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PROBLEMS IN 4-MANIFOLD TOPOLOGY

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0. Introduction

In the theory of manifold topology, dimension four stands alone. While a unified theory applies to all higher dimensions, 4-manifolds exhibit fundamentally different behavior. Topology in dimension four is characterized by a rich interplay between various categories of manifolds, such as the smooth ($DIFF$) and topological (TOP) categories. In dimensions less than four, this interplay does not exist, since all such categories are essentially the same (e.g., [Mo]). In dimensions greater than four, the theories of smooth and topological manifolds are quite similar to each other. While the categories are not identical, the differences between them (*i.e.*, exotic smooth structures and unsmoothable manifolds) are well-understood [KS]. In contrast, the theories of smooth and topological 4-manifolds are radically different from each other, resulting in complicated and poorly understood interactions. Several intermediate categories (Lipschitz and quasiconformal) admit uniquely 4-dimensional behavior and are largely unexplored. The smooth category also seems to be intimately linked to the holomorphic and algebraic categories in this dimension. We will discuss these categories and their interplay, with emphasis on some open problems in the area.

1. TOP

The topological category is the best understood category in dimension 4, primarily because of Freedman's breakthrough [F]. In particular, Freedman has completely classified simply

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connected, closed 4-manifolds (closed = compact with empty boundary). The general principle is that 4-manifolds behave just like their higher-dimensional counterparts, provided that the fundamental groups involved are not too "large." A good reference for this theory is [FQ]. The main unsolved problem is whether the restriction on fundamental groups is really necessary.

Another sort of problem arises from the fact that Freedman's main lemma is based on an extremely complicated infinite construction. How can we "see" any of Freedman's constructs? For example, is there a direct construction of Freedman's E_8-manifold, or $E_8 \# E_8$? Freedman proves that every homology sphere bounds a compact, contractible 4-manifold. Most of these must be unsmoothable. What do they look like? A knot in $S^3 = \partial B^4$ is called *slice* if it bounds a flat disk embedded in B^4. Freedman found a large class of knots (any knot with Alexander polynomial 1) which must be topologically slice. Many of these are not smoothly slice (see section 2). What do these topological disks in B^4 look like? Such a disk may be assumed smooth except at a single point. Deleting this point yields a smooth, proper embedding $S^1 \times [0,\infty) \hookrightarrow B^4 - 0 \approx S^3 \times [0,\infty)$. Draw an explicit (Morse) level picture for this.

2. DIFF

The smooth category in dimension 4 (which is the same as the PL category in this dimension) is radically different from the topological category. This follows from work of Donaldson, such as [D1], [D2]. Both existence and uniqueness of smooth structures on topological 4-manifolds may fail, even in the presence of predictions to the contrary by high-dimensional smoothing theory. For example $\mathbb{R}^n$ is contractible, so for any $n \neq 4$ it must admit a unique smooth structure. However, $\mathbb{R}^4$ admits exotic smooth structures, not diffeomorphic to $\mathbb{R}^4$ (by an argument of Casson, together with Freedman and Donaldson theory). In fact, there are uncountably many diffeomorphism types of smooth structures on $\mathbb{R}^4$, and these tend to

occur in families parametrized by continuous variables [G1], [G2], [DF].

Is there any hope of classifying exotic $\mathbb{R}^4$'s? A first step would be to construct an invariant (probably real-valued) to distinguish some exotic $\mathbb{R}^4$'s. (At present, no such invariant is known. The above families are distinguished by indirect proofs by contradiction.) The space of exotic $\mathbb{R}^4$'s admits a monoid structure ("end-sum") [G2] and (after modding out by a certain equivalence relation) a partial ordering and a metrizable topology with countable basis [G4]. Are these structures useful? What can be said about this space of exotic $\mathbb{R}^4$'s? Is there a better way to topologize the space? All known exotic $\mathbb{R}^4$'s are constructed by Freedman theory. How can an exotic $\mathbb{R}^4$ be constructed more explicitly? Is there any closed 4-manifold covered by an exotic $\mathbb{R}^4$?

Another basic question is the smooth Poincaré conjecture. Any homotopy 4-sphere is homeomorphic to S^4 by Freedman's work, but is it diffeomorphic to S^4? Cappell and Shaneson [CS1] constructed an explicit family of homotopy 4-spheres. It has been conjectured that some of these are exotic. Others have been shown to be standard [AR], [G6]. It may well be that all of these examples are standard, although no proof has yet turned up. Explicit descriptions of some undecided examples are given in [AR] and [G6]. Can these be trivialized directly?

