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THE TOPOLOGY OF THE PERIODIC POINT SET OF A FAMILY OF CIRCLE MAPS

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INTRODUCTION

In this paper we study the periodic orbits of a particular family, $\mathcal{F}$, of maps of the circle. These maps are two-to-one (a.e.), onto and discontinuous, in fact undefined, at a pair of antipodal points. They are described by:

(1) f maps $(0, 1/2)$ continuously, one-to-one, onto $\mathbf{S}^1 \backslash$point, this point, $y = \lim_{z \to 0^+} f(z) = \lim_{z \to 1/2^-} f(z)$;
(2) f maps $(1/2, 1)$ continuously, one-to-one, onto $\mathbf{S}^1 \backslash$ point, this point, $x = \lim_{z \to 1/2^+} f(z) = \lim_{z \to 1^-} f(z)$;
(3) f is orientation preserving.

These functions arose in the author's thesis **[K1]**, where it was shown that, roughly, the homeomorphism classes of a certain family of strange attractors were in one-to-one correspondence with the topological conjugacy classes of $\mathcal{F}$. This work is presented as a first step towards completing that classification. The first result is that, outside of a small exceptional set, if f' is always greater than the golden mean, $(1 + \sqrt{5})/2$, then f is topologically conjugate to some $g \in \mathcal{F}$ with $g' = 2$, i.e. g is piecewise linear.

We need some notation to describe the main result. If $z_1, z_2, \ldots, z_n$ are the distinct points of a periodic orbit $\mathcal{O}$ of f, i.e. $f(z_i) = z_{i+1}$ and $f(z_n) = z_1$, then we define the word associated with $\mathcal{O}$, $w(\mathcal{O})$ via:

*This work was performed while the author was a postdoctoral visitor at the University of Delaware, and he thanks that insitution and the members of the mathematics department for their support.

$$w_i(\mathcal{O}) = t \quad if\ z_i \in (0,\, 1/2)$$
$$w_i(\mathcal{O}) = s \quad if\ z_i \in (1/2,\, 1).$$

For example, if $\mathcal{O}$ is $1/3$, $2/3$, $3/4$, i.e. $f(1/3) = 2/3$, $f(2/3) = 3/4$ and $f(3/4) = 1/3$, then $w(\mathcal{O}) = tss$, which we will abbreviate ts^2. Of course $w(\mathcal{O})$ is only unique up to cyclic permutation since it depends on where we start watching.

Given a particular word $w = t^{n_1}s^{m_1}t^{n_2}s^{m_2}\ldots t^{n_p}s^{m_p}$, let $n = \sum_{i=1}^{p} n_i$ and $m = \sum_{i=1}^{p} m_i$. We let $d = \gcd(n, m)$.

If we restrict our attention to the collection of maps, call it $\mathcal{G}$, which satisfy $f' > (1+\sqrt{5})/2$ and which lie outside the exceptional set, then mod out by topological conjugacy our first result implies that each equivalence class has a representative contained in the collection of all $f \in \mathcal{F}$ with $f' = 2$ everywhere. The elements of this collection, call it F, are distinguished from each other by different values of x and/or y. So F is a two parameter family each parameter being a point on the circle. F therefore is modelled by the standard torus. Note that $\mathcal{G}$ mod topological conjugacy is contained in F. We conjecture that, modulo some obvious symmetries within F (described below), these two sets are equal.

The main result links the topology of this parameter space with the coding of the periodic orbits which exist for given parameters. Except for words which are repetitions of subwords of themselves, the collection of $f \in F$ for which there does *not* exist a periodic orbit described by a given word w is d disjoint annuli each of whose central curve is an $(m/d,\, n/d)$ curve on the torus. Since the complement of such a set is a set with exactly the same topology, deleting the word "not" in the above statement yields the same theorem. We choose to state it negatively for the following reason. For a given f it is possible to have more than one orbit described by the same word. As this can occur, for a particular f we might have two orbits, o_1 and o_2, described by the same word. It may be possible to find another function f_1 for which o_1 exists and o_2 does not. And still a third function f_2 for which o_2 exists and o_1 does not. Moreover, all three may be in the same annulus so that we can start at f_1 and continuously vary the parameters in such a way that we pass through f and arrive at f_2 without ever leaving this annulus. That is, while both f_1 and f_2 possess orbits with the required description, as does every function on the path between them, the orbits are in a very real sense different.

In figure 1 we show the toroidal parameter space cut open, along $x = 0 = 1$ and $y = 0 = 1$, into a square. The shaded regions are the parameter values for which there is no orbit described by t^2s^2. We see the two annuli of the theorem by reidentifying the edges to form the torus. Inside the four prominent quadrilateral regions labelled 2 are the parameter values for which there exist two distinct orbits described by t^2s^2.

In the next section we prove the first result and explain the symmetry of figure 1. In the third section we describe the algorithm used to draw figure 1 and describe the creation and destruction of periodic orbits. The final section contains the proof of the main result. The author would like to thank Bob Williams for his advice and encouragement and for maintaining interest in spite of the near glacial progress to this result.

Parameter Space

William Parry [**P**] has shown:

Theorem 1. *If $f : I \to I$ is piecewise continuous and strongly transitive, then f is topologically conjugate to some $g : I \to I$ where $g' = \pm constant$.*

Definition 1. *If $f : I \to I$ has the property that for every subinterval $J \subset I, \cup_{i=0}^{\infty} f^i(J) = I$, then we say that f is* strongly transitive.

We will show that most $f \in \mathcal{G}$ are strongly transitive. We then invoke Parry's theorem to conclude that each of these f's has a representative in its topological conjugacy class which lives on the torus F, described earlier. We begin by describing the small exceptional set.

We consider the collection of $f \in \mathcal{G}$ which satisfy:

(1) f has two fixed points, a and b, with $0 < a < 1/2$ and $1/2 < b < 1$;

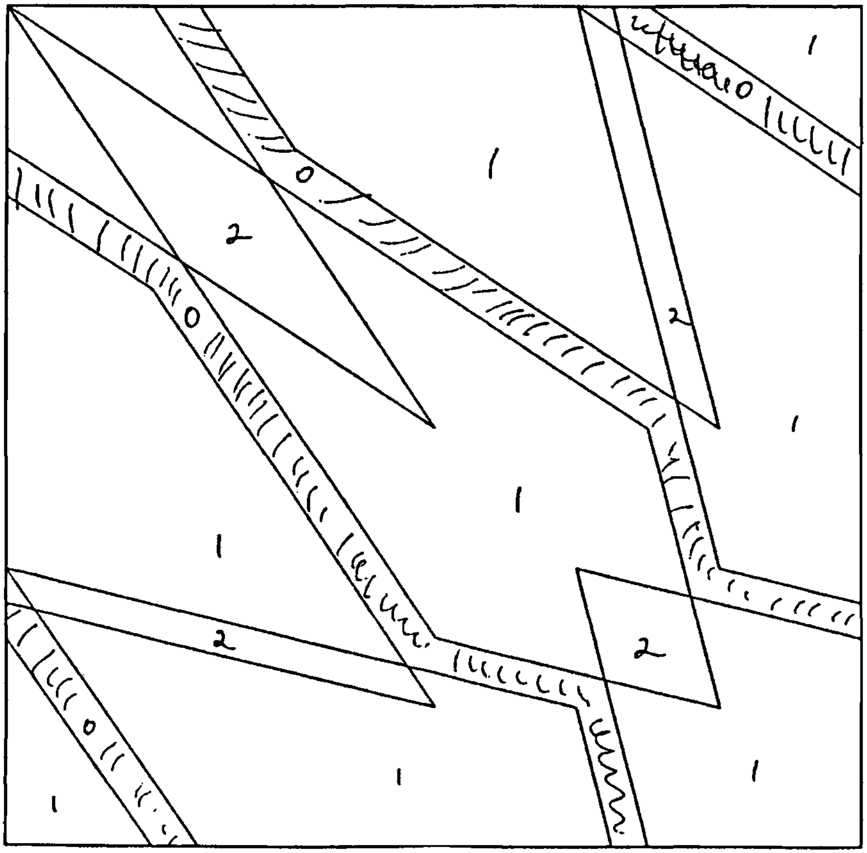


Figure 1 The regions in parameter space describing the occurrences of the word s^2t^2.

(2) $a \leq x < 1/2$;
(3) $1/2 < y \leq b$.

We call this set L. Let $\mathcal{D}$ be the subset of L for which equality holds in both 2) and 3), i.e. $a = x$ and$b = y$.

