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# THE PRODUCTS OF $K$-SPACES WITH POINT-COUNTABLE CLOSED $K$-NETWORKS


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# THE PRODUCTS OF K-SPACES WITH POINT-COUNTABLE CLOSED K-NETWORKS

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ABSTRACT. Using a technique of [2] we prove the theorem(CH +MC): Let $X$ and $Y$ be k-spaces with point-countable closed k-networks. Then $X \times Y$ is sequential if and only if one of the three properties below holds:
a) $X$ and $Y$ have point-countable bases.
b) $X$ or $Y$ is locally compact.
c) $X$ and $Y$ are locally $k_\omega$-spaces.

## 1. INTRODUCTION

Throughout this paper, we shall assume that all spaces are regular, and all maps are continuous surjections.

A cover $\mathcal{F}$ of a space is a k-network if for any $K \subset U$ with $K$ compact and $U$ open, $K \subset \cup\mathcal{F}' \subset U$ for some finite $\mathcal{F}' \subset \mathcal{F}$. A space $X$ is in class $\mathfrak{T}'$ [7] if $X$ has the weak topology with respect to a countable cover of closed locally compact subsets. Y. Tanaka [7, Theorem 3.1] has proven:

**Theorem 3.1.** *Let $X$ and $Y$ be k-and $\aleph$-spaces, then $X \times Y$ is a k-space if and only if one of the three properties below holds:*
*a) $X$ and $Y$ are metrizable spaces.*
*b) $X$ or $Y$ is a locally compact metrizable space.*
*c) $X$ and $Y$ are spaces of the class $\mathfrak{T}'$.*

Here an $\aleph$-space [6] is a space with a $\sigma$-locally finite k-network. A k-and $\aleph$-space is a quotient s-image of a metric space by [4, Theorem 6.1]. So it is desirable to consider the k-ness of the product $X \times Y$ of quotient s-images $X, Y$ of metric spaces. If $X$ and $Y$ are Fréchet, by [10, Theorem 9], it follows that $X \times Y$ is a k-space if and only if $X$ and $Y$ have point-countable bases. Otherwise b) or c) of the above theorem holds. Every quotient s-image of a metric space has a point-countable k-network by [4, theorem 6.1]. In this paper, under CH(continuum hypothesis) and MC( there exists measurable cardinal) we prove the result:

**Theorem 1.** *Let $X$ and $Y$ be k-spaces with point-countable closed k-networks, then $X \times Y$ is sequential if and only if one of the three properties below holds:*

*a) $X$ and $Y$ have point-countable bases.*
*b) $X$ or $Y$ is locally compact.*
*c) $X$ and $Y$ are locally $k_\omega$-spaces.*

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## 2. Lemmas

Recall that a space $X$ has countable tightness, $t(X) \leq \omega$, if $x \in \overline{A}$ in $X$, then $x \in \overline{C}$ for some countable $C \subset A$.

**Lemma 1. (10, Lemma 4).** *Suppose that $X \times Y$ has a k-system with $t(X) \leq \omega$, then the following condition $(C_1)$ or $(C_2)$ holds,*

*$(C_1)$. If $(A_n) \downarrow x$ in $X$, then there exists a nonclosed subset $\{a_n; n \in \omega\}$ of $X$ with $a_n \in A_n$.*

*$(C_2)$. If $(A_n)$ is a k-sequence in $Y$, then some $\overline{A_n}$ is countably compact.*

Here $(A_n) \downarrow x$ means a decreasing sequence $\{A_n;\ n \in \omega\}$ such that $x \in \overline{A_n \setminus \{x\}}$ for $n \in \omega$. A k-sequence [5] is a decreasing sequence $\{A_n;\ n \in \omega\}$ such that $C = \bigcap_{n\in\omega} A_n$ is compact and each neighborhood of $C$ contains some $A_n$.

**Lemma 2. (4, Lemma 1.7)** *If $f : X \to Y$ is a quotient map, and if $X$ is determined by the cover $\mathcal{P}$, then $Y$ is determined by $f(\mathcal{P}) = \{f(P); P \in \mathcal{P}\}$.*

We use "$X$ is determined by $\mathcal{P}$" just as "$X$ has the weak topology with respect to $\mathcal{P}$ ". The terminology is due to [4, note 2].

**Lemma 3.** *Let $Y$ be a quotient s-image of a metric space, and let $S_\omega \times Y$ be sequential. If $Y$ is determined by cover $\mathcal{P}$, then $S_\omega \times Y$ is determined by cover $\{S_n \times P;\ P \in \mathcal{P}$ and $S_n \in \varphi\}$.*

*Proof.* Let $f_1 : M_1 \to S_\omega$ be a quotient s-map, and let $f_2 : M_2 \to Y$ be a quotient s-map. Here $M_1 = \Sigma_n S_n$ and $M_2$ is a metric space. Let $\mathcal{B}$ be a $\sigma$-locally finite base of $M_2$ and let $\mathcal{P} = f_2(\mathcal{B})$. $M_1 \times M_2$ is a metric space and $M_1 \times M_2$ is determined by cover $\{S_n \times B;\ S_n \in \varphi$ and $B \in \mathcal{B}\}$. $f_1 \times f_2$ is a quotient s-map if $S_\omega \times Y$ is sequential. Then $S_\omega \times Y$ is determined by $\{S_n \times P;\ S_n \in$P$H$ and $P \in \mathcal{P}\}$ from Lemma 2.

**Lemma 4.** *Let $Y$ be a quotient s-image of a metric space $M$, and let $\mathcal{P} = f(\mathcal{B})$. If $S_\omega \times Y$ is sequential, the $\overline{P}$ is a compact metric subset of $Y$ for each $P \in \mathcal{P}$. Here $\mathcal{B}$ is a $\sigma$-locally finite base of $M$.*

*Proof.* $\mathcal{B} = \cup_{n<\omega}\mathcal{B}_n$ is a $\sigma$-locally finite base of $M$, Let $\mathcal{B}_x = \{B_n \in \mathcal{B};\ x \in B_n$ and $B_1 \supset B_2 \supset \cdots\}$ with $\{f(B_n);\ B_n \in \mathcal{B}_x\}$ is a k-sequence. Then some $\overline{f(B_n)}$ is a compact metric subset of $Y$ by [10, Lemma 6]. Thus we can suppose that $\overline{f(B)}$ is compact for each $B \in \mathcal{B}$ .

**Lemma 5 (CH).** *Let $Y$ be a quotient s-image of a metric s-pace. If $S_\omega \times Y$ is sequential, then there exists a subcollection $\mathcal{P}_y$ of $\mathcal{P}$ such that $|\ \mathcal{P}_y\ | \leq \aleph_0$ and $\cup\ \mathcal{P}_y$ is a neighborhood of $y$ for each $y \in Y$.*

*Proof.* Suppose there exists a point $y_0$ of $Y$ such that we take any subcollection $\mathcal{P}'$ of $\mathcal{P}$, if $|\ \mathcal{P}\ '\ | \leq \aleph_0$, then $\cup\mathcal{P}'$ is not a neighborhood of point $y_0$.

A. Let $N_0$ be a Moore-Smith net which converges to $y_0$, $y_0 \notin N_0$ and $|\ N_0\ | = \aleph_0$. Let $\mathcal{P}_0 = \{P \in$ P $;\ P \cap N_0 \neq \emptyset\}$, then $|\ \mathcal{P}_0\ | \leq \aleph_0$ by $\mathcal{P}$ point-countable. If we have defined a Moore-Smith net $N_\beta$ which converges to $y_0$ with $y_0 \notin N_\beta, |\ N_\beta\ | \leq \aleph_0$, $\mathcal{P}_\beta = \{P \in \mathcal{P}\ ;\ P \cap N_\beta \neq \emptyset\}$ and $|\ \mathcal{P}_\beta\ | \leq \aleph_0$ for all $\beta < \alpha$, here $\alpha < \omega_1$. Then $\bigcup(\cup_{\beta<\alpha}\ \mathcal{P}_\beta\ )$ is not a neighborhood of point $y_0$ by $|\ \cup_{\beta<\alpha}\ \mathcal{P}_\beta\ | \leq y\aleph_0$. So we can take a Moore-Smith net $N_\alpha$ which converges to $y_0$, $y_0 \notin N_\alpha$, $|\ N_\alpha\ | \leq \aleph_0$ and $N_\alpha \bigcap [\bigcup(\cup_{\beta<\alpha}\ \mathcal{P}_\beta\ )] = \emptyset$. Let $\mathcal{P}_\alpha = \{P \in \mathcal{P}\ ;\ P \cap N_\alpha \neq \emptyset\}$, then $|\ \mathcal{P}_\alpha$

$|\leq \aleph_0$ by $\mathcal{P}$ point-countable. Then, by induction, there exists a collection $\{N_\alpha;\ \alpha < \omega_1\}$ such that:

1) $N_\alpha$ converges to $y_0$, $y_0 \notin N_\alpha$ and $|\ N_\alpha\ |\leq \aleph_0$ for each $N_\alpha$.

2) $P$ meets only one $N_\alpha$ for each $P$ of $\mathcal{P}$.

