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# TOPOLOGY PROCEEDINGS



Volume 15, 1990

Pages 1–9

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## CONFLUENT MAPPINGS ON $[0, 1]$ AND INVERSE LIMITS

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### Topology Proceedings

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**ISSN:** 0146-4124

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# CONFLUENT MAPPINGS ON $[0,1]$ AND INVERSE LIMITS

JAMES F. DAVIS

ABSTRACT. Let  $I = [0,1]$ . In this paper, confluent mappings  $f : I \rightarrow I$  are characterized. The degree,  $\deg(f)$  of such a mapping is defined as (number of components of  $f^{-1}(0)$ ) + (number of components of  $f^{-1}(1)$ ) - 1. This definition agrees with the usual definition of  $\deg(f)$  in the case where  $f$  is open. It is shown that, if  $f_i : I \rightarrow I$  is confluent,  $g_i : I \rightarrow I$  is open and  $\deg(f_i) = \deg(g_i)$  for  $i = 1, 2, \dots$ , then  $\varprojlim \{I, f_i\}$  is homeomorphic with  $\varprojlim \{I, g_i\}$ .

The simplest indecomposable continua are those which can be constructed as inverse limits of  $I = [0, 1]$ , with open bonding maps. In this paper we show that inverse limits on  $I$  with confluent bonding maps are homeomorphic to inverse limits on  $I$  with open bonding maps, and we identify the particular inverse limit of the latter type that a given inverse limit with confluent bonding maps is homeomorphic to. This weakening of the condition on the bonding map is useful: it makes it much easier to construct mappings onto these continua (see [2] for example). To prove this result, we first obtain a characterization of confluent mappings from  $I$  onto  $I$  which is similar to the characterization of open mappings in [8]. From this characterization it follows that such confluent mappings are uniformly approximated by open mappings. Given an inverse limit,  $X$ , on  $I$  with confluent bonding maps we may apply a theorem of Morton Brown [1] to obtain a homeomorphism from  $X$  onto an inverse limit on  $I$  with open bonding maps. The degrees of the corresponding confluent mappings and open mappings are the same, and thus the inverse limit with open bonding maps

is identified.

A classification of inverse limits on  $I$  with a fixed open bonding map was obtained independently by W. Debski [3] and W.T. Watkins [8]. A classification in the more general case, allowing different bonding maps, was obtained by Debski. It follows as a corollary to the result discussed above that the inverse limits on  $I$  with confluent bonding maps are classified in the same way.

All spaces considered in this paper are metric. A *continuum* is a compact connected metric space and a *mapping* is a continuous function. A mapping  $f$  from the  $X$  onto the space  $Y$  is *confluent* provided that, for each subcontinuum  $K$  of  $Y$ , each component of  $f^{-1}(K)$  is mapped by  $f$  onto  $K$ . We will adopt the following notational conveniences: if  $a = b$ , we define  $[a, b] = \{a\}$ ; if  $H_1$  and  $H_2$  are mutually exclusive sets of real numbers then we say that  $H_2$  is *to the right of*  $H_1$  and write  $H_1 < H_2$  if it is true that  $s < t$  for all  $s \in H_1$  and  $t \in H_2$ ; if  $H_1, H_2$  and  $H_3$  are mutually exclusive sets of real numbers then we say that  $H_2$  is between  $H_1$  and  $H_3$  if either  $H_1 < H_2 < H_3$  or  $H_3 < H_2 < H_1$ .

Our first lemma follows easily from the Intermediate Value Theorem and the definition of confluent mappings.