If $g \in \mathcal{D}$, then we notice that $g[a, 1/2) = [a, y)$; since $g(a) = a$ and $\lim_{z \to 1/2^-} g(z) = y$ and g is orientation preserving. Similarly, $g(1/2, b] = (x, b]$. However, $g \in \mathcal{D}$ means $x = a$ and $y = b$ so $\cup_{i=0}^{\infty} g^i[a, 1/2) = [a, b)$. We see that g is not strongly transitive. Parenthetically, we remark g restricted to $[a, b]$ and g restricted to $\mathbf{S}^1 \backslash [a, b]$ are both strongly transitive – in fact, locally eventually onto. This is just the observation that both of these maps are Poincaré first return maps on the Lorenz template of Bob Williams. See **[K1**, chap. 3], **[K2]**, and **[W]**.

If $h \in L \backslash \mathcal{D}$, then either $x > a$ or $y < b$, or both. We assume $x > a$, the other cases are identical. We observe that $h[a, 1/2) = [a, y)$ and $h(1/2, b] = (x, b]$. However, $x < a$ and $y \leq b$, so $\cup_{i=0}^{\infty} h^i[a, 1/2) \subset [a, b]$. Thus again, h is not strongly transitive. As before, h restricted to (x, y) is strongly transitive (and locally eventually onto), but, h on the complement is not **[K1**, chap. 3]. We notice that reversing all our inequalities is equivalent to just turning the picture on the circle upside down. Therefore we define $\mathcal{L}$ to be all the functions in $\mathcal{G}$ which satisfy 1, 2, and 3 or 1, 2′, and 3′, where 2′ and 3′ are:

(2′) $0 < x \leq a$;
(3′) $b < y \leq 1$.

Lemma 1. *If a is a fixed point for $f \in \mathcal{G}$ and $J \subset \mathbf{S}^1$ an interval with $a \in Int(f^n(J))$ for some non-negative integer n, then $\cup_{i=0}^{\infty} f^i(J) = \mathbf{S}^1$.*

Proof. We assume that $0 < a < 1/2$ and let $(u, v) = f^n(J)$. Since f is orientation preserving, as long as $f^m(u, a]$ does not contain 0, $f^m(u, a] = (f^m(u), a]$. Also, since $f' > \phi$, $a - f^m(u) > \phi^m(a - u)$, where $\phi = (1 + \sqrt{5})/2$. Therefore, there exists an integer k such that $[0, a] \subset f^k(u, a]$. Similarly, there exists l such that $[a, 1/2] \subset f^l[a, v)$. Let $j = max(k, l)$ and we have $f^{(n+j)}(J) \supset [0, 1/2]$, and therefore $f^{(n+j+1)}(J) = \mathbf{S}^1$. In what follows we will say that an interval J "captures" the fixed point a if a and J satisfy the hypotheses of the lemma.

Theorem 2. *If $f \in \mathcal{G} - \mathcal{L}$, then f is strongly transitive.*

Proof. We will assume f is not strongly transitive and show that then $f \in \mathcal{L}$. First let us assume there exists $J \subset \mathbf{S}^1$ an interval such that $\cup_{i=0}^{\infty} f^i(J) \neq \mathbf{S}^1$. Then we define J_0 to be the largest subinterval of J which does not contain a point of discontinuity. And then we recursively define J_i, $i \in \mathcal{Z}^+$, to be the largest subinterval of $f(J_{i-1})$ which does not contain a point of discontinuity. We notice $\cup J_i \subset \cup f^i(J) \neq \mathbf{S}^1$. If $f(J_i)$ ever contains both points of discontinuity, then $f^2(J_i) = \mathbf{S}^1$. So $f(J_i)$ contains at most one point of discontinuity. Therefore, if we let $l(I)$ denote the length of the interval I, $l(J_i) > 1/2\ l(f(J_{i-1})) > \phi/2\ l(J_{i-1})$. There must exist j such that $f(J_j)$ and $f(J_{j+1})$ both contain a point of discontinuity. Otherwise, $l(J_n) > (\phi/\sqrt{2})^n l(J_0)$, which increases without bound hence J_n covers $\mathbf{S}^1$ for sufficiently large n.

For ease of notation we let $I = J_{j+1}$, then we have $\cup f^i(I) \neq \mathbf{S}^1$. We assume that $I = (u, 1)$; either 0 or 1/2 is in $f(I)$. First, we assume the latter. Since f is orientation preserving this means $x \in (1/2, 1)$ and $(1/2, x) \subset f(I)$. But, $\lim_{z \to 1^-} f(z) = \lim_{z \to 1/2^+} f(z) = x$, so $f(1/2, x) = (x, x_1)$ with $l(x, x_1) > \phi\, l(1/2, x)$. If $x_1 \geq u$, then $(1/2, 1) \subset \{I \cup f(I) \cup f^2(I)\}$ and f applied to this set covers $\mathbf{S}^1$. Thus, $x_1 < u$. But we can continue: $f(x, x_1) = (x_1, x_2)$ with $l(x_1, x_2) > \phi\, l(x, x_1)$, and again $x_1 < x_2 < u$, and so on. Plainly, as this can't continue, there must exist some $n \in \mathcal{Z}^+$ such that $u \in [x_n, x_{n+1}]$. At which point we have $(1/2, 1) \subset \cup_{i=0}^{n+2} f^i(I)$, and one more iteration of f covers $\mathbf{S}^1$.

Therefore, $0 \in f(I)$. If $(v, 1] \subset f(I)$, then $v > u$ since otherwise there exists a fixed point in $(u, 1)$ which is captured and by the lemma $f(u, 1)$ must eventually cover the circle. Now, $(0, x) \subset f(I)$ and $x \in (0, 1/2)$ which implies there exists a fixed point $b \in (1/2, 1)$. Let n be the least integer such that $f^n(0, x)$ contains a point of discontinuity. We consider separately the two cases:

I): $0 \in f^n(0, x)$;
II): $1/2 \in f^n(0, x)$.

Case I: If $n = 1$, then there exists a fixed point $a \in (0, 1/2)$.

We must have $0 < x \leq a$, else $f(I)$ captures a. We have already a fixed $b \in (1/2, 1)$, and we must have $b \leq y < 1$ else b is captured. This is the definition of $\mathcal{L}$.

If $n > 1$, then we want to show that $(0, x) \subset Int(f^n(0, x))$. In order to establish this we need only show that $f^n(0, x)$ is longer than $(0, x) \cup I$, since without loss of generality we can assume $I = [b, 1)$. Therefore, $f(I) = [0, x) \cup I$. It suffices to show this for $n = 2$. We want to show that it must be the case that:

$$lf^2(0,x) > l(0,x) + l(I) = lf(I). \qquad (*)$$

Since $lf^2(0, x) > \phi^2 l(0, x)$, the worst we could do is to minimize $(0, x)$. That is, $l(0, x) = (\phi - 1)l(I)$. And now (*) becomes:

$$\begin{aligned} \phi^2(\phi-1)l(I) &> \phi l(I) \\ \phi(\phi-1) &> 1 \\ \phi^2 - \phi - 1 &> 0 \\ \phi &> (1+\sqrt{5})/2. \end{aligned}$$

We see here that the bound on f' is sharp. It is easy to construct $f \in \mathcal{G}$ not strongly transitive if we allow $f' \leq \phi$. (See **[K1]**, **[K2]**.) All we need to do is build an f which satisfies (*).

We define $I_1 = f^n(0, x) \cap (0, 1/2)$ and recursively define $I_i = f^n(I_{i-1}) \cap (0, 1/2)$. In order for the lengths of the I_i's to be bounded, we must have positive integers m and k with $0 < m < n$ such that $f^m(I_k)$ contains a point of discontinuity. If that point is 0, then repeat the above argument with I_1 in the place of $(0, x)$ and m in the place of n. We obtain some interval $(0, w) \supset I_1$ and some $0 < m' < m$ with $f^{m'}(0, w)$ containing a point of discontinuity. We repeat this argument, as necessary, until either $m = 1$ and $f^m(0, w)$ contains 0, which as shown above means $f \in \mathcal{L}$, or until we find m and w with $f^m(0, w)$ containing $1/2$.

Case II: If $m = 1$, then because f is orientation preserving and $f(0, w)$ contains only one point of discontinuity, $y \in (0, 1/2)$ and $(y, 1/2) \subset f(0, w)$. If $y < w$, then $(0, w) \cup (y, 1/2) = (0, 1/2)$, and one more iteration of f covers the circle. Therefore, $y \geq w$, and we repeat an argument we've seen before. Since $\lim_{z \to 0^+} f(z) = \lim_{z \to 1/2^-} f(z) = y$, we have $f(y, 1/2) = (y_1, y)$, where $l(y_1, y) > \phi l(y, 1/2)$. Also, $f(y_1, y) = (y_2, y_1)$, where $l(y_2, y_1) > \phi l(y_1, y)$. Eventually we must get $w \in (y_n, y_{n-1})$ and then $\cup_{i=0}^{n+1} f^i(0, w) \supset (0, 1/2)$ and f is strongly transitive.