B. Each $N_\alpha = \{x_{\alpha 1}, x_{\alpha 2}, \dots, x_{\alpha n}, \dots\} = \{1_\alpha, 2_\alpha, \dots, n_\alpha, \dots\}$. For each $f_\alpha \in {}^\omega\omega$ let $H_\alpha = \cup_{n<\omega}(\{1_n, 2_n, \dots, f(n)_n\} \times \{1_\alpha, 2_\alpha, \dots, n_\alpha\}) \subset S_\omega \times Y$. Let $H = \cup_{\alpha<\omega_1} H_\alpha$. Here $S_n = \{1_n, 2_n, \dots\}$ is a convergent sequence of $S_\omega$.

a) $H\cap\ (S_n \times P\ )$ is closed in $S_n \times P$ for each $S_n \times P \in \{S_n \times P;\ S_n \in \varphi$ and $P \in \mathcal{P}\}$. In fact : $P$ meets only one $N_\alpha$ by property 2) of $\{N_\alpha;\ \alpha < \omega_1\}$. Then $(S_n \times P) \cap H = (S_n \times P\ ) \cap H_\alpha = (S_n \times P\ ) \cap (\cup_{i\leq n}\{1_i, 2_i, \dots, f(i)_i\} \times \{1_\alpha, 2_\alpha, \dots, i_\alpha\})$ has only finitely many points.

b) $H$ is not closed. We prove $(\infty, y_0) \in \overline{H} - H$. Here "$\infty$" denotes the nonisolated point in $S_\omega$. If $f \in {}^\omega\omega$, let $U_f$ be the neighborhood of point "$\infty$" in $S_\omega$ defined by $U_f = \{\infty\} \cup \{n_m;\ n \geq f(m)\}$. Let $U$ be a neighborhood of point $y_0$ in $Y$, then $N_\alpha \cap U \neq \emptyset$ for each $N_\alpha \in \{N_\alpha;\ \alpha < \omega_1\}$. Then there exists $n(\alpha)_\alpha \in N_\alpha \cap U$. Let $g(\alpha) = n(\alpha)$, then $g \in {}^{\omega_1}\omega$. By $f \in {}^\omega\omega$, there exists function $f_{\alpha_0} \in {}^\omega\omega$ such that $A = \{n \in \omega;\ f_{\alpha_0}(n) > f(n)\}$ is infinite. Because $g(\alpha_0) = n(\alpha_0)$, there exists $n' \in A$ with $n' > n(\alpha_0)$. Then $(f_{\alpha_0}(n')_{n'}, n(\alpha_0)_{\alpha_0}) \in \{1_{n'}, 2_{n'}, \dots, f_{\alpha_0}(n')_{n'}\} \times \{1_{\alpha_0}, 2_{\alpha_0}, \dots, n(\alpha_0)_{\alpha_0}, \dots, n'_{\alpha_0}\} \subset H$. On the other hand the $n' \in A$ gives $f_{\alpha_0}(n') > f(n')$, so $f_{\alpha_0}(n')_{n'} \in \{n_m; n \geq f(m)\} \subset U_f$ and $n(\alpha_0)_{\alpha_0} \in N_{\alpha_0} \cap U$. Then $(f_{\alpha_0}(n')_{n'}, n(\alpha_0)_{\alpha_0}) \in (U_f \times U) \cap H_{\alpha_0} \subset (U_f \times U) \cap H$ and $H$ is not closed. Then $S_\omega \times Y$ can not be determined by $\{\ S_n \times P;\ S_n \in \varphi$ and $P \in \mathcal{P}\ \}$. But $S_\omega \times Y$ is determined by $\{\ S_n \times P\ ;\ S_n \in \varphi$ and $P \in \mathcal{P}\ \}$ because of Lemma 3. This is a contradiction.

**Lemma 6.** *Let $Y$ be a quotient s-image of a metric space. If $S_\omega \times Y$ is sequential, then $Y$ satisfies the following condition $(C_3)$,*

*$(C_3)$. If $Y$ has a collection $\{C_{nm};\ n, m < \omega\}$ such that*

*1) $C_{nm}$ is a closed set for each $C_{nm} \in \{C_{nm};\ n, m < \omega\}$,* *2)* $\{C_{nm};\ m < \omega\}$ *is a discrete subcollection of* $\{C_{nm};\ n,m < \omega\}$ *for each* $n < \omega$, *then for each* $x \in \overline{\cup_{nm} C_{nm}}$, *there exists a function* $f \in {}^{\omega}\omega$ *with the* $x \notin \overline{\cup\{C_{nm}; m \geq f(n)\}}$.

*Proof.* If $Y$ does not satisfy $(C_3)$, then there exists a collection $\{C_{nm};\ n,m < \omega\}$ in $Y$ such that

1) $C_{nm}$ is closed for each $C_{nm} \in \{C_{nm};\ n,m < \omega\}$.

2) $\{C_{nm};\ m < \omega\}$ is a discrete subcollection of $\{C_{nm};\ n,m < \omega\}$ for each $n < \omega$. But there exists $x_0 \in \overline{\cup_{nm} C_{nm}}$, with the $x_0 \in \overline{\cup\{C_{nm};\ m \geq f(n)\}}$ for each $f \in {}^{\omega}\omega$.

Let $A = \cup_{nm}(\{m_n\} \times C_{nm})$. On the one hand; $(S_n \times P) \cap A = \cup_m(\{m_n\} \times C_{nm}) \cap (S_n \times P)$ is closed in $S_n \times P$ by 1) and 2) of $(C_3)$. On the other hand: $(U_f \times U) \cap A \neq \emptyset$ for each $f \in {}^{\omega}\omega$ and each neighborhood $U$ of the $x_0$. In fact: for each $f \in {}^{\omega}\omega$, if $x_0 \in \overline{\cup\{C_{nm};\ m \geq f(n)\}}$ then $U \cap (\cup\{C_{nm};\ m \geq f(n)\}) \neq \emptyset$. So, there exists $C_{nm} \in \{C_{nm};\ m \geq f(n)\}$ with $U \cap C_{nm} \neq \emptyset$. Then $(U_f \times U) \cap A \supset (U_f \times U) \cap (\{m_n\} \times C_{nm}) \neq \emptyset$. This implies that $S_\omega \times Y$ is not sequential. This is a contradiction.

**Lemma 7.** *Let* $(X\mathcal{T}_1)$ *and* $(X\mathcal{T}_2)$ *be regular, and let* $\mathcal{T}_2 \subset \mathcal{T}_1$. *If subset* $C$ *of* $X$ *is* $\mathcal{T}_1$*-compact, then* $C$ *is* $\mathcal{T}_2$*-compact and* $\mathcal{T}_2|C = \mathcal{T}_1|C$. *Here* $\mathcal{A}|B = \{A \cap B;\ A \in \mathcal{A}\}$.

*Proof.* We only prove $\mathcal{T}_2|C = \mathcal{T}_1|C$. $\mathcal{T}_2 \subset \mathcal{T}_1$ then $\mathcal{T}_2|C \subset \mathcal{T}_1|C$. On the other hand: if $O_1 \in \mathcal{T}_1$, $O_1 \cap C \in \mathcal{T}_1|C$ then $C-(O_1 \cap C)$ is a $\mathcal{T}_1$-compact subset of $X$. $C - (O_1 \cap C)$ is a $\mathcal{T}_2$-compact subset of $X$, then $C - (O_1 \cap C)$ is a $\mathcal{T}_2$-compact subset of $C$. Then $C - (O_1 \cap C)$ is a $\mathcal{T}_2$-closed subset of $C$. Then $C - (C - (O_1 \cap C)) = O_1 \cap C$ is a $\mathcal{T}_2$-open subset of $C$. Then there exists $O_2 \in \mathcal{T}_2$ with $O_2 \cap C = O_1 \cap C$. This implies $O_1 \cap C \in \mathcal{T}_2|C$.

**Lemma 8.** *Let* $Y$ *be sequential. If* $Y$ *is the union of countably many compact metric subsets of* $Y$, *then there exists a totally disconnected sequential space* $Z$ *which is the union of countably many compact metric subsets of* $Z$ *and there exists a perfect map* $f : Z \to Y$.

*Proof.* If $(Y\mathcal{T}_1)$ is a regular sequential space which is the union of countably many compact metric subsets of $Y$. Then $(Y\mathcal{T}_1)$

is a paracompact $\sigma$-space. Then $Y$ has a $G_\delta$-diagonal by [3, Theorem 4.6]. Then $(Y\mathcal{T}_1)$ is submetrizable by [3, Corollary 2.9]. There exists a topology $\mathcal{T}_2$ on $Y$ such that $\mathcal{T}_2 \subset \mathcal{T}_1$ and $(Y\mathcal{T}_2)$ is metrizable. $(Y\mathcal{T}_1)$ is the union of countably many compact subsets of $(Y\mathcal{T}_1)$, then $(Y\mathcal{T}_2)$ is the union of countably many compact subsets of $(Y\mathcal{T}_2)$ by Lemma 7. Then $(X\mathcal{T}_2)$ has a countable base $\mathcal{B}$. By [1, Chapter 6, 252], $(X\mathcal{T}_2)$ is an image of a subspace $(Z\mathcal{O}_2)$ of the Baire space $B(\aleph_0)$ under an irreducible perfect map $f$. The Baire space $B(\aleph_0)$ is totally disconnected, so is the subspace $(Z\mathcal{O}_2)$. Let $\mathcal{K} = \{f^{-1}[K];\ K$ is a compact subset of $(Y\mathcal{T}_1)\}$. Then $\mathcal{K}$ is a compact metrizable subset collection of $(Z\mathcal{O}_2)$ by Lemma 7 and $\mathcal{K}$ is a cover of $Z$. Let

$$\mathcal{O}_1 = \{A \subset Z;\ A \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K] \text{ for each } f^{-1}[K] \in \mathcal{K}\}.$$

We can prove these results:

A) $\mathcal{O}_2 \subset \mathcal{O}_1$ and $\mathcal{O}_1$ is a topology of $Z$.
In fact:

1) $O \in \mathcal{O}_2$ then $O \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$ for each $f^{-1}[K] \in \mathcal{K}$. This implies $O \in \mathcal{O}_1$. Then $X = \cup\mathcal{O}_2 \subset \cup\mathcal{O}_1 \subset X$.

