neq \emptyset$. Take any point $x \in \mathrm{Lim}_n F_n$ and any convex subcontinuum K of (Y,B) with x in its interior. Then $K \cap F_n \neq \emptyset$ for almost all n. Hence, since K is convex with respect to B, $E_n \subset K$ for almost all n. Therefore

$$p,\ (1,\pi/2),\ (1,\pi/4) \in \mathrm{Lim}_n E_n \subset K.$$

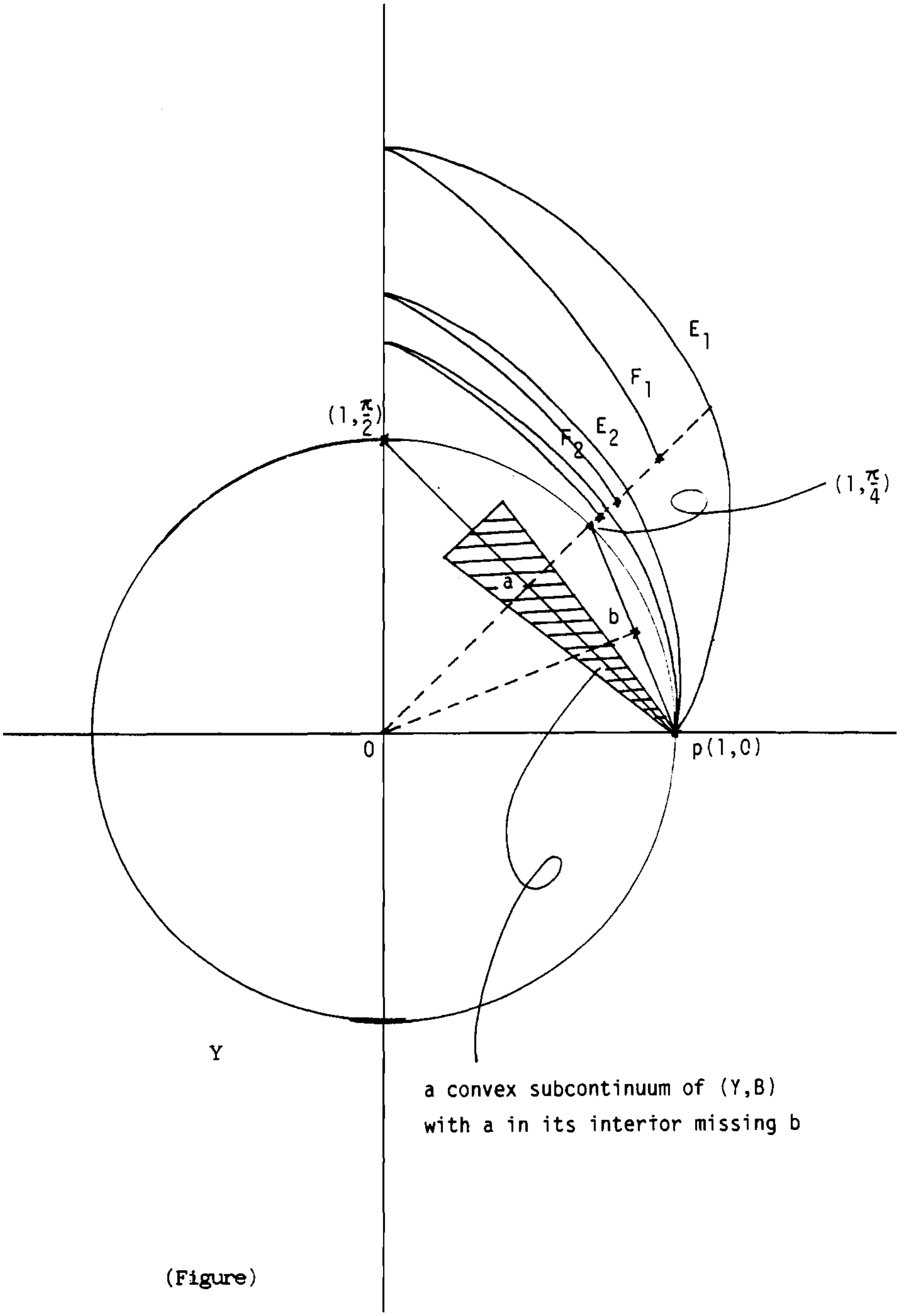
Since K is convex with respect to B,

$$a,\ b \in B(p,(1,\pi/2)) \cup B(p,(1,\pi/4)) \subset K.$$

Hence $x \in T_B(a) \cap T_B(b)$. Thus,

$$\emptyset \neq \mathrm{Lim}_n F_n \subset T_B(a) \cap T_B(b).$$

Therefore the pair (Y,B) and the points $a,b \in Y$ satisfy the required conditions. We note that, in fact, $T_B(a) \cap T_B(b) = \mathrm{Lim}_n F_n$.


E1
F1
E2
F2
(1,π/2)
(1,π/4)
a
b
0
p(1,0)
Y
a convex subcontinuum of (Y,B)
with a in its interior missing b

(Figure)

References

[1] J. J. Charatonik, *The set function* T *and homotopies*, Colloq. Math. 39(1978), 271-274.

[2] ____________________ and C. Eberhart, *On smooth dendroids*, Fund. Math. 67(1970), 297-322.

[3] ____________________ and Z. Grabowski, *Homotopically fixed arcs and the contractibility of dendroids*, Fund. Math. 100(1978), 229-237.

[4] J. B. Fugate, G. R. Gordh, Jr. and Lewis Lum, *On arc-smooth continua*, Top. Proc. 2(1977), 645-656.

[5] ____, *Arc-smooth continua*, Trans. Amer. Math. Soc. 265(1981), 545-561.

[6] A. Koyama, *Weakly arc-smooth continua* I, Glasnik Mat. 22(42)(1987), 171-185.

[7] ____________________, *Weakly arc-smooth continua* II, Glasnik Mat. 24(44)(1989), 133-138.

[8] K. Kuratowski, *Topology* I, Academic Press and PWN, 1966.

[9] ____________________, *Topology* II, Academic Press and PWN, 1968.

[10] L. Lum, *Weakly smooth dendroid*, Fund. Math. 83(1974), 111-120.

[11] ____________________, *Weakly smooth continua*, Trans. Amer. Math. Soc. 214(1975), 153-167.

Osaka Kyoiku University

Ikeda-city, Osaka, 563 JAPAN