Therefore $m > 1$. We define $L_0 = (0, w]$ and $L_i = f^i(0, w]$, for $i = 1, \dots, m$. We let a_i and b_i be the "forward orbits" of the open and closed ends of L_0, respectively. We let K_i, $i = 0, \dots, m$, be the complementary intervals of the circle, with K_i immediately counterclockwise of L_i. And we define $L' = L_m \cap (0, 1/2)$, which by hypothesis is not empty. Since $a_1 = y$, $f(L')$ is an interval which is immediately clockwise of L_1 and has non-empty intersection with the K_i which is there. The same is true for $f^j(L')$ and L_j and, in particular, $f^m(L')$ is immediately clockwise of L' and $l(f^m(L')) > \phi^m l(L')$. The forward orbit of $f^m(L') \cup L'$ has the same properties, and so on. Since we can't cover everything, we must capture an earlier pre-image of a point of discontinuity. That is, there exist $0 < q < m$ and $p \in (0, 1/2)$ such that $f^q(p, 1/2)$ contains a point of discontinuity and $(p, 1/2) \subset \cup_{i=0}^{\infty} f^i(0, w]$. Since the forward orbits of this interval always lie immediately clockwise of the L_i, $f^q(p, 1/2)$ must cover either K_m or K_0.

We assume $f^q(p, 1/2)$ covers K_m. If L_n is immediately counterclockwise of K_m, then $K_m = [b_m, a_n)$; we have already $[1/2, b_m] \in L_m$. Therefore, $[1/2, a_n)$ is in $\cup f^i(L_0)$. However, $\lim_{z \to 1/2^+} f(z) = x$ by definition, $x \in L_0$, and $f(a_n) = a_{n+1}$. Since K_0 lies between x and the first a_i counterclockwise of it, $K_0 \subset f(1/2, a_n)$. Notice that $f(K_i)$ covers K_{i+1}, for $i = 0, \dots, m-1$, and therefore $\cup f^i(L_0 \cup K_0) = \mathbf{S}^1$. Thus f is strongly transitive.

Now we invoke Parry's theorem to get that each $g \in \mathcal{G} \backslash \mathcal{L}$ is topologically conjugate to some $f \in F$. If we let F be represented by the unit square in the plane, we ask – do distinct choices of parameters $(x, y) \in F$ correspond to distinct topological conjugacy classes in $\mathcal{G}$?

There are two fairly obvious things to notice first. One, considering the set $S = \{(x,y) \mid y = 1-x,\ \ x \in (0,1/2)\}$, we see immediately that $S \subset \mathcal{D} \subset \mathcal{L}$. In fact, all of $\mathcal{D}$ (and therefore S, too) is a single topological conjugacy class of which any element of S is a "nice" representative. That any element $g \in \mathcal{D}$ is conjugate to an element of S follows from the earlier observation that any such g is strongly transitive on each of its invariant pieces. Simply apply Parry's theorem to each piece to build the conjugacy to an element of S. The conjugacy between elements of S sends the fixed points to the fixed points and uniformly expands and contracts the intervals between them. Second, reflection in the line $y = 1-x$ ($(x,y) \mapsto (1-y, 1-x)$) is a topological conjugacy. It corresponds to rotating the circle about the diameter through 0 and 1/2. This is why figure 1 is symmetric about $y = 1-x$. Of course this rotation interchanges s and t so that if we draw the diagram for s^2t and reflect in $y = 1-x$ we get the diagram for t^2s. Figure 1 is special because s^2t^2 is symmetric in s and t (up to cyclic permutation, which is all that matters). We will exploit this symmetry in the proof of the main theorem.

Creating an Orbit

We will try to describe what happens during the creation (or destruction) of a particular periodic orbit. That is, if f and f_1 are elements of F with associated parameters (x,y) and (x_1,y_1), γ is a path in F from (x,y) to (x_1,y_1), and f has a periodic orbit described by the word w and f_1 does not, can we characterize the set $\{g \mid g \in \gamma,\ g$ has an orbit described by $w\}$? What is the boundary of this set?

For any $f \in F$ if we know the values of x and y we can write a formula for f as a map of the interval to itself:

$$f(z) = \begin{cases} 2z+y & z \in (0, \frac{1-y}{2}] \\ 2z+y-1 & z \in (\frac{1-y}{2}, \frac{1}{2}) \\ 2z+x-1 & z \in (\frac{1}{2}, \frac{2-x}{2}] \\ 2z+x-2 & z \in (\frac{2-x}{2}, 1). \end{cases}$$

Now if we have a particular periodic orbit described by the word w, $\mathcal{O} = \{z_1, z_2, \ldots, z_n\}$, then it must be the case that $z_i = f^n(z_i)$. That is, $z_i = 2^n z_i + kx + ly - p_i$, where k and l depend only on w and p_i depends on which of the four subintervals each successive z_i is in. However, if we know that the z_i's exist, then we get $\frac{\partial z_i}{\partial x} = -\frac{k}{2^n-1}$ and $\frac{\partial z_i}{\partial y} = -\frac{l}{2^n-1}$. Thus we can tell in which direction and with what speed a point in a periodic orbit moves as we vary the parameters x and y.

Example: We let $(x, y) = (7/12, 5/12)$. Consulting figure 1 we see that there should exist a periodic orbit described by s^2t^2. A little computation shows $\mathcal{O} = \{39/60, 53/60, 21/60, 7/60\}$. These points are, respectively, the solutions to:

$$z_1 = 16z_1 + 12x + 3y - 18 \tag{1}$$
$$z_2 = 16z_2 + 9x + 6y - 21 \tag{2}$$
$$z_3 = 16z_3 + 3x + 12y - 12 \tag{3}$$
$$z_4 = 16z_4 + 6x + 9y - 9. \tag{4}$$

We get $\frac{\partial z_1}{\partial y} = \frac{-1}{5}$, $\frac{\partial z_2}{\partial y} = \frac{-2}{5}$, $\frac{\partial z_3}{\partial y} = \frac{-4}{5}$, and $\frac{\partial z_4}{\partial y} = \frac{-3}{5}$. Therefore, if we increase y and leave x fixed, then the locations of the z_i's move clockwise on the circle at these relative speeds. So that we see, for instance, if we increase y by $1/12$ to $(7/12, 1/2)$ the points of $\mathcal{O}$ have migrated to $\mathcal{O}' = \{38/60, 51/60, 17/60, 4/60\}$. The locations of these new z_i''s are simply $z_i' = z_i + \frac{\partial z_i}{\partial y}\Delta y$. We see immediately there remains an orbit described by s^2t^2 as long as the z_i''s remain on the correct semi-circles. When $\Delta y = 7/36$ we get $z_4' = 7/60 + (-3/5)7/36 = 0$, and we get the "orbit" of zero: $0 \mapsto 22/36 \mapsto 29/36 \mapsto 7/36 \mapsto 1$. We will call this the orbit of zero, i.e. 0 goes to $\lim_{z \to 0^+} f(z) = y$ and after that just apply f to get more points in the orbit. We define the orbit of one similarly as $1 \mapsto \lim_{z \to 1^-} f(z) = x$ and its forward images. The "orbits" as we approach $1/2$ from the left and right, respectively, are the same as these two. If we write the word describing the orbit of 0 we get ts^2t. When the orbit of 0 or 1 is finite, i.e. eventually gets back to either 0=1 or $1/2$, we will say we have a *saddle connection*, since that is exactly what we would get if we construct a suspension flow for f. Note the convention that the word describing the orbit of 0 (resp. 1) always begins with t (resp. s) since that is the side from which we are taking the limit.

Getting back to our example, we see that if we decrease y from $5/12$ the z_i move on the circle at the same relative speeds as before but in the opposite direction (i.e. counterclockwise). So that the first z_i to land on a point of discontinuity is z_3, and this happens when $-\frac{4}{5}\Delta y = \frac{9}{60}$. Thus $\Delta y = -9/48$ and $y = 11/48$, and, of course, x is still $7/12$. Now the orbit of 0, which is the same as the orbit of $1/2$ from the negative side, is $1/2 \mapsto 11/48 \mapsto 33/48 \mapsto 46/48 \mapsto 1/2$, which is described by the word t^2s^2. The point is that if we decrease y any more, this periodic orbit disappears and there is no longer a periodic orbit described by s^2t^2 at $(28/48, 109/480)$. This can be checked by simply solving the system of equations (1)-(4) subject to the conditions $z_1, z_2 \in (1/2, 1)$ and $z_3, z_4 \in (0, 1/2)$. There is one caveat, and it is part of what makes the problem interesting: the constants we subtract in that system depend upon the values of the z_i, and, therefore, we can't know what they are in advance. What we do in practice is try to solve that system for all possible choices of these constants.