2) If $A, B \in \mathcal{O}_1$, for each $f^{-1}[K], A \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$, then there exists $O_1 \in \mathcal{O}_2$ with $A \cap f^{-1}[K] = O_1 \cap f^{-1}[K]$. Also $B \cap f^{-1}[K] = O_2 \cap f^{-1}[K]$. Then $(A \cap B) \cap f^{-1}[K] = (O_1 \cap O_2) \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$, which implies $A \cap B \in \mathcal{O}_1$.

3) If $A_\alpha \in \mathcal{O}_1, \alpha \in \Lambda$, for each $f^{-1}[K] \in \mathcal{K}, A_\alpha \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$ then there exists $O_\alpha \in \mathcal{O}_2$ with $A_\alpha \cap f^{-1}[K] = O_\alpha \cap f^{-1}[K]$. Then $(\cup_{\alpha\in\Lambda}A_\alpha) \cap f^{-1}[K] = (\cup_{\alpha\in\Lambda}O_\alpha) \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$. This implies $\cup_{\alpha\in\Lambda}A_\alpha \in \mathcal{Q}_1$.

B) $\mathcal{O}_1|f^{-1}[K] = \mathcal{O}_2|f^{-1}[K]$ and $f^{-1}[K]$ is a $\mathcal{O}_1$-compact metric subset of $Z$ for each $f^{-1}[K] \in \mathcal{K}$.
In fact:

1) $\mathcal{O}_2 \subset \mathcal{O}_1$ then $\mathcal{O}_2|f^{-1}[K] \subset \mathcal{O}_1|f^{-1}[K]$. On the other hand $A \in \mathcal{O}_1|f^{-1}[K]$ then there exists $B \in \mathcal{O}_1$ with $A = B \cap f^{-1}[K]$. $B \in \mathcal{O}_1$ then $B \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$. Then there exists $O_2 \in \mathcal{O}_2$ with $B \cap f^{-1}[K] = O_2 \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K]$. This implies $B \cap f^{-1}[K] = A \in \mathcal{O}_2|f^{-1}[K]$.

2) $(Z\mathcal{O}_2)$ is a metric space. If $d$ is the metric, then $(f^{-1}[K]\, d)$ is a compact metric subspace of $(Z\mathcal{O}_1)$ by $\mathcal{O}_1|f^{-1}[K] = \mathcal{O}_2|f^{-1}[K]$.

C) $(Z\mathcal{O})$ is sequential.
In fact: $\mathcal{K}$ is a compact metric subset collection of $(Z\mathcal{O}_1)$ by B). On the other hand $\mathcal{O}_1 = \{A \subset Z;\ \text{for each } f^{-1}[K] \in \mathcal{K}, A \cap f^{-1}[K] \in \mathcal{O}_2|f^{-1}[K] = \mathcal{O}_1|f^{-1}[K]$ by B). This implies $(Z\mathcal{O}_1)$ is sequential.

D) $(Z\mathcal{O}_2)$ is totally disconnected. $\mathcal{O}_2 \subset \mathcal{O}_1$ then $(Z\mathcal{O}_1)$ is totally disconnected.

Above we have proven that $(Z\mathcal{O}_1)$ is a totally disconnected sequential space which is the union of countably many compact metric subsets of $(Z\mathcal{O}_1$. Now we prove that $f : (Z\mathcal{O}_1) \to (Y\mathcal{T}_1)$ is a continuous perfect map. In fact : K is a compact metric subset of $(Y\mathcal{T}_1)$ for each $f^{-1}[K] \in \mathcal{K}$. Let $O \subset K$, and let $O$ be open in $K$. Then $O \in \mathcal{T}_2|K$ by Lemma 7 $\mathcal{T}_1|K = \mathcal{T}_2|K$. We have known that $f : (Z\mathcal{O}_2) \to (Y\mathcal{T}_2)$ is a continuous perfect map. Then $f|f^{-1}[K] : f^{-1}[K] \to K$ is a continuous map. $(f|f^{-1}[K])^{-1}(0) \in \mathcal{O}_2|f^{-1}[K] = \mathcal{O}_1|f^{-1}[K]$ then $(f|f^{-1}[K])^{-1}(O)$ is an $\mathcal{O}_1|f^{-1}[K]$-open subset of $f^{-1}[K]$. If $O \in \mathcal{T}_1$, then for each $f^{-1}[K]$, $f^{-1}[K] \cap f^{-1}[O] = f^{-1}[K \cap O] = (f|f^{-1}[K])^{-1}[O \cap K]$ is $\mathcal{O}_1|f^{-1}[K]$-open in $f^{-1}[K]$. $(Z\mathcal{O}_1)$ is determined by $\mathcal{K}$ then $f^{-1}[O] \in \mathcal{O}_1$. This implies that $f : (Z\mathcal{O}_1) \to (Y\mathcal{T}_1)$ is a continuous map. On the other hand $f^{-1}(x) \in \mathcal{K}$ then $f^{-1}(x)$ is $\mathcal{O}_1$-compact. If we take a closed subset $B$ of $(Z\mathcal{O}_1)$ then $f^{-1}[K] \cap B$ is $\mathcal{O}_1$-compact. $f$ is $\mathcal{O}_1$-continuous so $f[f^{-1}[K] \cap B] = K \cap f[B]$ is a $\mathcal{T}_1$-compact subset of $Y$. Then $K \cap f[B]$ is $\mathcal{T}_1$-closed. $(Y\mathcal{T}_1)$ is sequential hence $f[B]$ is $\mathcal{T}_1$-closed. This implies that $f : (Z\mathcal{O}_1) \to (Y\mathcal{T}_1)$ is a continuous perfect map.

**Lemma 9.** *Let $Y$ be a quotient s-image of a metric space, and let $S_\omega \times Y$ be a sequential space. Suppose that there exists $\mathcal{P}' = \{P_n \in \mathcal{P};\ n < \omega\}$ and $\{S_n;\ n < \omega\}$ such that for each convergent sequence $S \cup \{x\}$ of $Y$ and each $P_n \in \mathcal{P}'$, if $|(S \cup \{x\}) \cap P_n| < \aleph_0$ then $(S \cup \{x\}) \setminus (\cup\{S_n : n < \omega\})$ is finite. Then $Y$ is a $k_\omega$-space.*

*Proof.* Let $\mathcal{P}'' = \{P \in \mathcal{P};\ P \cap (\cup_{n<\omega} S_n) \neq \emptyset\ \}$ then $|\mathcal{P}''| \leq \aleph_0$ since $\mathcal{P}$ point-countable and $|\cup_{n<\omega} S_n| \leq \aleph_0$. Suppose that $B$ is not closed then there exists a convergent sequence $S \cup \{x\}$ such that $(S \cup \{x\}) \cap B$ is not closed in $S \cup \{x\}$. We may assume $S \subset B$ without loss of generality. If there exists a $P_n \in \mathcal{P}'$ with $|P_n \cap S| = \aleph_0$, then $\overline{P}_n \cap B$ is not closed in $\overline{P}_n$. If $|P_n \cap S| < \aleph_0$ for each $P_n \in \mathcal{P}'$, then $S \cup \{x\} \setminus (\cup\{S_n : n < \omega\})$ is finite. $S$ is not closed then there exists a $P$ such that $P \cap S$ is not closed in $P$. This $P \in \mathcal{P}''$ and $B \cap \overline{P}$ are not closed in $\overline{P}$. We prove that $Y$ is determined by $\{\overline{P};\ P \in \mathcal{P}'$ or $P \in \mathcal{P}''\}$. Then $Y$ is a $k_\omega$-space by Lemma 4.

**Lemma 10.** *Let $f : Z \to Y$ be a perfect map. If $Z$ is sequential then $S_\omega \times Z$ is sequential if and only if $S_\omega \times Y$ is sequential.*

*Proof.* Let $I_{s_\omega} : S_\omega \to S_\omega$ be an identity map. Then $I_{S_\omega} \times f : S_\omega \times Z \to S_\omega \times Y$ is a perfect map. If $S_\omega \times Z$ is sequential then $S_\omega \times Y$ is sequential. On the other hand: If $S_\omega \times Y$ is sequential then $S_\omega \times Z$ is a k-space. Since each compact subset of $S_\omega \times Z$ is a metric subset then $S_\omega \times Z$ is sequential.

**Remark.** In order to show the following Lemma 11, we shall use the assumption MC( there exists measurable cardinal) and a technique of [2]. The author does not know whether the assumption MC of Lemma 11 can be omitted. We say that $C$ is a Cantor set if $C$ is homeomorphic to $\{0,1\}^\omega$.