**Lemma 1.** *Suppose that  $f : I \rightarrow I$  is confluent,  $a$  and  $b$  are in  $I$ ,  $a < b$  and  $f(a) = f(b)$ .*

(1) *If there is a number  $x$  between  $a$  and  $b$  such that  $f(x) > f(a)$  then there is a number  $c$  between  $a$  and  $b$  such that  $f(c) = 1$ .*

(2) *If there is a number  $x$  between  $a$  and  $b$  such that  $f(x) < f(a)$  then there is a number  $c$  between  $a$  and  $b$  such that  $f(c) = 0$ .*

**Lemma 2.** *Suppose that  $f : I \rightarrow I$  is confluent,  $a$  and  $b$  are in  $I$ ,  $a < b$  and  $f(a) = f(b)$ .*

(1) *If  $f(x) > f(a)$  for all  $x$  in  $(a, b)$  then there is just one component of  $f^{-1}(1)$  which is between  $a$  and  $b$ .*

(2) *If  $f(x) < f(b)$  for all  $x$  in  $(a, b)$  then there is just one component of  $f^{-1}(0)$  which is between  $a$  and  $b$ .*

*Proof.* This is a straight forward application of Lemma 1.

From Lemma 2 and the uniform continuity of continuous functions on  $I$  we obtain the following lemma.

**Lemma 3.** *Suppose that  $f : I \rightarrow I$  is confluent. Then there exists  $\epsilon > 0$  such that if  $a$  and  $b$  are in  $I$  and  $f(a) = f(b) = 0$  or  $f(a) = f(b) = 1$  and  $f$  is not constant on  $[a, b]$  then  $|b - a| > \epsilon$ .*

**Corollary** *Suppose that  $f : I \rightarrow I$  is confluent. Then  $f^{-1}(0)$  and  $f^{-1}(1)$  each have only finitely many components.*

**Lemma 4.** *If  $f : I \rightarrow I$  is confluent then either  $f(0) = 0$  or  $f(0) = 1$  and either  $f(1) = 0$  or  $f(1) = 1$ .*

*Proof.* We will prove the first conclusion only. Suppose  $f(0) \neq 0$  and  $f(0) \neq 1$ . Let

$$x = \text{g.l.b.}(f^{-1}(0) \cup f^{-1}(1))$$

Then either  $f(x) = 0$  or  $f(x) = 1$ . In either case  $x > 0$ . Suppose that  $f(x) = 0$ . Let  $K = [f(0)/2, 1]$ . Let  $L$  be the component of  $f^{-1}(K)$  which contains 0. Since  $f$  is confluent, there is a point  $c$  in  $L$  such that  $f(c) = 1$ . Since  $f(0)/2 > 0 = f(x)$  we have that  $L \subset [0, x]$ , and thus  $c < x$ . This is inconsistent with the definition of  $x$ .

**Lemma 5.** *Suppose that  $f : I \rightarrow I$  is confluent,  $f^{-1}(0)$  has  $n$  components,  $L_1 < L_2 < L_3 < \dots < L_n$ , and  $f^{-1}(1)$  has  $m$  components,  $H_1 < H_2 < H_3 < \dots < H_m$ . Then the components of  $f^{-1}(0)$  and  $f^{-1}(1)$  alternate, and*

- (1) *if  $0 \in L_1$  and  $1 \in H_m$  then  $m = n$ ,*
- (2) *if  $0 \in H_1$  and  $1 \in L_n$  then  $m = n$ ,*
- (3) *if  $0 \in L_1$  and  $1 \in L_n$  then  $n = m + 1$ , and*
- (4) *if  $0 \in H_1$  and  $1 \in H_m$  then  $m = n + 1$ .*

*Proof:* That the components of  $f^{-1}(0)$  and  $f^{-1}(1)$  alternate follows from Lemma 2. Suppose 0 is in  $L_1$ . Thus no  $H_i$  is to the left of  $L_1$ . Since  $H_1, H_2, \dots, H_m$  alternates with  $L_1, L_2, L_3, \dots, L_n$ , there are  $n - 1$   $H_i$ 's to the left of  $L_n$ . There can be no more than one  $H_i$  to the right of  $L_n$ . If 1 is in  $L_n$ , there are no  $H_i$ 's to the right of  $L_n$ . Thus if 1 is in  $L_n$ , then  $m = n - 1$  and we have (3). If 1 is not in  $L_n$ , then 1 is in  $H_m$  by Lemma