One more example, if we decrease x and y at the same speed (i.e., move along the line $x - y = 1/6$ in parameter space), then the motion of the z_i is described by the directional derivative. We get $D_{(-1,-1)}z_i = 1/\sqrt{2}$, for all i, which means that z_2 is the first to run into a point of discontinuity. This happens when $(1 - 53/60) = 1/\sqrt{2} * \sqrt{(\Delta x)^2 + (\Delta y)^2} \Rightarrow 7/60 = \Delta x = \Delta y$. So at the point $(14/30, 9/30)$ we should find that the orbit of 1 is a saddle connection. Computing, we see $1 \mapsto 14/30 \mapsto 7/30 \mapsto 23/30 \mapsto 1$, which is described by ts^2t. If we move the parameters any small amount more in this direction, there does not exist an orbit described by t^2s^2.

These observations lead to:

Proposition 1. *If there exist n distinct orbits described by the word w for the point (a, b) in parameter space and there exist (n-1) distinct orbits described by w for (c, d), then there exists a point (x, y) on the line joining (a, b) to (c, d) in parameter space such that one of the following is true:*

i): *The orbit of zero (or one) is a saddle connection described by some cyclic permutation of w;*
ii): *Both the orbit of zero and the orbit of one are saddle connections, described by w_0 and w_1 respectively, and the concatenated word w_0w_1 is a cyclic permutation of w; or*
iii): *Both the orbit of zero and the orbit of one are saddle connections, described by w_0 and w_1 respectively, and some cyclic permutation of w is a concatenation of w_i's in which one or both occurs more than once.*

Proof. We let m be the length (# of letters) of w. There are at most four pre-images of the points of discontinuity under f. These four points have, at most, eight pre-images. Consider the collection, D_m, of all pre-images of the points of discontinuity up to order $m-1$. Then

$$D_m = \{x \in \mathbf{S}^1 \mid f^n(x) = 0 \ \text{or}\ 1/2 \ \text{or}\ 1 \ \text{and}\ 0 \leq n \leq m-1\}.$$

Let I_i be the collection of open intervals joining adjacent points in D_m, in no particular order. There are at most $2(2^m + 1)$ such intervals. Each I_i can be uniquely described by a word of length m in s and t, which describes its location under the iterates of f, and a non-negative integer p which is the integral part of $2^m z + kx + ly$. Here z is any point of I_i, and k and l are the coefficients determined by iterating f on I_i. These coefficients are determined by w in a manner described below.

Now, $f^m(z) = 2^m z + kx + ly - p$ is continuous as a map of I_i to $\mathbf{S}^1$. Moreover, there exists a periodic orbit with word w' iff there exists an i such that I_i is described by w' and $I_i \subset f^m(I_i)$. At most one point of the periodic orbit is in any I_i. We also notice that if there exists more than one orbit with word w', then only one point of the totality of points in the orbits is in any given I_i. As noted before the locations of periodic points change continuously (at least for a while) in x and y, so, too, do the locations of the points of D_m. If there exists an orbit, $\{p_0, p_1, \cdots, p_{m-1}\}$, described by w, then, as we change x and y, eventually some p_i moves onto the endpoints of its I_j, i.e., it becomes a pre-image of a point of discontinuity.

If we assume that $f^j(p_i)$ = point of discontinuity, and realize that $f^j(p_i) = p_{i+j}$, where $i+j$ is taken mod m (as are all subscripts that follow), then this implies that p_{i+j+1} is either x or y. However, $f^{(m-j-1)}(p_{i+j+1}) = p_{i+m} = p_i$, by the periodicity. Therefore the orbit of p_{i+j} is the orbit of either 0 or 1 and forms a saddle connection described by w. This is true unless there exists k $(k < m-j)$ such that p_{i+j+k} is a point of discontinuity. In this case p_{i+j} is again a saddle connection, but now it is described by some word w' of length k. Clearly, w' is a truncation of a cyclic permutation of w. If we consider $p_{i+j+k+1}$, this is either x or y. And, in fact, it is whichever of these two p_{i+j+1} is not. Otherwise we would just keep repeating w' infinitely often and never get back to p_i, which we must do as p_i was periodic. So the orbit of p_{i+j+k} is either the orbit of 0 or the orbit of 1, and it is whichever of these two that the orbit of p_{i+j} is not. This orbit must contain p_i, as there is no other way of ever getting back to it. Since $f^j(p_i)$ is a point of discontinuity, this orbit is also a saddle connection and it is described by w''. It is clear that $w'w'' = w$. This is case ii) of the proposition.

There is another "catastrophe" which can occur to destroy a particular periodic orbit. As p_i converges onto the endpoint of its I_i, some forward iterate of p_i, say p_i', could be converging onto that same pre-image of a point of discontinuity from its other side, so that p_i and p_i' coalesce. If this happens, since p_i and p_i' are separated by a point of D_m, then after some number of iterates p_i and p_i' are on opposite sides of a point of discontinuity. As p_i and p_i' converge on each other these image points converge on the point of discontinuity. That means their orbits converge onto the orbits of 0 and 1, both of which must form saddle connections, since they must pass through the coalesced p's (because of the periodicity) and return by definition. Now, as before, the concatenated word describing "the orbit of 0 followed by the orbit of 1" is w or some cyclic permutation of it. In tracing out this orbit one goes through some points more than once, precisely those points upon which two points of the periodic orbit converged, in fact.

This generalizes to case iii) where we might have more than two points converging onto a single point. If this happens, then we must have 2 or more points of D_m converging to that point as well, because all points in a periodic orbit are separated by points of D_m. Just as above, this gives saddle connections for both the orbit of 0 and the orbit of 1. Now, however, following these two saddle connections successively is not necessarily described by w. We may have to repeat one or both one or more times so that you go through every coalesced point exactly as many times as there are points which coalesced onto it.

Cases ii) and iii) are extremely rare; in fact, they can occur only at finitely many points for any given word w. This is easy to see. To have a saddle connection for a given word w (length n) means that $z = 0$ or $z = 1/2$ is a solution to: $f^n(z) = i$, where i is an integer or $1/2$ plus an integer. This equation looks like $2^n z + kx + ly = i$, where k and l are determined by w. If the first letter of w is s, then one iteration of f yields $2z + x$, and every subsequent iteration multiplies by two. Thus the coefficient on x is at least 2^{n-1} and is exactly 2^{n-1} if there are no more occurrences of s in w. If there are more s's in w and the next is the i^{th} letter, then $f^{i-1}(z) = 2^{i-1}z + 2^{i-2}x + l'y$, where l' is some as yet unknown integer. Then $f^i(z) = 2f^{i-1}(z) + x = 2^i z + 2^{i-1}x + 2l'y + x$, all these equations are to be taken mod 1, of course. The coefficient of x is now $2^{i-1} + 1$ and this will get doubled by every subsequent iteration of f. If there are no more occurrences of s in w then we will get: $f^n(z) = 2^n z + 2^{n-i}(2^{i-1}+1)x + 2^{n-i}(2l'y) + y$; and the coefficient on x is $2^{n-1} + 2^{n-i}$. The 2^{n-1} is the result of the $n-1$ doublings of the coefficient of the first x and the 2^{n-i} is the result of the $n - i$ doublings of the i^{th} x. Arguing inductively we see that $k = \sum_{j \in J} 2^{n-j}$, where $J = \{j \in \{1, 2, \dots, n\} \mid$ the j^{th} letter of w is $s\}$. The coefficient on y is computed the same way: $l = \sum_{i \in I} 2^{n-i}$, $I = \{i \in \{1, 2, \dots, n\} \mid$ the i^{th} letter of w is $t\}$. I and J are, of course, complementary subsets of $\{1, 2, \dots, n\}$ so that $k + l = \sum_{i=1}^{n} 2^{n-i} = 2^n - 1$. So, for example, if $w = ststtssst$, then $k = 256 + 64 + 8 + 4 + 2 = 334$ and $l = 128 + 32 + 16 + 1 = 177$ and $334 + 177 = 511 = 2^9 - 1$. To get a saddle connection we must have $kx + ly = i$, where

i is as before (since z is one of 0, 1/2 or 1 we can put the $2^n z$ in with the i). Thus, saddle connections for a given word occur on line segments of slope $-k/l$. The form of i implies there are only finitely many such line segments in parameter space for a given word. Also if the first letter of w is s, then $k \geq 2^{n-1}$ and $-k/l < -1$. Similarly, if the first letter of w is t then $-1 < -k/l < 0$. Now, if the orbit of 0 is a saddle connection, the word describing it begins with t, and if the orbit of 1 is a saddle connection, the word describing it begins with s. Cases ii) and iii) of the above proposition occur when both of these orbits are saddle connections described by one of the finitely many truncations of the finitely many cyclic permutations of the given word w. Therefore, a point (x, y) where this happens is on 2 line segments: one of slope less than -1, the other of slope greater than -1. There are only finitely many such line segments describing each choice of truncation and permutation, so there are only finitely many such points (x, y).