**Lemma 11. (CH + MC).** *Let $Y$ be a k-space with a point-countable closed k-network $\{P_\alpha;\ \alpha \in \Lambda\}$. If $S_\omega \times Y$ is a k-space, then $Y$ is a locally $k_\omega$-space.*

*Proof.* We may assume $Y$ is a quotient s-image of a metric space by [4, Theorem 6.1]. Firstly; each point $y$ of $Y$ has a neighborhood $U$ which is the union of countably many compact metric subsets of $Y$ by Lemma 5. Let $U$ be a closed neighborhood. $S_\omega \times Y$ is a k-space then $S_\omega \times Y$ is sequential. Hence $S_\omega \times U$ is a sequential subspace. Without loss of generality, let $Y = U = \cup_{n<\omega} P_n$. We prove that $Y$ is a $k_\omega$-space.

Suppose that $Y$ is not a $k_\omega$-space.

Let $S_0 = \emptyset$, $\mathcal{P}_0{}' = \{P_n;\ \cup_{n<\omega} P_n = Y\}$, then there exists a sequence $S_1$ converging to $x_1$ and $|P_n \cap \overline{S}_1| < \aleph_0$ for each $P_n \in \mathcal{P}_0'$. Here $\overline{S}_1 = S_1 \cup \{x_1\}$. Let $\mathcal{P}_1' = \{P \in \mathcal{P};\ P \cap \overline{S}_1 \neq \emptyset\}.|\mathcal{P}_1'| \leq \aleph_0$ since $\mathcal{P}$ point-countable. Let $\alpha < \omega_1$. Suppose that for each $\beta < \alpha$ we have taken $S_\beta$ such that

1) $S_\beta$ converges to $x_\beta$, $x_\beta \notin S_\beta$ and $\overline{S}_\beta = S_\beta \cup \{x_\beta\}$.

2) $|P \cap S_\beta| < \aleph_0$ for each $P \in \cup_{\delta<\beta}\mathcal{P}_\delta$.

3) $\mathcal{P}_\beta{}' = \{P \in \mathcal{P};\ P \cap \overline{S}_\beta \neq \emptyset\}$. Because of $|\cup_{\beta<\alpha} \mathcal{P}_\beta{}'| \leq \aleph_0$ $|\{S_\beta;\ \beta < \alpha\}| \leq \aleph_0$ and Lemma 9 there exists sequence $S_\alpha$ converging to $x_\alpha$ such that 1) $|P \cap \overline{S}_\alpha| < \aleph_0$ for each $P \in \cup_{\beta<\alpha}\mathcal{P}_\beta{}'$. 2) $S_\alpha \cap (\cup_{\beta<\alpha} S_\beta) = \emptyset$. Let $\mathcal{P}_\alpha' = \{P \in \mathcal{P}\,;\, P \cap \overline{S}_\alpha \neq \emptyset\}$ then $|\mathcal{P}_\alpha{}'| \leq \aleph_0$. Then, by induction, there exists a collection $\varphi' = \{S_\alpha;\ \alpha < \omega_1\}$ such that:

$(C^*)$ :

1) $S_\alpha$ converges to $x_\alpha, x_\alpha \notin S_\alpha$ and $\overline{S}_\alpha = S_\alpha \cup \{x_\alpha\}$ for each $S_\alpha \in \varphi'$.

2) $|S_\alpha \cap P| < \aleph_0$ for each $P \in \cup_{\beta<\alpha} \mathcal{P}_\beta{}'$.

3) If $\beta < \alpha$ then $S_\beta \cap S_\alpha = \emptyset$.

4) Let $E = \{x_\alpha;\ S_\alpha$ converges to $x_\alpha$ and $\alpha < \omega_1\}$. Then $|E| = \aleph_1$, so we may assume $x_\beta \neq x_\alpha$ for $\beta < \alpha$.

In fact: If $|E| = \aleph_0$ then there exists an $x_{\alpha_0}$ with $|\{\alpha_\beta : x_{\alpha_\beta} = x_{\alpha_0}, x_{\alpha_\beta} \in E\}| = \aleph_1$. $S_{\alpha_0}$ is not closed, so there exists a $P_{\alpha_0} \in \mathcal{P}$ such that $P_{\alpha_0} \cap S_{\alpha_0}$ is not closed in $P_{\alpha_0}$. Then $|P_{\alpha_0} \cap S_{\alpha_0}| = \aleph_0$ and the $x_{\alpha_0} \in P_{\alpha_0} \in \mathcal{P}'_{\alpha_0}$. Let $\beta < \omega_1$. For each $\delta < \beta$, $P_{\alpha_\delta} \cap S_{\alpha_\delta}$ is not closed in $P_{\alpha_\delta}$, $|P_{\alpha_\delta} \cap S_{\alpha_\delta}| = \aleph_0$ and $x_{\alpha_0} = x_{\alpha_\delta} \in P_{\alpha_\delta} \in \mathcal{P}'_{\alpha_\delta}$. Since $S_{\alpha_\beta}$ is not closed, then there exists a $P \in \mathcal{P}$ such that $P \cap S_{\alpha_\beta}$ is not closed in $P$. Then $|P \cap S_{\alpha_\beta}| = \aleph_0$ and $x_{\alpha_0} = x_{\alpha_\beta} \in P_{\alpha_\beta} \in \mathcal{P}'_{\alpha_\beta}$. Since for each $\delta < \beta, |P_{\alpha_\delta} \cap S_{\alpha_\delta} = \aleph_0$ hence $P_{\alpha_\delta} \in \mathcal{P}'_{\alpha_\delta}$ and $\mathcal{P}'_{\alpha_\delta} \subset \cup_{\delta'<\alpha_\beta} \mathcal{P}'_{\delta'}$. Then $|P_{\alpha_\delta} \cap S_{\alpha_\beta}| < \aleph_0$ and $P_{\alpha_\beta} \notin \{P_{\alpha_\delta};\ \delta < \beta\}$ by $|P_{\alpha_\beta} \cap S_{\alpha_\beta}| = \aleph_0$. By induction there exists a collection $\{P_{\alpha_\beta};\ \beta < \omega_1\}$ such that $|\{P_{\alpha_\beta}; \beta < \omega_1\}| = \aleph_1$ and $x_{\alpha_0} \in \cap_{\beta<\omega_1} P_{\alpha_\beta}$. This contradicts the point-countable of $\mathcal{P}$. Thus we any assume $x_\beta \neq x_\alpha$ for $\beta < \alpha$. The following shows that we may assume $E$ contains a Cantor set C.

As $Y = \cup_{n<\omega} P_n$ , by Lemma 8, there exists a sequential space $Z$ which is the union of countable many compact metric subsets of $Z$ and there exists a perfect map $f : Z \to Y$ such that $f^{-1}[Y] = Z$ is determined by $\{f^{-1}[P];\ P \in \mathcal{P}\}$. $\{f^{-1}[P]; P \in \mathcal{P}\}$ is a point-countable cover of $Z$ and $f^{-1}[\overline{P}] = \overline{f^{-1}[P]}$ is compact. $Y$ is not a $k_\omega$-spcae, so $Z$ is not a $k_\omega$-space. Analogously we can prove that there exists a collection $\{S_\alpha;\ \alpha < \omega_1\}$ with property $(C^*)$. $f^{-1}(\overline{P}_0)$ is a totally disconnected compact metrizable subset of $Z$ and $|f^{-1}[\overline{P}]_0 \cap E| = \aleph_1$ so there exists a Cantor set $C$ with $|C \cap E \cap f^{-1}[\overline{P}_0]| = \aleph_1$. MC implies that every uncountable subset of Cantor set contains a Cantor set. Then $E$ contains a Cantor set C. By Lemma 10, $S_\omega \times Y$ is sequential if and only if $S_\omega \times Z$ is sequential. Then we amy assume $E$ contains a Cantor set $C$. $Y$ is a submetric space, then there exists an $(Y\mathcal{T}_2)$ which is a metric space with $\mathcal{T}_2 \subset \mathcal{T}_1$. Here $(Y\mathcal{T}_1)$ is a sequential space. Let $d$ be a metric for $(Y\mathcal{T}_2)$. Then $d$-open ball of $(Y\mathcal{T}_2)$ is open in $(Y\mathcal{T}_1)$ and every $\mathcal{T}_1$-compact set is a $\mathcal{T}_2$-compact set by Lemma 7.

A) Let $C = \{x_\alpha;\ \alpha < \omega_1\}$ be a Cantor subset of $E$. Let $\varphi = \{S_\alpha \in \varphi';\ S_\alpha$ converges to $x_\alpha$ and $x_\alpha \in C\}$.