4. Therefore  $m = n$  and we have (1). Similar reasoning yields (2) and (4).

**Definition.** Suppose that  $f : I \rightarrow I$  is confluent,  $m$  is the number of components of  $f^{-1}(0)$ , and  $n$  is the number of components of  $f^{-1}(1)$ . The *degree* of  $f$ , denoted  $\deg(f)$ , is the integer  $m + n - 1$ . Note that if  $f$  is an open mapping, then  $\deg(f)$  as defined in [3, p. 204] agrees with this definition.

**Theorem 1.** *A mapping  $f : I \rightarrow I$  is confluent if and only if there is a positive integer  $n$  and sequences  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_n$  with*

$$0 = a_0 \leq b_0 < a_1 \leq b_1 < \dots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1,$$

such that, for  $i = 1, 2, \dots, n$ ,  $f$  restricted to  $[b_{i-1}, a_i]$  is a monotone mapping of  $[b_{i-1}, a_i]$  onto  $I$ , and, for  $i = 0, 1, 2, \dots, n$ ,  $f$  restricted to  $[a_i, b_i]$  is constant and equal to 0 or 1.

*Proof.* To prove the sufficiency of the stated condition suppose that  $n$  is a positive integer,  $0 = a_0 \leq b_0 < a_1 \leq b_1 \dots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1$ , and  $f : I \rightarrow I$  is a mapping satisfying the condition above with respect to  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_n$ . Suppose that  $K = [c, d]$  is a subinterval of  $I$ , and that  $L$  is a component of  $f^{-1}(K)$ . Suppose  $L \cap [b_{i-1}, a_i] \neq \emptyset$  for some  $i$ . Since  $f$  is monotone on  $[b_{i-1}, a_i]$ , there is a subinterval,  $J$ , of  $[b_{i-1}, a_i]$  such that  $f(J) = K$ . Also by the monotonicity of  $f$  on  $[b_{i-1}, a_i]$ ,  $J \cap L \neq \emptyset$ . Since  $L$  is a component of  $f^{-1}(K)$ ,  $J \subset L$ . Hence  $f(L) = K$ . If  $L \cap [b_{i-1}, a_i] = \emptyset$  for all  $i$ , then  $L \subset [a_i, b_i]$  for some  $i$ . But, since  $L$  is a component of  $f^{-1}(K)$ ,  $L = [a_i, b_i]$  and thus  $L \cap [b_{i-1}, a_i] \neq \emptyset$ , a contradiction. Therefore  $f$  is confluent.

Now suppose that  $f : I \rightarrow I$  is confluent. Suppose that  $f^{-1}(0)$  has  $n$  components,  $L_1 < L_2 \dots < L_n$ , and  $f^{-1}(1)$  has  $m$  components,  $H_1 < H_2 < \dots < H_m$ . Let  $k = \deg(f) = m + n - 1$ . Either  $0 \in L_1$  or  $0 \in H_1$  by Lemma 4. Suppose the former. Define  $a_0$  and  $b_0$  to be the right and left endpoints of  $L_1$  respectively;  $L_1 = [a_0, b_0]$ . For  $i > 0$  define  $a_i$  and  $b_i$  to be the right and left endpoints, respectively, of  $H_{(i+1)/2}$  if  $i$  is odd, or to be the right and left endpoints, respectively, of  $L_{(i/2+1)}$  if  $i$  is even. We need only show that  $f$  is monotone on  $[b_{i-1}, a_i]$

for  $0 < i < k$ . Suppose that  $i > 0$  and that  $i$  is odd. Let  $j = (i + 1)/2$ . Then  $[a_{i-1}, b_{i-1}] = L_j$  and  $[a_i, b_i] = H_j$ . Suppose that  $t$  is in  $I$  and that  $f^{-1}(t)$  has two components,  $C_1$  and  $C_2$ , lying in  $[b_{i-1}, a_i]$ . By Lemma 2, there is a component of either  $f^{-1}(0)$  or  $f^{-1}(1)$  lying between  $C_1$  and  $C_2$ . But  $L_j$  and  $H_j$  are consecutive components of  $f^{-1}(0) \cup f^{-1}(1)$ , so we have reached a contradiction. A similar contradiction arises if  $i$  is even.