It should be pointed out that for the existence of a saddle described by w, we need to satisfy more than $kx + ly = i$. We also must have the points in the orbit making the saddle lying in the correct semi-circle. For example, to get a saddle described by s^2t^2 we need $12x + 3y = i$. We also must have $x \in (1/2, 1)$, since the second letter of w is s. We also need $f(x) \in (0, 1/2)$, since the next letter is t. This gives $3x \in (j, j + 1/2)$ for some integer j. All we can say about j is that it is no greater than 2 since $x \in (0, 1)$. Applying f to this point must keep us in $(0, 1/2)$, so $6x + y \in (j, j + 1/2)$. This time j is no greater than 6, since both x and y are no bigger than 1. Applying f to this point and insisting that we get a saddle gives $12x + 3y = i$. This gives a system of 1 equation and $2(n - 1)$ inequalities, for any particular choice of the j's and choice of i, that needs to be satisfied to get a particular saddle. The algorithm used to draw figure 1 is to take all possible choices of the j and i and take all the cyclic permutations of w, and solve all the resulting systems. This is a natural problem to give to a machine since with even moderate word lengths the amount of computation is fearsome. It takes about eight hours on a Sun workstation to solve the problem for a word of length ten, and the resulting figure is often incomprehensible. The distance separating two line segments in the solution can be as small as $2^{-(n+1)}$. I would like to thank Kevin Woerner for his help with the graphics portion of this program and the computer czars at Northwestern for the use of their facilities.

In light of the above we can restate the proposition:

Proposition 2. *If there exist n distinct orbits described by w for the point (a, b) in parameter space and there exist $(n-1)$ distinct orbits described by w for (c, d), then there exists a piecewise linear path, γ from (a, b) to (c, d) in parameter space so that there exists $(x, y) \in \gamma$ with the orbit of 0 (or 1) at (x, y) a saddle connection described by w.*

Simply choose any path which misses all of the finitely many pathological points.

At this point it would be helpful to understand the geometry of the set D_m. To that end we define, for $i = 1, 2, \ldots,$ sets

$$T^i = \{z \in \mathbf{S}^1 \mid f^j(z) \in (0, 1/2) \ j = 0, 1, \ldots, i-1$$
$$\text{and } f^i(z) \in (1/2, 1)\}$$
$$S^i = \{z \in \mathbf{S}^1 \mid f^j(z) \in (1/2, 1) \ j = 0, 1, \ldots, i-1$$
$$\text{and } f^i(z) \in (0, 1/2)\}.$$

Then T^3, for example, is all the points in (0,1/2) whose images remain in (0,1/2) under f and f^2 but not under f^3. The word describing any such point necessarily begins $ttts\ldots$, hence the name.

Proposition 3. (1) *$T^i \cap T^j = \emptyset$, $S^i \cap S^j = \emptyset$ for $i \neq j$, and $T^i \cap S^j = \emptyset$ $\forall i, j$*

(2) $\mathbf{S}^1 = \{\cup_{i=1}^{\infty} cl(T^i) \cup cl(S^i)\}$

(3) *T^i (likewise S^i) is one or two intervals; in fact, either T^i is one interval for all i or there exists $j \in \mathcal{Z}^+$ such that T^i is two intervals for $i \geq j$ and T^i is one interval for $i < j$.*

Proof. The first two assertions are obvious; they amount to no more than the fact that the function $z \to w(z)$ is well-defined on the complement of D_m. We prove the third inductively on i. If $y \in [0, 1/2]$, then the interval $(1/4 - y/2, 1/2 - y/2)$ maps to $(1/2, 1)$, hence is T^1. If, however, $y \in (1/2, 1)$, then the interval $(0, 1/2 - y/2)$ maps to $(y, 1)$, the interval $(3/4 - y/2, 1/2)$ maps to $(1/2, y)$, and T^1 is the union of these two disjoint intervals. In this latter case, T^2 the inverse image of T^1, is $(1/2 - y/2, 3/4 - 3y/4) \cup (7/8 - 3y/4, 3/4 - y/2)$, and these are disjoint for $y \in (1/2, 1)$. T^3 is two disjoint intervals that abut the intervals forming T^2 and so on for T^n (figure 2). The successive T^n's are $1/2$ as long as the T^{n-1}, and they converge from both sides to a fixed point at $(1 - y)$, whose orbit can be described t^∞. In the former case, i.e., T^1 is one interval, either $y \in T^1$ or not. If not, then T^2 is one interval, $(1/8 - 3y/4, 1/4 - 3y/4)$ or $(5/8 - 3y/4, 3/4 - 3y/4)$ depending on whether $y < 1/4 - y/2$ or $y > 1/2 - y/2$. If, on the other hand, $y \in T^1$, then there exist intervals $T^{2,1}$ and $T^{2,2}$ of the form $(0, a)$ and $(b, 1/2)$ respectively, with $f(a) = 1/2 - y/2$ and $f(b) = 1/4 - y/2$. T^3 must now be two disjoint intervals, and in fact we can tell exactly where they are: $T^3 = (1/2 - y/2, a_2) \cup (b_2, 1/4 - y/2)$, where $f(a_1) = a$ and $f(b_1) = b$. Now it is clear T^4 is two intervals that abut the ends of T^2 since $f(T^{i+1}) = T^i$.

Now we suppose $T^1, \ldots, T^n$ are each a single interval. Either $y \in T^n$ or not, and $y \notin \cup_{i=1}^{n-1} T^i$. If $y \in T^n$, then $T^{n+1} = (0, a) \cup (b, 1/2)$ where $f(a)$ and $f(b)$ are the endpoints of T^n. Suppose T^m, $m > n$ has more than two disjoint components. Under $m - n$ iterations of f these components map to the single interval T^n. But the only way to get two disjoint intervals mapped to a single interval is to have them each have a point of discontinuity as endpoint. This can only happen once, as $T^i \cap T^j = \emptyset$. The assertion follows.

In this situation, $y \in T^n$ and T^{n+2} is 2 disjoint intervals. Since $f(T^{n+2}) = T^{n+1}$, the stuff between these intervals must map to [1/2,1]. That is, T^{n+2} is 2 intervals: one immediately clockwise and the other immediately counterclockwise of T^1. Similarly, T^{n+3} is two disjoint intervals bracketing T^2. T^{2n+2}

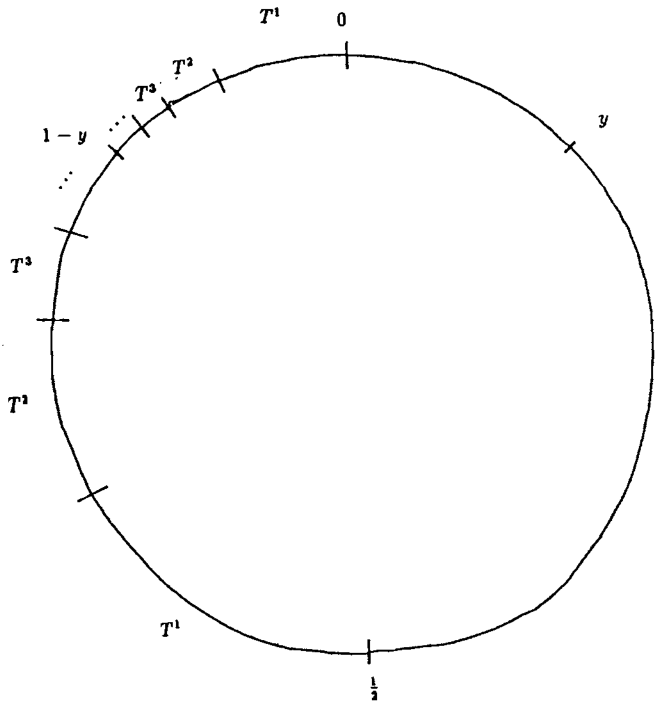


Figure 2. The positions of the T^i for $y \in (1/2, 1)$.

is two intervals abutting the free ends of the T^{n+1}. T^{2n+3} is two intervals abutting the ends of the single interval formed by $T^1 \cup T^{n+2}$ (figure 3).

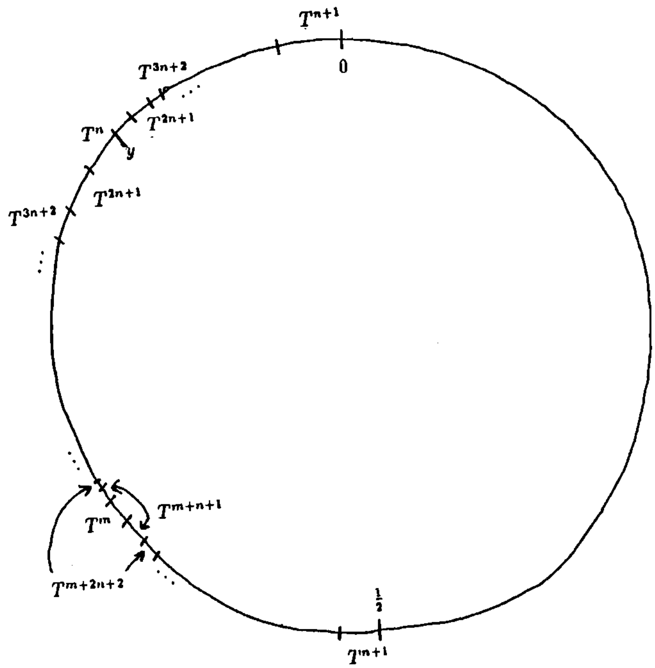


Figure 3. The positions of the T^i for $y \in T^n \subset (0, 1/2)$.