Since $C$ is a compact metric subset and $\mathcal{P}$ is a point-countable collection, there are only countably many $P$ with $O_C(P \cap C) \neq \emptyset$. Here $O_C(P \cap C)$ denotes the interior of $P \cap C$ in the subspace $C$. Let $\mathcal{P}'_0 = \{P \in \mathcal{P}\ ;\ O_C(P \cap\ C) \neq \emptyset\} = \{P_n;\ n < \omega\}$. Then for each $P \in \mathcal{P} \setminus \mathcal{P}'_0$, $O_C(P \cap C) = \emptyset$ that is, $P \cap C$ is nowhere dense in $C$ since $\mathcal{P}$ is a collection of closed subsets of $Y$. Now pick a $y_0 \in C$. Let $\mathcal{P}_0 = \{P \in \mathcal{P} \setminus \mathcal{P}'_0\ ;\ y_0 \in P\} = \{P_{1n};\ n < \omega\}$. If $y_\beta$ and $\mathcal{P}_\beta$ have been defined for all $\beta < \alpha$, where $\alpha < \omega_1$, pick a $y_\alpha \in C \setminus [\cup(\cup_{\beta<\alpha} \mathcal{P}_\beta\ )]$. Because the Cantor set $C$ can not be denoted as the union of countably many nowhere dense subsets of $C$. Let $\mathcal{P}_\alpha = \{P \in \mathcal{P} \setminus \mathcal{P}'_0;\ y_\alpha \in P\} = \{P_{\alpha n};\ n < \omega\}$. Then, by induction, there exists a subset $A = \{y_\alpha;\ \alpha < \omega_1\}$ of $C$ with $|A| = \aleph_1$. Then there exists a Cantor set $C_0$ of $A$ with $|P \cap C_0| \leq 1$ for each $P \in \mathcal{P} \setminus \mathcal{P}'_0$. If $P_n \in \mathcal{P}'_0$, then at most there exists one $S_{\alpha_n} \in \varphi$ with $|S_{\alpha_n} \cap P_n| = \aleph_0$ by 2) of $(C^*)$. Let $\alpha_0 = sup\{\alpha_n;\ |S_{\alpha_n} \cap P_n| = \aleph_0$ and $P_n \in \mathcal{P}'_0\}$. If $\alpha > \alpha_0$ then $|S_\alpha \cap P_n| < \aleph_0$. Thus we may assume without loss of generality that

1) $|P \cap C| \leq 1$ for each $P \in \mathcal{P} \setminus \mathcal{P}'_0$.

2) $|P_n \cap S_\alpha| < \aleph_0$ for each $S_\alpha \in \varphi$ and each $P_n \in \mathcal{P}'_0$.

B) Let $C(0)$ and $C(1)$ be two Cantor subsets of $C$ with $C(0) \cap C(1) = \emptyset$. Let $d(C(0), C(1)) = r_1 > 2/n_1$, and let $V(\delta_1)$ be a $1/n_1$ open ball of $C(\delta_1)$ in $(Y\mathcal{T}_2)$. Let $\varphi(\delta_1) = \{(S_\alpha \cap V(\delta_1)) \setminus (P_1 \cup C_0);\ S_\alpha \in \varphi,\ S_\alpha$ converges to $x_\alpha$ and $x_\alpha \in C(\delta_1)\}$. Here $P_1 \in \mathcal{P}_0$, $\delta_1 = 0, 1$. If $x_\alpha \in C(\delta_1)$, then $(S_\alpha \cap V(\delta_1)) \setminus (P_1 \cup C_0)$ is a sequence which converges to $x_\alpha$. Then $|\varphi(\delta_1)| = \aleph_1$ and $\cup\varphi(\delta_1) \subset V(\delta_1) \setminus (P_1 \cup C_0)$. $V(\delta_1) \setminus (P_1 \cup C_0)$ is an open subset of $Y$ which is the union of countably many compact metric subsets of $Y$. Then there exists a compact metric subset $K(\delta_1)$ which is a subset of $V(\delta_1) \setminus (P_1 \cup C_0)$ such that $K(\delta_1)$ meets $\aleph_1$-many sequences in $\varphi(\delta_1)$. Let

$$D(\delta_1) = \{x_\alpha \in C(\delta_1);\ S_\alpha \in \varphi(\delta_1),\ \ S_\alpha \cap K(\delta_1) \neq \emptyset \text{ and } S_\alpha \text{ converges to } x_\alpha\},$$

then $|D(\delta_1)| = \aleph_1$.

If it has been defined that :

a) $\{C(\delta_1\delta_2 \ldots \delta_n);\ \delta_i = 0, 1.\ i = 1, 2, \ldots, n\}$ is a collection of mutually disjoint Cantor sets.

b) $\{V(\delta_1\delta_2 \ldots \delta_n);\ \delta_i = 0, 1.\ i = 1, 2, \ldots, n\}$ is a collection of mutually disjoint open balls such that

$$V(\delta_1\delta_2 \ldots \delta_n) \cap [\cup_{j \leq n-1}(\cup\{K(\delta_1\delta_2 \ldots \delta_j);\ \delta_i = 0, 1.\ i = 1, 2, \ldots, j\})] = \emptyset.$$

Here $V(\delta_1\delta_2 \ldots \delta_n) = \{y;\ d(y,\ C(\delta_1\delta_2 \ldots \delta_n)) < 1/n_n\}$.

c) $\{\varphi(\delta_1\delta_2 \ldots \delta_n);\ \delta_i = 0, 1, .\ i = 1, 2, \ldots, n\}$ such that $\varphi(\delta_1 \ldots \delta_n) = \{(S_\alpha \cap V(\delta_1 \ldots \delta_n)) - (P_1 \cup \ldots \cup P_n \cup C_0);\ S_\alpha \in \varphi(\delta_1 \ldots \delta_{n-1}),\ S_\alpha$ converges to $x_\alpha$ and $x_\alpha \in C(\delta_1 \ldots \delta_n)\}$ and $|\varphi(\delta_1 \ldots \delta_n)| = \aleph_1$, here $P_i \in \mathcal{P}'_0 = \{P_n \in \mathcal{P};\ O_C(P_n \cap C) \neq \emptyset\}$.

d) $\{K(\delta_1 \ldots \delta_n);\ \delta_i = 0, 1.\ i = 1, 2, \ldots, n\}$ is a collection of compact subsets such that $K(\delta_1 \ldots \delta_n)$ is a subset of open subset $V(\delta_1 \ldots \delta_n) \setminus (P_1 \cup \ldots \cup P_n \cup C_0)$ and $|\{S_\alpha \in \varphi(\delta_1 \ldots \delta_n);\ S_\alpha \cap K(\delta_1 \ldots \delta_n) \neq \emptyset\}| = \aleph_1$.

e) $\{D(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n\}$, where $D(\delta_1 \dots \delta_n) = \{x_\alpha \in C(\delta_1 \dots \delta_n);\ S_\alpha \in \varphi(\delta_1\delta_2 \dots \delta_n), S_\alpha$ converges to $x_\alpha$ and $S_\alpha \cap K(\delta_1\delta_2 \dots \delta_n) \neq \emptyset\} \subset C(\delta_1\delta_2 \dots \delta_n)$.

Take $C(\delta_1 \dots \delta_n 0)$ and $C(\delta_1 \dots \delta_n 1)$ which are two disjoint Cantor subsets of $D(\delta_1 \dots \delta_n)$. By e), $K(\delta_1 \dots \delta_n)$ and $C_0$ are $\mathcal{T}_1$-compact, thus $K(\delta_1 \dots \delta_n)$ and $C_0$ are $\mathcal{T}_2$-compact. Then $d(C, \cup_{j \le n}[\cup\{K(\delta_1 \dots \delta_j);\ \delta_i = 0,1.\ i = 1,2,\dots,j\}]) = r_{n+1} > 0$ by b) and d) of B). Choose $n_{n+1}$ with $(\frac{2}{n_{n+1}}) < r_{n+1}$ such that $V(\delta_1 \dots \delta_{n+1}) = \{y;\ d(y, C(\delta_1 \dots \delta_{n+1})) < \frac{1}{n_{n+1}}\} \subset \overline{V(\delta_1 \dots \delta_{n+1})} \subset V(\delta_1\delta_2 \dots \delta_n)$ and $V(\delta_1 \dots \delta_n 0) \cap V(\delta_1 \dots \delta_n 1) = \emptyset$, then $\{V(\delta_1 \dots \delta_{n+1});\ \delta_i = 0,1.\ i = 1,2,\dots,n+1\}$ is a collection of mutually disjoint open balls. Let $\varphi(\delta_1 \dots \delta_{n+1}) = \{(S_\alpha \cap V(\delta_1 \dots \delta_{n+1})) - (P_1 \cup \dots \cup P_{n+1} \cup C_0);\ S_\alpha \in \varphi(\delta_1 \dots \delta_n),\ S_\alpha$ converges to $x_\alpha$ and $x_\alpha \in C(\delta_1 \dots \delta_n)\}$, where $P_{n+1} \in \mathcal{P}_0$. Then $(S_\alpha \cap V(\delta_1 \dots \delta_{n+1})) \setminus (P_1 \cup \dots \cup P_{n+1} \cup C_0)$ is a sequence which converges to $x_\alpha$ for each $x_\alpha \in C(\delta_1\delta_2 \dots \delta_{n+1})$ and $|\varphi(\delta_1 \dots \delta_{n+1})| = \aleph_1$. $V(\delta_1 \dots \delta_{n+1}) \setminus (P_1 \cup \dots \cup P_{n+1} \cup C_0)$ is an open subset of $Y$ which is the union of countably many compact metric subsets of $Y$, so there exists a compact subset $K(\delta_1\delta_2 \dots \delta_{n+1}) \subset V(\delta_1\delta_2 \dots \delta_{n+1}) \setminus (P_1 \cup \dots \cup P_{n+1} \cup C_0)$ such that $K(\delta_1 \dots \delta_{n+1})$ meets $\aleph_1$ many sequences in $\varphi(\delta_1\delta_2 \dots \delta_{n+1})$. If $D(\delta_1 \dots \delta_{n+1}) = \{x_\alpha \in C(\delta_1 \dots \delta_{n+1});\ S_\alpha \in \varphi(\delta_1 \dots \delta_{n+1}),\ S_\alpha \cap K(\delta_1 \dots \delta_{n+1}) \neq \emptyset$ and $S_\alpha$ converges to $x_\alpha\}$, then $D(\delta_1 \dots \delta_{n+1}) \subset C(\delta_1 \dots \delta_{n+1}) \subset D(\delta_1 \dots \delta_n)$ and $|D(\delta_1 \dots \delta_{n+1})| = \aleph_1$. Then, by induction, there exist:

1) $\mathcal{K} = \{K(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n.\ n > 0.\}$ with $[\cup\{K(\delta_1 \dots \delta_j);\ \delta_i = 0,1.\ i = 1,2,\dots,j.\ j \ge n\}] \cap P_n = \emptyset$ for each $P_n \in \mathcal{P}_0$.