For mappings between Peano continua, light confluent mappings are open (see [6, Theorem 13.23]). The special case of this result where the continuum is  $[0,1]$  follows as an easy corollary of Theorem 1.

**Corollary** *If  $f : I \rightarrow I$  is a light confluent mapping then  $f$  is open.*

*Proof.* Suppose that  $f : I \rightarrow I$  is confluent. Let  $n = \text{deg}(f)$  and let  $a_0 \leq b_0 < a_1 \leq b_1 < \dots < a_n \leq b_n$  be the sequences given by Theorem 1. Since  $f^{-1}(0)$  and  $f^{-1}(1)$  are discrete,  $a_i = b_i$  for  $i = 1, 2, \dots, n$ . Since  $f$  is monotone and light on  $[a_{i-1}, a_i]$ ,  $f$  is a homeomorphism from  $[a_{i-1}, a_i]$  onto  $I$  for  $i = 1, 2, \dots, n$ . Thus, by [7, Lemma 1, p.453],  $f$  is open.

**Theorem 2.** *The uniform closure of the space of all open mappings from  $I$  onto  $I$  is the space of all confluent mappings from  $I$  onto  $I$ .*

*Proof.* Let  $\mathcal{C}$  (respectively,  $\mathcal{O}$ ) denote the space of all confluent (resp., open) mappings from  $I$  onto  $I$ . Denote the supremum norm of a mapping  $f : I \rightarrow I$  by  $\|f\|$ . We first note that  $\mathcal{C}$  is uniformly closed. This can be seen from either [4, Theorem 5.48, p. 41] or [5, 3.1]. Thus,  $cl(\mathcal{O}) \subset \mathcal{C}$ . Suppose that  $f : I \rightarrow I$  is confluent. Let  $n = \text{deg}(f)$  and let  $a_0 \leq b_0 < a_1 \leq b_1 < \dots < a_n \leq b_n$  be the sequences given by Theorem 1. Let  $g : I \rightarrow I$  be the open mapping such that  $g(a_0) = f(a_0)$ , and  $g(b_i) = f(b_i)$  for  $1 < i < n$ , and such that  $g$  is linear on  $[a_0, b_1]$  and on  $[b_i, b_{i+1}]$  for  $1 < i < n - 1$ . For  $k = 1, 2, 3, \dots$  define  $g_k : I \rightarrow I$  by

$$g_k(x) = \frac{1}{k}((k - 1)f(x) + g(x)).$$

Now  $g$  is increasing on each of the intervals  $[a_0, b_1], [b_1, b_2], \dots, [b_{n-1}, b_n]$  on which  $f$  is nondecreasing and  $g$  is decreasing on each of those intervals on which  $f$  is nonincreasing. Consequently  $g_k$  is either increasing or decreasing on those same intervals. Thus  $g_k$  is an open mapping for all  $k$ . Clearly

$$\|g_k - f\| = \frac{1}{k} \|g - f\|,$$

so  $\lim_{n \rightarrow \infty} g_k = f$  uniformly. Therefore  $cl(\mathcal{O}) = \mathcal{C}$ .