The dynamics of this picture as y increases (i.e. moves counterclockwise) are controlled by the behavior of the inverse images of the points of discontinuity. Every inverse image of a point of discontinuity moves (for fixed x) in the direction opposite the direction y is moving with speed $(2^n - 1)/2^n$ times the speed of y (n is minimal such that $f^n(z)$ is a point of discontinuity). To see this we just take partial derivatives. This means that higher order inverse images overtake those of lower order. This happens as follows: when $T^n = (0, a)$, T^m and T^{m+n}, for all $m < n$, share an endpoint (the counterclockwise end of T^m and the clockwise end of T^{m+n}). This point is an m^{th} pre-image of 0. Also, T^{m+n} and T^{m+2n} share an endpoint, which is an $(m+n)^{th}$ pre-image of 0. A similar statement is true for T^{m+kn} and $T^{m+(k+1)n}$. Thus, $\bigcup_{i=0}^{\infty} T^{m+in}$

is a single interval, with the superscripts in increasing order reading counterclockwise. The point y is the clockwise endpoint of T^{n-1}. If y is increased slightly it becomes an interior point of T^{n-1}. T^n then becomes two intervals, $(0, a')$ and $(b, 1/2)$. T^m, $\forall m < n$, remains a single interval, but T^{m+n} is now two intervals abutting the ends of T^m. T^{m+2n} is two intervals abutting the ends of the interval $T^m \cup T^{m+n}$. Again, $\cup_{i=0}^{\infty} T^{m+in}$ is a single interval, but now the superscripts start with m in the middle and increase in both directions $(\dots, m+3n, m+2n, m+n, m, m+n, m+2n, m+3n, \dots)$. Now we increase y until it coincides with the counterclockwise endpoint of T^{n-1}. At this point, a' becomes zero and T^n is a single interval $(b, 1/2)$. We still have that $\cup_{i=0}^{\infty} T^{m+in}$ is a single interval for all $m < n$. However, now the superscripts increase as we read in the clockwise direction. We still have to describe the sets T^{in}. When T^n is a single interval $(0, a)$, T^{2n} is the interval $(a, a + a/2^n)$, since $f^n(T^n) = (1/2, 1)$ we can conclude that f^n of the stuff immediately counterclockwise of T^n is T^n. For the same reason, T^{3n} is immediately counterclockwise of T^{2n}, and so on. So, $\cup_{i=1}^{\infty} T^{in}$ is a single interval with superscripts in increasing order reading counterclockwise. When T^n is two intervals, $(0, a')$ and $(b, 1/2)$, then T^{2n} is two intervals, one abutting each of these two. T^{3n} is two intervals, one abutting each component of $T^n \cup T^{2n}$, and so on. When $a' = 0$ and T^n is $(b, 1/2)$, then $\cup_{i=1}^{\infty} T^{in}$ is a single interval with one endpoint at $1/2$ and superscripts reading in increasing order clockwise.

The dynamics of the T^i if $y \in [1/2, 1]$ are much simpler. They are in fact the above T^{in} with $n = 1$. If $y = 1/2$, then $T^1 = (0, 1/4)$, $T^2 = (1/4, 3/8), \dots, T^n = (1/2 - 1/2^n, 1/2 - 1/2^{n+1})$. If $y = 1$, then $T^1 = (1/4, 1/2)$, $T^2 = (1/8, 1/4), \dots$, $T^n = (1/2^{n+1}, 1/2^n)$. If $y \in (1/2, 1)$, then the T^i are as in figure 2, with the length of the upper T^i shrinking and the lower growing as y goes from $1/2$ to 1.

The Main Result

We begin the proof with some lemmata. The first lemma says that while we know that to destroy an orbit we need to go through a saddle connection, if we have only one orbit of a given description and one of the parameters is zero, then the only saddles which can destroy that orbit have particular cyclic permutations describing them.

Lemma 2. *If* $x = 0$ *and* $w = t^{n_1}s^{m_1}t^{n_2}s^{m_2}\dots t^{n_p}s^{m_p}$*, then there exists a periodic orbit described by* w *if and only if the orbit of 0 does not begin* $t^k s^{m_i} t^{n_{i+1}} s^{m_{i+1}} \dots t^{n_1} s^{m_1} \dots t^{n_{i-1}} s^{m_{i-1}} t^d$*, where, for some* $i \in \{1, 2, \dots, p\}$*, either* $k = n_i$ *and* $d \geq 1$*, or* k *divides* n_i *and* d *is a multiple of* k *less than* n_i*.*

Proof. Since $x = 0$, $S^1 = (1/2, 3/4)$, $S^2 = (3/4, 7/8), \dots$, $S^n = ((2^n - 1)/2^n, (2^{n+1} - 1)/2^{n+1})$. We define sets: $T^{i_1}S^{j_1}T^{i_2}S^{j_2}\dots T^{i_q}S^{j_q} = \{z \mid w(z) = t^{i_1}s^{j_1}\dots t^{i_q}s^{j_q}t*\}$, where $*$ is any word (finite or infinite) in s and t. This set is the set of all points in (0,1/2) whose orbits begin with the corresponding word, and end in (0,1/2) after $\sum_{l=1}^{q}(i_l + j_l)$ iterations of f. This set, for any choice of i's and j's, is a finite collection of intervals. This is easy to see: S^{j_q} is a single interval and T^{i_q} is at most two intervals. Therefore, $T^{i_q}S^{j_q}$ is at most two intervals. Similarly, $T^{i_{q-1}}S^{i_{q-1}}$ is at most two intervals, so $T^{i_{q-1}}S^{i_{q-1}}T^{i_q}S^{i_q}$ is at most three intervals. And so on, until we get that the set $T^{i_1}S^{j_1}T^{i_2}S^{j_2}\dots T^{i_q}S^{j_q}$ is at most $(q+1)$ intervals. We observe that in general S^j can be at most two intervals, and then we get that for any parameter values an upper bound for the number of periodic orbits having a given description is one more than the number of letters in the word.

Suppose that the orbit of zero is not as described in the statement of the lemma, then the set $T^{n_1}S^{m_1}T^{n_2}S^{m_2}\dots T^{n_p}S^{m_p}$ is at most $(p+1)$ intervals. We assume there are $j \leq p+1$, and let I_i be these intervals numbered counterclockwise starting at zero. If $j = 1$, then I_1 is an interval on which $f^{(n+m)}$ is continuous ($f^{(n+m)}$ is continuous on each I_i since they, by definition contain no pre-images of points of discontinuity of order less than $n + m$). Also, $f^{(n+m)}(I_1) = (0, 1/2) \supset I_1$; therefore there exists a fixed point of $f^{(n+m)}$ in I_1 which necessarily has the correct orbit.

If we assume $j > 1$, this implies that the orbit of zero begins t^q, with $q < n_i$, for some i. We can assume, without loss of generality, that $i = 1$. Now, consider two adjacent intervals, I_r and I_{r+1}. From the descriptions in the previous section we know that the interval, J, which lies between them, consists of points all of whose orbits begin t^l, $l \leq n_1$, (l, of course, is not constant on J but every value it takes on satisfies this inequality). This fails to be true when the orbit of zero has the described form. Partition the points of J into intervals according to the location of their first $n + m$ iterates. Let J_0 and J_1 be the end intervals. That is, J_0 (resp. J_1) is that part of J which abuts I_r (resp. I_{r+1}) and all the points of J_0 (resp. J_1) are on the same semi-circle for $n + m$ iterations. If J_0 and J_1 begin t^l with $l < n_1$, then $f^{(n_1-l)}(I_r) = (b, 1/2)$ and $f^{(n_1-l)}(I_{r+1}) = (0, a)$. And, therefore, one more iteration of f gives a single interval with order preserved, in the sense that the part of this interval which is the image of I_r is clockwise of the part which comes from I_{r+1}. If J_0 and J_1 begin T^{n_1} , then $f^{n_1}(I_r), f^{n_1}(J_0), f^{n_1}(J_1)$ and $f^{n_1}(I_{r+1})$ must occur in this order reading counterclockwise within S^{m_1}. We now apply f^{m_1} and their images remain in this order within (0,1/2). But $f^{(n_1+m_1)}(I_r)$ and $f^{(n_1+m_1)}(I_{r+1})$ are both subsets of T^{n_2}, $f^{(n_1+m_1)}(J_0)$ and $f^{(n_1+m_1)}(J_1)$ lie between them and hence begin T^{l_1}, where $l_1 \leq n_2$. If $l_1 < n_2$, then, as before, $f^{(n_1+m_1+n_2-l_1+1)}(I_r \cup I_{r+1})$ is a single interval, still in the right order. If $l_1 = n_2$, then we iterate the argument, eventually J_0 and J_1 must have orbit descriptions which differ from those of I_r and I_{r+1}, by their definitions this must happen before $n + m$ iterations. Therefore, $f^{(n+m)}(\cup_{i=1}^{j} I_i)$ is the single interval (0,1/2) and the images of the I_i are in order of increasing subscript reading counterclockwise. This implies that one of them covers itself, giving the desired fixed point.