2) $\mathcal{C} = \{C(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n.\ n > 0\}$ and $\{V(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n.\ n > 0\}$ such that $\cap_{n>0}(\cup\{C(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n\}) = \cap_{n>0}(\cup\{V(\delta_1 \dots \delta_n);\ \delta_i = 0,1.\ i = 1,2,\dots,n.\}) = C^*$ and $C^*$ is a Cantor set. If $C^* = \{x(\delta_1\delta_2 \dots);\ \delta_i = 0,1.\ i = 1,2,\dots,n.\ n > 0\} = \{x_\alpha;\ \alpha < \omega_1\}$, then $x(\delta_1\delta_2 \dots \delta_n \dots) = C(\delta_1) \cap C(\delta_1\delta_2) \cap \dots = V(\delta_1) \cap V(\delta_1\delta_2) \cap \dots$ and $|C^* \cap P| \le 1$ for each $P \in \mathcal{P} \setminus \mathcal{P}_0$ .

3) If $\varphi_0 = \{S_\alpha \in \varphi;\ S_\alpha$ converges to $x_\alpha$ and $x_\alpha \in C^*\}$,

then $|\varphi_0| = \aleph_1$. If $S_\alpha \in \varphi_0$, $S_\alpha$ converges to $x_\alpha$ and $x_\alpha = x(\delta_1\delta_2\dots) \in C^*$, then $S_\alpha \cap K(\delta_1\dots\delta_n) \neq \emptyset$ for $n > 0$.

C) We use the results of B) to construct a collection $\{C_{nm};\ n, m < \omega\}$ which does not satisfy condition $(C_3)$.

i) Let $A_1 = \{x(00\dots)\}$, and let $\mathcal{C}_1 = \{P \in \mathcal{P} \setminus \mathcal{P}_0\ ;\ P \cap \overline{A}_1 \neq \emptyset\} = \{P_{1n};\ n < \omega\}$. For each $n < \omega$; let $\mathcal{K}(0\dots0\overset{n}{1}) = \{K(0\dots0\overset{n}{1}\ \delta_1\delta_2\dots\delta_m);\ K(0\dots0\overset{n}{1}\ \delta_1\dots\delta_m) \subset V(0\dots0\overset{n}{1})\}$ thus $(P_{11}\cup\dots\cup P_{1n})$ meets only finitely many sets in $\mathcal{K}(0\dots0\overset{n}{1})$.

In fact $P_{11}\cup\dots\cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\overset{n}{1})$, so $P_{11}\cup\dots\cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\overset{n}{1}\ 0)$ or in $\mathcal{K}(0\dots0\overset{n}{1}\ 1)$ by $\mathcal{K}(0\dots0\overset{n}{1}) = \{K(0\dots01)\} \cup \mathcal{K}(0\dots0\overset{n}{1}\ 0) \cup \mathcal{K}(0\dots0\overset{n}{1}\ 1)$. Then we may assume $P_{11} \cup \dots \cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\overset{n}{1}\ \delta_1)$. Because $\mathcal{K}(0\dots0\overset{n}{1}\ \delta_1) = \{K(0\dots0\ \overset{n}{1}\ \delta_1)\} \cup \mathcal{K}(0\dots0\ \overset{n}{1}\ \delta_1 0) \cup \mathcal{K}(0\dots0\ \overset{n}{1}\ \delta_1 1)$, then $P_{11} \cup \dots \cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\overset{n}{1}\ \delta_1 0)$, or in $\mathcal{K}(0\dots0\ \overset{n}{1}\ \delta_1 1)$. We may assume $P_{11} \cup \dots \cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\ \overset{n}{1}\ \delta_1\delta_2)$. Then, by induction, there exist $\mathcal{K}(0\dots0\overset{n}{1}\ \delta_1) \supset \mathcal{K}(0\dots0\overset{n}{1}\ \delta_1\delta_2) \supset \dots$ such that $P_{11} \cup \dots \cup P_{1n}$ meets infinitely many sets in $\mathcal{K}(0\dots0\overset{n}{1}\ \delta_1\dots\delta_m)$. $\cup\, \mathcal{K}(0\dots0\ \overset{n}{1}\ \delta_1\dots\delta_m) \subset V(0\dots0\ \overset{n}{1}\ \delta_1\dots\delta_m)$ so $(P_{11} \cup \dots \cup P_{1n}) \cap V(0\dots0\overset{n}{1}\ \delta_1\dots\delta_m) \neq \emptyset$. $\cap_{m>0} V(0\dots0\overset{n}{1}\ \delta_1\dots\delta_m) = \{x(0\dots0\ \overset{n}{1}\ \delta_1\delta_2\dots)\}$ so $x(0\dots0\ \overset{n}{1}\ \delta_1\delta_2\dots) \in \overline{P_{11} \cup \dots \cup P_{1n}} = P_{11} \cap \dots P_{1n}$. This is a contradiction to $P_{1i} \in \mathcal{C}_1$ and $|P_{1i} \cap C^*| \leq 1$ for $i = 1, 2, \dots, n$. Then there exists an $n_n$ and $\mathcal{K}_{1n} = \{K(0\dots0\overset{n}{1}\ \delta_1\dots\delta_n);\ \delta_i = 0, 1.\ i = 1, 2, \dots n_n\}$ such that:

(1) $(P_{11} \cup \dots \cup P_{1n}) \cap (\cup\mathcal{K}_{1n}) = \emptyset$.
(2) If $S_\alpha$ converges to $x_\alpha$, $S_\alpha \in \varphi_0$ and $x_\alpha = x(0\dots0\overset{n}{1}\ \delta_1\delta_2, \dots) \notin \overline{A}_1$ then $S_\alpha \cap (\cup\mathcal{K}_{1n}) \neq \emptyset$.

Let $\mathcal{K}_1 = \cup_{n>0}\mathcal{K}_{1n}$, then

(1) $P$ meets only finitely many sets in $\mathcal{K}_1$ for each $P \in \mathcal{C}_1$.
(2) $P$ meets only finitely many sets in $\mathcal{K}_1$ for each $P \in \mathcal{P}$.

(3) If $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(0\ldots0\overset{n}{1}\,\delta_1\delta_2,\ldots)$ and $x_\alpha \notin \overline{A}_1$, then $S_\alpha \cap (\cup\mathcal{K}_{1n}) \neq \emptyset$ for each $S_\alpha \in \varphi_0$.

In fact :

1) We omit the proof.

2) For each $P \in \mathcal{P}$.

a) if $P \cap C^* = \emptyset$, then there exists an $n$ with $P \cap (\cup \{V(\delta_1\ldots\delta_n);\ \delta_i = 0,1.\ i = 1,2,\ldots,n\}) = \emptyset$ by $C^* = \cap_{n>0}(\cup \{V(\delta_1\ldots\delta_n);\ \delta_i = 0,1.\ i = 1,\ldots,n\})$. Then $P$ meets only finitely many sets in $\mathcal{K}_1$.

b) if $P \cap C^* \neq \emptyset$ and $P \in \mathcal{P}_0$, then $P$ meets only finitely many sets in $\mathcal{K}_1$. If $P \notin \mathcal{P}_0$, then $|P \cap C^*| \leq 1$. Suppose that there exists a infinite subcollection $\{K(0\ldots0\overset{n}{1}\,\delta_{n1}\ldots\delta_{nn_n});\ n < \omega\}$ of $\mathcal{K}_1$ such that $P \cap K(0\ldots0\overset{n}{1}\,\delta_{n1}\ldots\delta_{nn_n}) \neq \emptyset$ for each $K(0\ldots0\overset{n}{1}\,\delta_{n1}\ldots\delta_{nn_n})$. Because $K(0\ldots0\overset{n}{1}\,\delta_{n1}\ldots\delta_{nn_n}) \subset V(0\ldots\overset{n-1}{0}), P \cap V(0\ldots\overset{n-1}{0}) \neq \emptyset$. $V(0) \supset V(00) \supset \ldots$ and $\cap_{n>0}V(0\ldots\overset{n}{0}) = \{x(00\ldots)\}$ so $x(00\ldots) \in \overline{P} = P$ and $P \in \mathcal{C}_1$. This is a contradiction to 1).

3) We omit the proof.

If $C_{1n} = \cup_{i\geq n}(\cup\mathcal{K}_{1i})$, then $\{C_{1n};\ 1 \leq n < \omega\}$ has the properties:

1) $P$ meets only finitely many sets in $\{C_{1n};\ 1 \leq n < \omega\}$ for each $P \in \mathcal{P}$.