Examples of exotic smooth structures on closed, simply connected 4-manifolds are provided by Donaldson's invariants. There are homeomorphism types of such manifolds which are realized by infinitely many diffeomorphism types [FM]. (Specifically, these are families of algebraic surfaces which are homeomorphic to each other by Freedman's work, but distinguished by Donaldson's invariants.) What is the smallest closed, simply connected 4-manifold which admits several nondiffeomorphic smooth structures? Infinitely many? (The smallest known examples have Betti number $b_2 = 9$ [B],[K] and 10 [FM], respectively.) Exotic structures on closed, simply connected 4-manifolds cannot be distinguished by classical invariants since high dimensional theory predicts a bijective correspondence between smooth structures on M^4 (up to isotopy) and

$H_1(M^4; Z_2)$ (for M^4 compact and smoothable). Are there exotic smooth structures on closed, oriented 4-manifolds which are detected by the classical invariant? For example, does such an exotic structure (or *any* exotic structure) exist on $S^3 \times S^1$? (The classical invariant is realized if the universal cover is a smoothing of $S^3 \times \mathbb{R}$ with a nontrivial μ-invariant in the sense of μ-invariants of homology spheres. Such smoothings of $S^3 \times \mathbb{R}$ exist [F], but most will not cover $S^3 \times S^1$.) It is known that such "classical" exotic smoothings exist on oriented, compact 4-manifolds with boundary [G3], and on closed, nonorientable manifolds (such as $\mathbb{R}P^4$ [CS2]). Are there exotic smoothings on closed, nonorientable manifolds which are *not* detected by classical invariants? (It does not work to connected sum Donaldson's examples with nonorientable manifolds, since the loss of orientability destroys the exoticness [G7], c.f. section 4.)

Armed with Donaldson's theorems and elementary topological arguments, one may obtain results about diverse questions regarding dimensions 3 and 4. Which homology spheres bound contractible 4-manifolds? (In TOP they all do, but in $DIFF$ many do not.) Which bound 4-manifolds with definite intersection forms? Which knots in S^3 are slice? (Many Alexander polynomial 1 knots are not smoothly slice, for example, the double of the trefoil knot.) Can the double of a non-slice knot ever be (smoothly) slice? (It is always topologically slice.) For a given knot K, what is the minimal number of self-intersections of an immersed disk in B^4 bounded by K? Many techniques and partial results are given in [CG], for example, but there is undoubtedly much room for stronger results.

3. LIP AND QC

What categories lie between TOP and $DIFF$? In arbitrary dimensions, there is a *Lipschitz* category (LIP), where each manifold has an atlas of charts whose overlap functions are Lipschitz maps with Lipschitz inverses. (The Lipschitz condition puts a bound on how much distances may be stretched.) Similarly, there is a *quasiconformal* category (QC) made with quasiconformal maps (which do not stretch angles too much).

In general, we have $TOP \leq QC \leq LIP \leq DIFF$, where $A \leq B$ means that each B-structure has essentially a unique underlying A-structure. (Smooth maps are automatically Lipschitz (locally), and Lipschitz maps are quasiconformal.) In dimensions ≤ 3, these categories are all essentially the same (see [Mo] for $TOP = PL$), and in dimensions ≥ 5, Sullivan showed that $TOP = QC = LIP$ [S]. In dimension 4, however, Donaldson and Sullivan [DS] showed that much of Donaldson's work goes through in QC. In particular, both existence and uniqueness of QC structures fail on topological 4-manifolds, so 4 is the unique dimension in which $TOP \neq QC$. What can be said about QC and LIP in dimension 4? Not much is known. It is conceivable that $QC = LIP = DIFF$, but these three categories may also be quite different. Is there a category between TOP and QC in which Freedman's techniques work? (We would need to allow much stretching, but possibly not arbitrary homeomorphisms.) This category should then be equivalent to TOP, in the sense that $QC = TOP$ in dimensions $\neq 4$.

4. ALG.

We may form other categories by imposing additional structure on smooth manifolds. For example, we may form the *complex* category (in even dimensions) by identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and requiring the overlap functions in our atlases to be holomorphic (with holomorphic inverses). An even more restrictive condition is to require our manifolds to be *algebraic varieties,* that is, zero sets in $\mathbb{C}P^N$ of collections of homogeneous complex polynomials. This forms an *algebraic category* (ALG), where the morphisms are given by rational functions. Although the algebraic category is in general much more restrictive than the holomorphic category, we will be primarily concerned with closed, simply connected 4-manifolds up to diffeomorphism, and in this case the two categories are essentially the same. (Any complex structure on such a 4-manifold may be deformed into an algebraic one, by Kodaira's classification theorem. See, for example, [BPV].)

Algebraic structures are quite rigid compared to smooth structures. Nevertheless, the algebraic category has much to tell us about the smooth category, at least in low (even) dimensions. For example, all closed, oriented surfaces are realized as algebraic curves (*complex* dimension one, or *real* dimension two). Similarly, many examples of closed 4-manifolds are provided by algebraic surfaces (complex dimension two, or real dimension four). We have already seen (section 2) that these provide us with infinite families of nondiffeomorphic smooth structures on various simply connected topological 4-manifolds. The diffeomorphism classification of simply connected algebraic surfaces promises to be a complicated and difficult problem. What are the smallest examples of such manifolds? If we rule out the particularly simple class of *rational surfaces* (such as $\mathbb{C}P^2$ and $S^2 \times S^2$), the smallest remaining known example has second Betti number equal to nine [B]. Can we improve on this?