Now we prove the other implication. We suppose, without loss of generality, that $k = n_1$ or k divides n_1, then the clockwise end of I_1 and the counterclockwise end of I_j get identified under f^{n_1}. This by itself is not enough to guarantee the nonexistence of a fixed point. Recall the manner in which the T^i move on the circle as y increases, and assume first that $k = n_1$. If the orbit of zero begins as described, then for some $y' < y$ it is the case that $T^{n_1} = (0, a)$, which implies T^i is a single interval for all i. Therefore, there exists exactly one periodic orbit described by w, since each S^i is also one interval. The only way to destroy this periodic orbit is to go through a saddle of the right orbit description, by the earlier proposition. This does not happen until the clockwise end of $T^{n_1}S^{m_1}\dots T^{n_p}S^{m_p} \subset T^{n_1}$ coincides with zero. It is not *a priori* true that this set is a single interval when this happens, though we will see that this is exactly the case. We suppose it is more than one interval and number them I_i as before. When I_1 is of the form $(\epsilon, a+\epsilon)$, just before we create the saddle, there exists a single periodic orbit for w. But, as before, the order in which the I_i lie is preserved by $f^{(n+m)}$. This implies, $f^{(n+m)}(I_1) = (0, b)$, and this interval is 2^{n+m} times as long as I_i. Hence the periodic orbit for w has a point in I_1 for small ϵ. When ϵ goes to zero, this point also goes to zero, and the saddle we create destroys the orbit (actually the orbit *becomes* the saddle). We increase y slightly, and I_1 splits into two intervals of the forms $(0, a)$ and $(d, 1/2)$, and $f^{(n+m)}(d, 1/2) = (0, c)$. This implies that there can not exist an I_2. If there were, we could increase y still more until all of I_1 has been "dragged across zero" and reappeared down near $1/2$ as $(c, 1/2 - \delta)$. The clockwise endpoint of this interval would map to zero under $f^{(n+m)}$. If there were another interval I_2 with the same orbit description, by the proof of the other implication of this lemma, this clockwise endpoint would have to identified with the counterclockwise endpoint of the I_j with the largest index. This is a contradiction.

Now we assume $k < n_1$, and k is a divisor of n_1. Then, because T^k is two intervals $(0, a)$ and $(b, 1/2)$, the set $T^{n_1}S^{m_1}\dots T^{n_p}S^{m_p}$ is two intervals (a_1, a_2) and (b_1, b_2), and $f^{(n_1-k)}(a_1, a_2) = (0, a)$, $f^{(n_1-k)}(b_1, b_2) = (b, 1/2)$. Now, by definition, $f^{(n+m-n_1+k)}((0, a) \cup (b, 1/2)) = (0, 1/2)$. We call this power u. We notice that $f^{(n+m)}(a_1, a_2) = f^u(0, a) = (f^{(u-1)}(x), 1/2)$ and $f^{(n+m)}(b_1, b_2) = (0, f^{(u-1)}(x))$. Notice that (a_1, a_2) covers itself under $f^{(n+m)}$ if and only if $f^{u-1}(x) < a_1$. Similarly, (b_1, b_2) covers itself if and only if $f^{u-1}(x) > b_2$. In either case, the orbit of zero begins $t^k s^{m_1}\dots t^{n_p}s^{m_p}t^d$, where d is a multiple of k less than n_1, since the intervals lying between T^{n_1} and T^k are exactly these T^d. Therefore, if d is anything other than one of these multiples there does not exist an orbit.

Definition 2. *Call the word w* primitive *if no cyclic permutation of it can be written $uuu\dots u$ for some subword u of w.*

For example, $s^2ts^3t^2st^7$ is primitive while $s^2ts^3t^2s^2ts^3t^2$ is not.

Lemma 3. *If $x = 0$ and w is a primitive word, then there exist n disjoint intervals in $\{y \in (0,1)\}$ such that there does not exist a periodic orbit described by w.*

Proof. In light of the preceding lemma and its proof, this is simply a matter of counting how many times T^k contains the orbit of zero as y goes from zero to one. As shown before, as y increases, T^k moves clockwise on the circle, sometimes splitting into two pieces but later rejoining and always moving. When the clockwise end of T^k gets to zero, if we increase y a little, T^k splits into two intervals of the forms $(0, a)$ and $(b, 1/2)$. As we increase y, a decreases until all of T^k is of the form $(b-a, 1/2)$, which interval then starts another journey towards zero. The only exception to this is when k has a divisor, say q. T^k does not become $(0, a) \cup (b, 1/2)$, but $(a_1, a_2) \cup (b_1, b_2)$. These intervals abut the two intervals T^{k-q}, which in turn abut T^{k-2q}, and so on down to the two intervals T^q, which are of the forms $(0, a)$ and $(b, 1/2)$. As the part of T^q up near zero shrinks to reappear down near one-half, so too does (a_1, a_2) shrink and (b_1, b_2) grow and then start moving towards zero again. We want to count how many times either of these two things happens. If $y = 0$, then $T^1 = (1/4, 1/2)$. Since the endpoints of T^1 move at exactly $1/2$ the speed of y, when $y = 1/2$, $T^1 = (0, 1/4)$. For all the values of $y \in (0, 1/2)$, T^1 is a single interval. If $y \in (1/2, 1)$, then T^1 is of the forms $(0, a)$ and $(b, 1/2)$ and stays split until $y = 1$. T^1 is partitioned into T^1S^m, for $m = 1, 2, \dots$, and these intervals occur in the order in which we see the S^m in $(1/2, 1)$. Thus, as y goes from zero to one, each T^1S^m gets dragged across zero once, and hence any word beginning $ts^mt\dots$ disappears and reappears exactly once. Proceeding inductively, suppose we know that T^k gets dragged across zero exactly k times. One of those times is the one that happens as a result of T^1 being dragged across zero. So assume that T^k is dragged across zero on its own, or by some proper divisor, exactly $(k-1)$ times. Then T^{k+1} is two intervals which have T^1 between them every one of those $(k-1)$ times. When we increase y, eventually the two T^{k+1} unite and T^{k+1} is located clockwise of T^1. Therefore, T^{k+1} passes from the counterclockwise to the clockwise side of T^1 $(k-1)$ times as y goes from zero to one. Once it is on the clockwise side it must get back to the counterclockwise side so that it can bracket T^1 the next time T^k gets dragged across zero. Since things are always moving clockwise, T^{k+1} gets dragged across zero exactly once for every one of these $(k-1)$ times. Also, when $y = 0$, T^{k+1} is clockwise of T^k; after the $(k-1)$ cycles just described it is still clockwise of T^k. However, when $y = 1/2$, it is counterclockwise of T^k, hence, it must get dragged across at least one more time, for a total of k. It cannot do it any more since to get dragged across zero again it would have to go from down near one-half to up near zero and hence overtake and bracket T^1 again. That can not happen without forcing T^k across zero an extra time.

Consider the collection of intervals of the form $T^{n_i}S^{m_i}\dots T^{n_{i-1}}S^{m_{i-1}}$: these are the cyclic permutations of w which begin with t and end in s. If w is primitive, these intervals are disjoint for all values of y. As y goes from zero to one, each of them gets dragged across zero n_i times, thus giving n intervals on the line segment $x = 0$ in parameter space for which there does not exist a periodic orbit described by w.

Theorem 3. *If w is the primitive word $w = t^{n_1}s^{m_1}t^{n_2}s^{m_2}\dots t^{n_p}s^{m_p}$ and $d = gcd(n, m)$, then the region of parameter space for which there does not exist a periodic orbit described by w is d annuli, with piecewise linear boundaries, each with central curve an $(n/d, m/d)$ curve on the torus.*

Proof. Recall that parameter space is symmetric with respect to the transformation, $(x, y) \mapsto (1-y, 1-x)$ and s and t are interchanged. Also we recall that zero is identified with one for both x and y. Therefore, there are n intervals on $x = 1$ and m intervals on $y = 0 = 1$ for which w does not exist. Almost all we have left to show to conclude the theorem is that the endpoints of these intervals match up in the right way. That is, if, from the lower endpoint of the first of the n intervals on $x = 0$, there extended a PL arc to the first endpoint of the first interval on $y = 0$ and if, from the next endpoint on $x = 0$ there extended a PL arc to the next endpoint on $y = 0$, and so on all the way up $x = 0$ and across $y = 1$, then we would know that the set with no orbits at least contained a set with the right topology. We know that there is a PL arc emanating from each of these endpoints. We also know that they do not just stop because of Proposition 1. If we eliminate the possibility of these arcs intersecting in any of the ways shown in figure 4, then they'd have to match up as we wish.