A mapping which is the uniform limit of onto homeomorphisms is called a near-homeomorphism (see [1]). Likewise we define a mapping  $f : X \rightarrow Y$  of continua to be *near-open* provided that it is the uniform limit of open mappings. With this terminology Theorem 2 may be restated as follows:

**Theorem 2a.** *Every confluent mapping from  $I$  onto  $I$  is near-open.*

**Lemma 6.** *Suppose that  $f : I \rightarrow I$  and  $g : I \rightarrow I$  are confluent and  $\|f - g\| < \frac{1}{2}$ .*

(1) *If  $f(a) = f(b) = 0$  then either  $f$  is constant on  $[a, b]$  or there is a number  $c$  between  $a$  and  $b$  such that  $g(c) = 1$ .*

(2) *If  $f(a) = f(b) = 1$  then either  $f$  is constant on  $[a, b]$  or there is a number  $c$  between  $a$  and  $b$  such that  $g(c) = 0$ .*

*Proof.* To prove (1), suppose that  $f$  is not constant on  $[a, b]$ . Then by Lemma 1 there is a number  $c'$ , such that  $a < c' < b$ , and  $f(c') = 1$ . Since  $\|f - g\| < \frac{1}{2}$ , it follows that  $g(c') > \frac{1}{2}$ ,  $g(a) < \frac{1}{2}$ , and  $g(b) < \frac{1}{2}$ . By the Intermediate Value Theorem, there are numbers  $a'$  and  $b'$  such that  $a < a' < c' < b' < b$ ,  $g(a') = \frac{1}{2}$ , and  $g(b') = \frac{1}{2}$ . Then, by Lemma 1, there is a number  $c$ , with  $a' < c < b'$ , such that  $g(c) = 1$ .

**Lemma 7.** *Suppose that  $f : I \rightarrow I$  and  $g : I \rightarrow I$  are confluent and  $\|f - g\| < \frac{1}{2}$ .*

(1) *Between consecutive components of  $f^{-1}(0)$  there is just one component of  $g^{-1}(1)$ .*

(2) *Between consecutive components of  $f^{-1}(1)$  there is just one component of  $g^{-1}(0)$ .*

*Proof.* Suppose that  $L_1$  and  $L_2$  are consecutive components of  $f^{-1}(0)$ . By Lemma 6 there is at least one component of  $g^{-1}(1)$  between  $L_1$  and  $L_2$ . Suppose that  $[a, b]$  and  $[c, d]$  are components of  $g^{-1}(1)$  which lie between  $L_1$  and  $L_2$  and that  $b < c$ . Then, by Lemma 6, there is a component of  $f^{-1}(0)$  between  $b$  and  $c$ , and consequently between  $L_1$  and  $L_2$ . This is a contradiction.

**Theorem 3.** *Suppose that  $f : I \rightarrow I$  and  $g : I \rightarrow I$  are confluent and that  $\|f - g\| < \frac{1}{2}$ . Then  $\text{deg}(f) = \text{deg}(g)$ .*

*Proof.* Suppose that  $f^{-1}(0)$  has  $n_f$  components,  $f^{-1}(1)$  has  $m_f$  components,  $g^{-1}(0)$  has  $n_g$  components and that  $g^{-1}(1)$  has  $m_g$  components. Suppose that  $f(0) = 0$ . Since  $\|f - g\| < \frac{1}{2}$  and  $g$  is confluent,  $g(0) = 0$ . All components of  $g^{-1}(1)$  lie in the complement of  $f^{-1}(0)$ , again since  $\|f - g\| < \frac{1}{2}$ . If  $f(1) = 0$ , then  $g(1) = 0$  and all components of  $g^{-1}(1)$  lie between components of  $f^{-1}(0)$ . Thus, in this case,  $m_f = m_g$  by Lemmas 2 and 7. Now suppose that  $f(1) = 1$ . Let  $[a, b]$  denote the right most component of  $f^{-1}(0)$ . There is at least one component of  $g^{-1}(1)$  to the right of  $b$  since  $g(1) = 1$ . If there were two components of  $g^{-1}(1)$  to the right of  $b$ , by Lemma 7 there would be a component of  $f^{-1}(0)$  to the right of  $b$ . Hence there is just one component of  $g^{-1}(1)$  to the right of  $b$ . By similar reasoning, there is just one component of  $f^{-1}(1)$  to the right of  $b$ . All components of  $g^{-1}(1)$  to the left of  $a$  lie between two components of  $f^{-1}(0)$ . Hence, by Lemmas 2 and 7,  $g^{-1}(1)$  and  $f^{-1}(1)$  have the same number of components to the left of  $a$ . Therefore  $m_f = m_g$ . Similar reasoning yields that  $n_f = n_g$ . Therefore  $\text{deg}(f) = \text{deg}(g)$ .