Not all closed, simply connected, smooth 4-manifolds admit algebraic structures. For example, S^4 and the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ of two $\mathbb{C}P^2$'s cannot admit complex structures since their tangent bundles do not reduce to complex vector bundles, by a simple characteristic class argument. Until recently, however, some people have conjectured that all simply connected 4-manifolds are *connected sums* of algebraic surfaces (where S^4 is the trivial connected sum). There are now known to be infinite families of counterexamples to this [GM]. What is the smallest possible counterexample? The smallest homeomorphism type realizing infinitely many counterexamples? (The examples in [GM] have $b_2 = 22$.)

How many manifolds can we construct by forming connected sums of simply connected algebraic surfaces? It seems a good conjecture that if M and N are simply connected algebraic surfaces $\neq \mathbb{C}P^2$, and if we form the connected sum $M\#\overline{N}$ which is *not* compatible with the canonical orientations of M and N, then the result will always decompose as a connected sum of simple pieces, namely $\mathbb{C}P^2$'s (with both orientations) or $S^2 \times S^2$'s and $K3$ surfaces. This is true for M and N in a large class of algebraic surfaces [G5], [G7]. Can it be proven

in other cases? The incompatibility of the sum with orientations seems crucial here. Are there any examples of connected sums of irrational algebraic surfaces (summed compatibly with the canonical orientations) which have such simple decompositions? Perhaps connected sum decompositions are unique in this setting. The algebraic surfaces for which partial results are known all have nonpositive signature. It is possible to construct simply connected, irrational algebraic surfaces with positive signature [MT]. How do these fit into the picture? Do all simply connected algebraic surfaces split into simple pieces after connected sum with a nonorientable manifold? (Elliptic surfaces do [G7].) What can be said about more general simply connected 4-manifolds? (The examples in [GM] behave like elliptic surfaces for these purposes.) An informal exposition of some of these phenomena appears in [G8].

5. BETWEEN DIFF AND ALG

Any algebraic surface has an underlying *almost complex* structure, or reduction of the tangent bundle to a complex vector bundle. As we have seen, not every 4-manifold admits such a structure, but the existence and classification of almost complex structures (up to fiber homotopy) on a 4-manifold is merely an exercise in homotopy theory. A much harder and more interesting question is: Which almost complex structures are fiber homotopic to *integrable* almost complex structures, *i.e.*, ones which come from complex structures? Not every almost complex structure has this property. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ admits almost-complex structures, but no complex structure (by work of Donaldson [D3] or Yau [Y]).

Several interesting types of structure lie between almost complex and algebraic. A *symplectic* structure is a differential 2-form ω which is closed ($d\omega = 0$) and nondegenerate (as a bilinear form on each tangent space). If we drop the "integrability condition" $d\omega = 0$ to consider "almost symplectic" structures (nondegenerate 2-forms) we find that the classification problem is identical to that for almost complex structures. (In each case, we reduce the structure group to a group whose

maximal compact subgroup is $U(2)$.) Thus, the delicate question, once again, is integrability: which almost symplectic (= almost complex) structures are fiber homotopic to those with $d\omega = 0$? Note that any algebraic structure has an underlying symplectic structure (namely, the Kähler form). Non-Kähler examples are scarce, particularly for simply connected (closed) manifolds. One reference is [Mc].

Another type of intermediate structure is called a *Lefschetz fibration.* This structure is more topological in nature than symplectic structures – in fact, it is in some sense a complex analogue of a Morse function. A Lefschetz fibration is a smooth map $\varphi : M \to F$, where M is our given smooth, closed, oriented 4-manifold and F is a connected, oriented surface (usually S^2.) We require each critical point of φ to be complex quadratic. This is, there must exist orientation-preserving local coordinates in which φ becomes the function $z_1^2 + z_2^2$ (or equivalently, $z_1 z_2$) from a neighborhood in $\mathbb{C}^2$ to $\mathbb{C}$. Thus, there are only finitely many critical points, and away from the critical values we have a fiber bundle with fibers closed, oriented surfaces of fixed genus. The topology near the singular fibers is also completely determined. Just as we can construct a Morse function on any manifold by embedding it in $\mathbb{R}^N$ and projecting onto a generic line, we can construct a Lefschetz fibration on any algebraic surface (after some "blow-ups") by projecting it from $\mathbb{C}P^N$ onto a generic $\mathbb{C}P^1$. (We may need to blow up some "points at infinity" to resolve singularities.) See, for example, [L].

It can be shown that most Lefschetz fibrations admit compatible symplectic structures. (Specifically, a Lefschetz fibration admits a symplectic structure with all fibers symplectic submanifolds, if and only if the homology class of a fiber has infinite order.) Do symplectic manifolds necessarily admit Lefschetz fibrations (after blowing up)? Are manifolds with Lefschetz fibrations necessarily algebraic? What do these structures tell us about smooth 4-manifolds? This topic, like the rest of 4-manifold theory, presents far more questions than answers.

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