2) If $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(0\ldots0\overset{n}{1}\,\delta_1,\delta_2\ldots)$ and $x_\alpha \notin \overline{A}_1$, then $S_\alpha \cap C_{1n} \neq \emptyset$ for each $S_\alpha \in \varphi_0$.

ii) If $A_{n-1} = \{x(\delta_1,\delta_2\ldots);$ there exist $n-2$ many 1's in $\delta_1,\delta_2,\ldots\}$ and $\{C_{n-1\ m};\ n-1 \leq m < \omega\}$ have been defined so that:

1) $P$ meets only finitely many sets in $\{C_{n-1\ m};\ n-1 \leq m < \omega\}$ for each $P \in \mathcal{P}$.

2) For each $S_\alpha \in \varphi_0$, if $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}'\ldots\delta'_{m'}, 10\ldots0\overset{m'+j+1}{1}\,\delta_1\delta_2\ldots) \notin \overline{A}_{n-1}, x'_\alpha = x(\delta_1{}'\ldots\delta'_{m'}100\ldots) \in A_{n-1}$ and $j \geq m$, then $S_\alpha \cap C_{n-1\ m} \neq \emptyset$.

Let $A_n = \{x(\delta_1\delta_2\ldots)$; there exist $n-1$ many 1's in $\delta_1,\delta_2\ldots\}$ then $\overline{A}_n = A_n \cup \ldots \cup A_2 \cup A_1$ and $|\overline{A}_n| = \aleph_0$. Let $\overline{A}_n =$

$\{x_1, x_2, \dots\}$, and let $\mathcal{C}_n = \{P \in \mathcal{P} \setminus \mathcal{P}_0';\ P \cap \overline{A}_n \neq \emptyset\} = \{P_{ni};\ i < \omega\}$. Pick $x_m \in \overline{A}_n$. Then $x_m = x(\delta_1{}'\delta_2{}'\dots) = x(\delta_1{}'\dots\delta'_{m'}100\dots)$.

For each $j < \omega$, let $\mathcal{K}(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}) = \{K(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}\delta_1\dots\delta_n);\ K(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}\delta_1\dots\delta_n) \subset V(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1})\}$. Take $P_{ni} \in \mathcal{C}_n$, $i = 1, 2, \dots, m+j+1$. Then, as in the proof of i), there exists an n(j) and a subcollection $\mathcal{K}_{nj}(x_m) = \{K(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}\delta_{n1}\dots\delta_{n\,n(j)});\ \delta_{ni} = 0,1.\ i = 1, 2, \dots, n(j)\}$ of $\mathcal{K}(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1})$ such that;

1) $(P_{n1} \cup \dots \cup P_{m+j+1}) \cap (\cup\mathcal{K}_{nj}(x_m)) = \emptyset$.

2) For each $S_\alpha \in \varphi_0$, if $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}\delta_1\delta_2\dots) \notin \overline{A}_n$ and $x'_\alpha = x(\delta_1{}'\dots\delta'_{m'}100\dots) = x_m \in A_n$, then $S_\alpha \cap (\cup\mathcal{K}_{nj}(x_m)) \neq \emptyset$.

Let $\mathcal{K}_n(x_m) = \cup_{j>0}\mathcal{K}_{nj}(x_m)$. Then $\mathcal{K}_n(x_m)$ satisfies:

1) $(P_{n1} \cup \dots \cup P_{n\ m+j}) \cap [\cup\mathcal{K}_{nj}(x_m)] = \emptyset$, here $P_{ni} \in \mathcal{C}_n$ for $i = 1, 2, \dots, m+j$.

2) $P$ meets only finitely many sets in $\mathcal{K}_n(x_m)$ for each $P \in \mathcal{P}$.

3) For each $S_\alpha \in \varphi_0$, if $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j+1}{1}\delta_1\delta_2\dots) \notin \overline{A}_n$ and $x'_\alpha = x(\delta_1{}'\dots\delta'_{m'}100\dots) = x_m \in A_n$, then $S_\alpha \cap (\cup\mathcal{K}_{nj}(x_m)) \neq \emptyset$.

We prove only 2). In fact, if $P \in \mathcal{P} \setminus \mathcal{P}_0$', suppose that $P$ meets infinitely many sets in $\mathcal{K}_n(x_m)$. Then there exist infinitely many $\mathcal{K}_{nj}(x_m)$ such that $P$ meets sets in $\mathcal{K}_{nj}(x_m)$ because $\mathcal{K}_n(x_m) = \cup_{j>0}\mathcal{K}_{nj}(x_m)$ and $|\mathcal{K}_{nj}(x_m)| < \aleph_0$. $\cup\mathcal{K}_{nj}(x_m) \subset V(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j}{0})$ so $P \cap V(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j}{0}) \neq \emptyset$. Then $\{x_m\} = \cap_{j>0}V(\delta_1{}'\dots\delta'_{m'}10\dots0\overset{m'+j}{0}) \subset P$ and $P \in \mathcal{C}_n$. This is a contradiction to 1).

Let $\mathcal{K}_n = \cup\{\mathcal{K}_n(x_m);\ x_m \in \overline{A}_n\}$. Then $\mathcal{K}_n$ satisfies:

1) $P_{nm} \cap [\cup\cup\{\mathcal{K}_{nj}(x_i);\ j \geq m,\ i > 0\}] = \emptyset$ for each $P_{nm} \in \mathcal{C}_n$.

2) $P$ meets only finitely many sets in $\mathcal{K}_n$ for each $P \in \mathcal{P}$.

3) For each $S_\alpha \in \varphi_0$, $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}' \dots \delta'_{m'} 10 \dots 0 \overset{m+j+1}{1} \delta_1\delta_2 \dots) \notin \overline{A}_n$, $x'_\alpha = x(\delta_1{}' \dots \delta'_{m'} 100 \dots) = x_m \in A_n$ and $j \geq m$ so $S_\alpha \cap [\cup \cup \{\mathcal{K}_{nj}(x_i);\ j \geq m,\ i > 0\}] \neq \emptyset$.

In fact:

1) for each $P \in \mathcal{P} \setminus \mathcal{P}_0$, if $j \geq m$, then $P_{nm} \cap (\cup \mathcal{K}_{nj}(x_i)) = \emptyset$ for $i > 0$ by $(P_{n1} \cup \dots \cup P_{n\ m+j}) \cap (\cup \mathcal{K}_{nj}(x_m)) = \emptyset$. Then $P_{nm} \cap [\cup(\cup_{j \geq m} \mathcal{K}_{nj}(x_i))] = \emptyset$ for $i > 0$. Then $P_{nm} \cap [\cup\{\cup \mathcal{K}_{nj}(x_i);\ j \geq m,\ i > 0\}] = \emptyset$.

2) Let $P \in \mathcal{P} \setminus \mathcal{P}_0$. If $P$ meets infinitely many sets in $\mathcal{K}_n = \cup\{\mathcal{K}_n(x_m);\ x_m \in A_m\}$, then there exists a sets $\{m_1, m_2, \dots\}$ such that $P$ meets sets in $\mathcal{K}_n(x_{m_i})$ for each $m_i$ by property 2) of $\mathcal{K}_n(x_{m_i})$. $\{x_{m_i};\ i > 0\} \subset A_n$, so we may assume $\{x_{m_i}; i > 0\}$ is a sequence which converges to $x$ and $x = x(\delta_1\delta_2 \dots) \in \overline{A}_n$. We may also assume $\{x_{m_i};\ i \geq n\} \subset V(\delta_1 \dots \delta_n)$, so $V_n(x_{m_i}) = V(\delta_1 \dots \delta_n)$ for each $i \geq n$. Here $x_{m_i} = x(\varepsilon_1\varepsilon_2 \dots), V(\varepsilon_1) \supset V(\varepsilon_1\varepsilon_2) \supset \dots$ and $V_n(x_{m_i}) = V(\varepsilon_1\varepsilon_2 \dots \varepsilon_n) = V(\delta_1\delta_2 \dots \delta_n)$ so $x_{m_i} = x(\delta_1 \dots \delta_n \delta_{n1} \dots \delta_{n\ n_i} 000 \dots)$, $\cup \mathcal{K}_n(x_{m_i}) \subset V(\delta_1\delta_2 \dots \delta_n)$ and $P \cap V(\delta_1\delta_2 \dots \delta_n) \neq \emptyset$. $P$ is compact, so $x(\delta_1\delta_2 \dots) \in P$ and $P \in \mathcal{C}_n$. This is a contradiction to property 1) of $\mathcal{K}_n$.

3) $S_\alpha \in \varphi_0$, $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}' \dots \delta'_{m'} 10 \dots 0 \overset{m'+j+1}{1} \delta_1\delta_2 \dots) \notin \overline{A}_n$ and $x'_\alpha = x(\delta_1{}' \dots \delta'_{m'} 1000 \dots) = x_n \in A_n$, so $S_\alpha \cap (\cup \mathcal{K}_{nj}(x_n)) \neq \emptyset$. If $j \geq m$ , then $S_\alpha \cap (\cup \cup \{\mathcal{K}_{nj}(x_i);\ j \geq m,\ i > 0\}) \neq \emptyset$.