An *inverse sequence* is a pair  $\{X_i, f_i\}$  whose first term is a sequence of spaces and whose second term is a sequence of mappings  $f_i : X_{i+1} \rightarrow X_i$ , called the *bonding maps* of the sequence. The *inverse limit* of the sequence  $\{X_i, f_i\}$  is the set

$$\varprojlim \{X_i, f_i\} = \{(x_1, x_2, x_3, \dots) \mid x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i\},$$



with the topology induced by the metric

$$d(\underline{x}, \underline{y}) = \sum_{i>0} \frac{d_i(x_i, y_i)}{2^i}$$

where  $\underline{x} = (x_1, x_2, x_3, \dots)$ , and  $\underline{y} = (y_1, y_2, y_3, \dots)$  are in  $\varprojlim\{X_i, f_i\}$  and  $d_i$  is the metric for  $X_i$ .

**Theorem 4.** *Suppose that  $f_i : I \rightarrow I$  is confluent for  $i = 1, 2, \dots$ . Then there is a sequence  $g_i$  of open mappings from  $I$  onto  $I$  such that  $\deg(g_i) = \deg(f_i)$  and  $\varprojlim\{I, f_i\}$  is homeomorphic to  $\varprojlim\{I, g_i\}$ .*

*Proof.* For each  $i$  there exists a sequence  $\{h_{ij}\}_{j=1}^{\infty}$  of open mappings of  $I$  onto  $I$  which converges uniformly to  $f_i$  and which has the property that  $\|f_i - h_{ij}\| < \frac{1}{2}$  for all  $j$ . By Theorem 3,  $\deg(f_i) = \deg(h_{ij})$  for all  $i$  and  $j$ . Let

$$K_i = \{h_{ij} | j = 1, 2, \dots\}.$$

By [1, Theorem 3, p.481], there is a sequence  $\{g_i\}_{i=1}^{\infty}$  such that  $g_i$  is in  $K_i$  and  $\varprojlim\{g_i, I\}$  is homeomorphic to  $\varprojlim\{f_i, I\}$ .

If  $n$  is a positive integer, we define the *standard open mapping of degree  $n$* ,  $w_n : I \rightarrow I$ , to be the mapping such that, for  $i = 1, \dots, n$ ,

$$w_n(i/n) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd,} \end{cases}$$

and which is linear on the intervals  $[(i-1)/n, i/n]$  (see [3, p. 203]).

**Corollary 1.** *Suppose that  $f_i; I \rightarrow I$  is confluent for  $i = 1, 2, \dots$  and that  $n_i = \deg(f_i)$ . Then  $\varprojlim\{f_i, I\}$  is homeomorphic to  $\varprojlim\{I, w_{n_i}\}$ .*

*Proof.* This follows from Theorem 4 and [3, Lemma 4, p. 204].

**Corollary 2.** *If  $f : I \rightarrow I$  and  $g : I \rightarrow I$  are confluent then  $\varprojlim\{I, f\}$  and  $\varprojlim\{I, g\}$  are homeomorphic if and only if  $\deg(f)$  and  $\deg(g)$  have the same prime factors.*

*Proof.* This follows from Corollary 1 and [8, Thm. 5, p. 599].

*Acknowledgments.* The author thanks Sam Nadler for suggesting the inclusion of Theorem 2a and the corollary following Theorem 1.

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