At each point of intersection shown in figure 4 we have two line segments, each of which represents parameter values for which we have a saddle connection. These points then must correspond to parameter values for which we have two saddles. Therefore, one of the segments represents a saddle in the orbit of zero, and the other represents a saddle in the orbit of one. The former segments all have slopes greater than -1, and the latter, slopes less than -1. Therefore, the dotted lines shown can be chosen to have slope equal to -1.

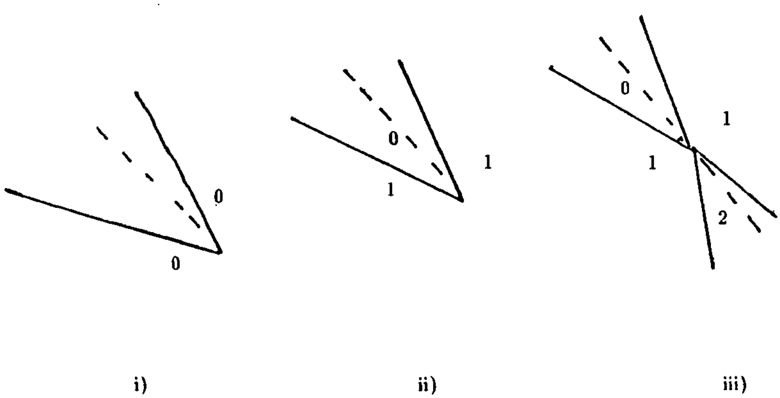


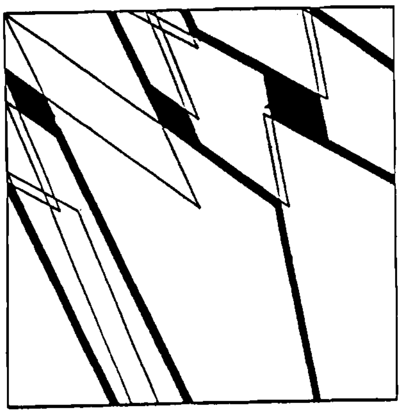
Figure 4. Impossible intersections of arcs representing saddles.

We consider case i) first. For parameter values inside the wedge-shaped region there must exist exactly one orbit for w. We choose parameter values in this region on the dotted line very close to the "tip". If from this point we hold y fixed and increase x, then we will get a saddle which is a permutation of w and lies on a line of slope less than -1. Hence this permutation begins with s and describes the orbit of one. On the other hand, if we fix x and decrease y, we get a saddle on the orbit of zero. If we take directional derivatives, we find that moving in parameter space along the dotted line in the direction of increasing x causes periodic points in $(0,1/2)$ to move counterclockwise and periodic points in $(1/2,1)$ to move clockwise. This means that there are points $p \in (0,1/2)$ and $q \in (1/2,1)$ in the orbit which both converge to $1/2$ as the parameters converge to the tip. We assume that $f^l(p) = q$; p is contained in an interval I which is described $T^k S^{m_i} \dots T^{n_{i-1}} S^{m_{i-1}} T^d$. Also, q is contained in an interval J, which is described $S^1 T^{n_j} S^{m_j} \dots T^{n_{j-1}} S^{m_{(j-1)}-1}$, and $f^l(I) \supset J$. As we change the parameters to the tip, p converges onto $1/2$ which must be the counterclockwise endpoint of I at this point. For the same reason, q must be the clockwise endpoint of J at these parameters. Then we have $J \subset f^l(I) \subset (1/2,1)$, and the counterclockwise end of I is mapped to the clockwise end of J. This forces f^l to reverse orientation on I, a contradiction.

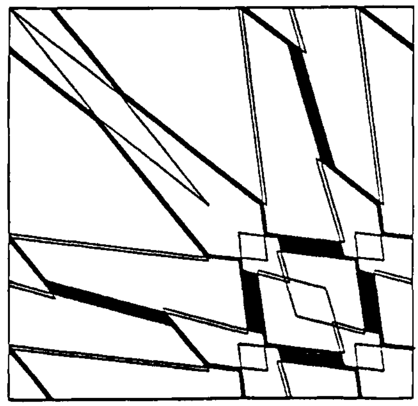
Next we consider case ii). The proof is identical. We choose parameter values on the dotted line just below the tip. There exists a single periodic orbit for w. As we move along the dotted line, two points of this orbit converge on zero, one from each side. The same geometry, and hence the same contradiction, results.

In case iii), we choose parameter values on the dotted line for which there exist two distinct orbits for w. As we change parameter values towards the intersection again, points in $(0,1/2)$ and points in $(1/2,1)$ move towards zero. If we are at parameter values very close to the point of intersection and move straight up we get a saddle on a line segment of slope less than -1; hence, it is the orbit of zero. If we fix y and decrease x, we get a saddle on the orbit of one. Thus moving along the dotted line, we get at least 2 points converging on zero, one from each side. They cannot be from the same orbit or the geometry above would result. If we are at a point near the intersection at which we have two orbits and we increase y a little, the point in $(0,1/2)$ converges on zero. Then there exists a so that $(0,a)$ has orbit description $T^k S^{m_i} \dots T^{n_{i-1}} S^{m_{i-1}} T^{n_i - k}$. We let (x_1, y_1) be a point on the line describing this saddle, and let the I_i be all the intervals in $(0,1/2)$ with this orbit description, numbered counterclockwise. There must be at least two intervals because we still have a periodic point described by this word and it can not be in $(0,a) = I_1$. Let I_j be the one which contains this periodic point. Now, from (x_1, y_1) we move, in parameter space, along the line segment giving the saddle on the orbit of zero to the point (x,y), where the orbit of one becomes a saddle. At this point we have an interval $(b,1)$ whose orbit is a cyclic permutation of w and, if you take the appropriate limit, whose counterclockwise end maps to itself under $f^{(n+m)}$. However, the orbit of one must go through the point onto which the periodic point in I_j converged. This must then be the counterclockwise end of I_j. This point can not be $1/2$; the simplest way to see this is that we know this point was moving clockwise. All this implies that $I_j = (c,d)$, $d < 1/2$, and $f^{(n+m)}(d) = d$. We know that $f^{(n+m)}(\cup I_i) = (0,1/2)$, and the orbit of zero is not an interior point of any I_i, i.e., there does not exist e such that $(e,1/2) \subset I_i$, for any i. This means that the I_i are mapped to $(0,1/2)$ with order preserved. Thus, there must be at least one interval $I_i \subset (d,1/2)$. Moreover, the image(s) of any such I_i cover $(d,1/2)$ under $f^{(n+m)}$ with order preserved. Therefore, one of them covers itself and contains a periodic point with the right orbit description. This orbit can be preserved for any sufficiently small perturbation of (x,y). This is a contradiction: from (x,y) there are arbitrarily small perturbations for which there are no orbits described by w.

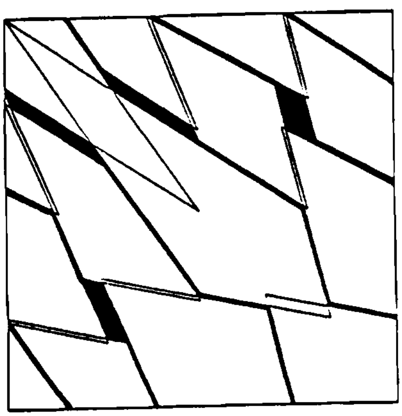
We are almost done: the boundaries match up correctly and therefore the set containing no orbits contains the tori of the theorem. If there are any more pieces of this set, then they have PL boundaries, all with negative slopes, and they do not intersect the boundary of the unit square. Therefore, they are polygons in its interior and must have a corner that looks like either ii) or iii), but that can not occur.



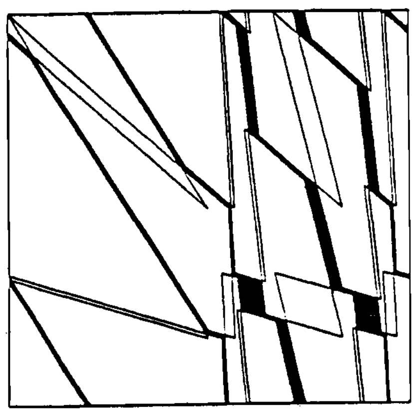
s^2tst



s^3t^3



s^2t^2st



s^4t^2

Figure 5 Some analogues of Figure 1.

As we have mentioned, this work grew out of an attempt to characterize the topological conjugacy classes of $\mathcal{F}$. This characterization has yet to be completed, though C. Tresser and R. Galeeva have made some progress in a recent (1991) preprint, *Piecewise Linear Discontinuous Double Coverings of* *the Circle.*

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