Let $C_{nm} = \cup \cup \{P\mathcal{K}_{nj}(x_i);\ j \geq m,\ i > 0\}$ for $m \geq n$ . Then $\{C_{nm};\ m \geq n\}$ satisfies:

1) $P$ meets only finitely many sets in $\{C_{nm};\ m \geq n\}$ for each $P \in \mathcal{P}$.

2) For each $S_\alpha \in \varphi_0$, if $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(\delta_1{}' \dots \delta'_{m'} 10 \dots \overset{m'+j+1}{1} \delta_1\delta_2 \dots) \notin \overline{A}_n$, $x'_\alpha = x(\delta_1{}' \dots \delta'_{m'} 1000 \dots) \in A_n$ and $j \geq m$, then $S_\alpha \cap C_{nm} \neq \emptyset$.

Then, by induction, there exists a collection $\{C_{nm};\ n \leq m < \omega\}$ such that:

1) $C_{nm}$ is closed in $(Y\mathcal{T}_1)$ for each $C_{nm} \in \{C_{nm}; n \leq m < \omega\}$.

2) For each $n, \{C_{nm};\ m \geq n\}$ is a discrete collection.

Now we prove $C^* \cap \overline{(\bigcup\{C_{nm};\ m \geq f(n)\})} \neq \emptyset$ and $C^* \cap (\bigcup\{C_{nm}; m \geq f(n)\}) = \emptyset$ for each $f \in {}^\omega\omega$.

In fact: If $f \in {}^\omega\omega$, let $f(i) = n_i$ and let $0 < n_1 < n_2 < n_3 \ldots$. Pick $x_\alpha = x(0\ldots0 \overset{m_1}{1}\ 0\ldots0 \overset{m_2}{1}\ 0\ldots) \in C^*$, where $m_i = n_1 + n_2 + \ldots + n_i > i$. Take $S_\alpha \in \varphi_0$ such that $S_\alpha$ converges to $x_\alpha$, $x_\alpha = x(0\ldots0 \overset{m_1}{1}\ 0\ldots0 \overset{m_2}{1} \ldots 0 \overset{m_i}{1}\ 0\ldots) \notin \overline{A}_i$ and $x'_\alpha = x(0\ldots0 \overset{m_1}{1}\ 0\ldots0 \overset{m_2}{1} \ldots 0 \overset{m_{i-1}}{1}\ 000\ldots) = x_m \in A_i$, so $S_\alpha \cap (\bigcup\mathcal{K}_{in_i}(x_m)) \neq \emptyset$ and $S_\alpha \cap C_{in_i} \neq \emptyset$ since $C_{i\,n_i} = \underline{\bigcup\bigcup\{\mathcal{K}_{ij}(x_m);\ j \geq n_i,\ m > 0\}}$ for $i \geq 1$. Then $x_\alpha \in C^* \cap \overline{\bigcup\{C_{in_i};\ i < \omega\}} \neq \emptyset$, hence $x_\alpha \in C^* \cap \overline{\bigcup\{C_{nm};\ m \geq f(n)\}} \neq \emptyset$.

D) Let $g : Y \to Y/\{C^*\}$. Then $g$ is a continuous perfect map. Then $\{g[C_{nm}];\ m \geq n\}$ satisfies:

1) $g[C_{nm}]$ is closed for each $g[C_{nm}] \in \{g[C_{nm}];\ n \leq m < \omega\}$ in $Y/\{C^*\}$.

2) $\{g[C_{nm}];\ m < \omega\}$ is a discrete subcollection of $\{g[C_{nm}];\ m \geq n\}$ for each $n < \omega$.

3) There exists a $C^* \in \overline{\bigcup\{g[C_{nm}];\ m \geq n\}}$ with $C^* \in \overline{\bigcup\{g[C_{nm}];\ m \geq f(n)\}}$ for each $f \in {}^\omega\omega$.

Then $S_\omega \times (Y/\{C^*\})$ is not sequential by Lemma 6. Thus $S_\omega \times Y$ is not sequential by Lemma 10. This is a contradiction to the conclusion of Lemma 10. Then $Y$ is a $k_\omega$-space.

## Results (CH + MC).

**Theorem 1.** *Let $X$ and $Y$ be k-spaces with point-countable closed k-networks. Then $X \times Y$ is a k-space if and only if one of the three properties below holds:*

*a) $X$ and $Y$ have point-countable bases.*

*b) $X$ or $Y$ is locally compact.*

*c) $X$ and $Y$ are locally $k_\omega$-spaces.*

*Proof.* If $X$ contains a copy of $S_2$ and $X$ contains no copy of $S_\omega$, then the perfect image $X/\{S_0\}$ of $X$ contains a copy of $S_\omega$, where $S_0$ is the converging sequence of $S_2$ which has no isolated point. By Lemma 10, $(X/\{S_0\}) \times Y$ is sequential if and only if $X \times Y$ is sequential.

"only if":

1) If $X$ and $Y$ contain copies of $S_\omega$ or $S_2$. Then $X$ and $Y$ are locally $k_\omega$-spaces by lemma 11.

2) If $X$ contains a copy of $S_\omega$ or $S_2$, and $Y$ contains no copy of $S_\omega$ and $S_2$. $X \times Y$ is sequential then $S_\omega \times Y$ is sequential. Then $P$ is compact metrizable for each $P \in \mathcal{P}$ by Lemma 4. Then $Y$ has a point countable base by [11, Corollary 4.5]. $S_\omega$ is not strongly Fréchet then $Y$ is locally compact by [8, Theorem 1.1].

3) $X$ and $Y$ contain no copies of $S_\omega$ and $S_2$ . $Y \times X$ is sequential and $t(Y) \leq \omega$ then $Y$ satisfies $(C_1)$ or $X$ satisfies $(C_2)$ by Lemma 1.

At the same time $X \times Y$ is sequential and $t(X) \leq \omega$ so $X$ satisfies $(C_1)$ or $Y$ satisfies $(C_2)$ by Lemma 1. Then there exist four cases:

case 1. $X$ satisfies $(C_1)$ and $Y$ satisfies $(C_1)$. Then $X$ and $Y$ have point-countable bases by [5, Theorem 9.8].

case 2. $X$ satisfies $(C_1)$ and $X$ satisfies $(C_2)$. Then $X$ has a point-countable base by $(C_1)$. Then $X$ is locally compact by $(C_2)$.

case 3. $Y$ satisfies $(C_1)$ and $Y$ satisfies $(C_2)$. Same as case 2.

case 4. $X$ satisfies $(C_2)$ and $Y$ satisfies $(C_2)$. If $X$ satisfies $(C_2)$ then $P$ is a compact metrizable set for each $P \in \mathcal{P}$. Then $X$ has a point-countable base by [11, lemma 4.1]. So does $Y$.

"if" we omit the straightforward proof.

**Corollary.** *Let $X$ and $Y$ be quotient s-images of locally compact metric spaces. Then $X \times Y$ is sequential if and only if one of the three properties below holds:*

*a) $X$ and $Y$ have point-countable bases.*

*b) $X$ or $Y$ is locally compact.*

*c) $X$ and $Y$ are locally $k_\omega$-spaces.*

*Proof.* A quotient s-image of a locally compact metric space has a point-countable closed k-network.

**Theorem 2.** *Let $X$ and $Y$ be closed images of metric spaces. Then $X \times Y$ is sequential if and only if one of three properties below holds:*

*a) $X$ and $Y$ have point-countable bases.*

*b) $X$ or $Y$ is locally compact.*

*c) $X$ and $Y$ are locally $k_\omega$-spaces.*

*Proof.* "only if"

1) If $X$ contains a copy of $S_\omega$ then $S_\omega \times Y$ is sequential. If $Y$ is a closed image of a metric space, then $\partial f^{-1}(y)$ is locally compact and Lindelöff for every $y \in Y$ by [ 9, Proposition 2.4]. We may assume without loss of generality that $Y$ is a closed s-image of a metric space. Then $Y$ has a closed point-countable k-network. Then $Y$ is a locally $k_\omega$-space by Lemma 11. So does $X$ .

2) $X$ contains a copy of $S_\omega$ and $Y$ contains no copy of $S_\omega$, then $S_\omega \times Y$ is sequential. As in the proof 1) we may assume $Y$ is a closed s-image of a metric space. As in the proof 2) of "only if " of Theorem 1, $Y$ is locally compact.

3) $X$ and $Y$ contain no copies of $S_\omega$. Then we may assume that $X$ and $Y$ are closed s-images of metric spaces by [11, Theorem 1.7 ii]. Then $X$ and $Y$ have closed point-countable k-networks. As in the proof 3, of "only if" of Theorem 1, then $X$ is locally compact or $Y$ is locally compact or $X$ and $Y$ have point-countable bases.

"if" We omit the straightforword proof.

The above theorem 2 is analogous to Theorem 1.1 of [9]. The following Theorem 3 is analogous to Theorem 3.1 of [7].

**Theorem 3.** *Let $X$ and $Y$ be k-and $\aleph$-spaces. Then $X \times Y$ is a k-and $\aleph$-space if and only if one of the three properties holds:*

*a) $X$ and $Y$ have point-countable bases.*

*b) $X$ or $Y$ is locally compact.*

*c) $X$ and $Y$ are locally $k_\omega$-spaces.*

*Proof.* Every k-and $\aleph$-space is a k-space with $\sigma$-locally finite k-network, then every k-and $\aleph$-space has a closed point-countable k-network. Then the Theorem 3 is a Corollary of Theorem 1.

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