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ABSTRACT. We present a short proof of the following result due to P. Fletcher and W. Hunsaker: Each totally bounded quiet quasi-uniformity is a uniformity.

Among other things, in [4] P. Fletcher and W. Hunsaker establish the surprising result that each totally bounded quiet quasi-uniformity is a uniformity. Their argument relies heavily on the theory of completing quiet quasi-uniformities in the sense of D. Doitchinov [1,2]. In the present note we give a short proof of the aforementioned result, which does not rely upon results of [1,2].

We recall the following notation and definitions. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ and $\mathcal{G}$ be filters on X. We write $(\mathcal{F}, G) \to 0$ (see [1]) provided that for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. A quasi-uniformity $\mathcal{U}$ on a set X is said to be *totally bounded* [3, p.12] provided that the coarsest uniformity finer than $\mathcal{U}$ on X is totally bounded and it is said to be *quiet* [1,2] provided that for each $U \in \mathcal{U}$ there exists an entourage $V \in \mathcal{U}$ such that for any filters $\mathcal{F}$ and $\mathcal{G}$ on X such that $(\mathcal{F}, \mathcal{G}) \to 0$ and for any points x and y of X such that $V^{-1}(y) \in \mathcal{F}$ and $V(x) \in \mathcal{G}$, we have that $(x, y) \in U$. If V fulfills the above condition, one says that V is *quiet for* U.

Proposition 1. [4] *Each totally bounded quiet quasi-uniformity is a uniformity.*

Proof. Let $\mathcal{V}$ be a totally bounded quiet quasi-uniformity on a set X and let α be the quasi-proximity induced by $\mathcal{V}$ on X

[3, p.12]. We suppose that α is not a proximity and choose $A, B \subseteq X$ such that $A\alpha B$, but $B\overline{\alpha}A$. Then $(X \times X)\backslash(B \times A) \in \mathcal{V}$ [3, Theorem 1.33]. Since $A\alpha B$, we have that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\mathcal{M} = \{(\mathcal{F}_1, \mathcal{F}_2)$: $\mathcal{F}_1, \mathcal{F}_2$ are filters on X such that $A \in \mathcal{F}_1, B \in \mathcal{F}_2$ and such that $C \in \mathcal{F}_1, D \in \mathcal{F}_2$ imply that $C\alpha D\}$. Let us define a partial order on $\mathcal{M}$. Given any $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ belonging to $\mathcal{M}$, we set $(\mathcal{F}_1, \mathcal{F}_2) \leq (\mathcal{G}_1, \mathcal{G}_2)$ provided that $\mathcal{F}_1 \subseteq \mathcal{G}_1$ and $\mathcal{F}_2 \subseteq \mathcal{G}_2$. It is easy to check that whenever $\mathcal{K}$ is a nonempty linearly ordered subset of the nonempty partially ordered set $(\mathcal{M}, \leq)$, then $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ (where $\tilde{\mathcal{F}}_1 = \cup\{\mathcal{F}_1 : (\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{K}\}$ and $\tilde{\mathcal{F}}_2 = \cup\{\mathcal{F}_2 : (\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{K}\}$) is an upper bound of $\mathcal{K}$ in $\mathcal{M}$. By Zorn's Lemma we conclude that $(\mathcal{M}, \leq)$ has a maximal element $(\mathcal{H}_1, \mathcal{H}_2)$. Let us show that $\mathcal{H}_1$ and $\mathcal{H}_2$ are ultrafilters on X.

Suppose that $\mathcal{H}_1$ is not an ultrafilter on X. Then there is an $E \subseteq X$ such that $E \notin \mathcal{H}_1$ and $X\backslash E \notin \mathcal{H}_1$. Let $\mathcal{K}_1$ be the filter generated by $\mathcal{H}_1 \cup \{E\}$ on X and let $\mathcal{K}_2$ be the filter generated by $\mathcal{H}_1 \cup \{X\backslash E\}$ on X. Since $(\mathcal{H}_1, \mathcal{H}_2)$ is maximal in $(\mathcal{M}, \leq)$ and $\mathcal{K}_1$ and $\mathcal{K}_2$ are strictly finer than $\mathcal{H}_1$, there are $H_1, H_1' \in \mathcal{H}_1$ and $H_2, H_2' \in \mathcal{H}_2$ such that $H_1 \cap E\overline{\alpha}H_2$ and $H_1' \cap (X\backslash E)\overline{\alpha}H_2'$.

It follows that $H_1 \cap H_1' \cap E\overline{\alpha}H_2 \cap H_2'$ and $H_1 \cap H_1' \cap (X\backslash E)\overline{\alpha}H_2 \cap H_2'$. Thus $H_1 \cap H_1'\overline{\alpha}H_2 \cap H_2'$ - contradiction. Hence $\mathcal{H}_1$ is an ultrafilter on X. Similarly, one proves that $\mathcal{H}_2$ is an ultrafilter on X.

Next we show that $(\mathcal{H}_1, \mathcal{H}_2) \to 0$. Assume the contrary. Then by [3, Theorem 1.33] it is clear that there are $C, D \subseteq X$ such that $C\overline{\alpha}D$, but $(H_1 \times H_2) \cap (C \times D) \neq \emptyset$ whenever $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$. Hence $C \in \mathcal{H}_1$ and $D \in \mathcal{H}_2$, because $\mathcal{H}_1$ and $\mathcal{H}_2$ are ultrafilters - contradicting the fact that $(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{M}$. Consequently $(\mathcal{H}_1, \mathcal{H}_2) \to 0$.

Finally let us choose $V \in \mathcal{V}$ such that V is quiet for the entourage $(X \times X)\backslash(B \times A)$ of $\mathcal{V}$. Since $A \in \mathcal{H}_1, B \in \mathcal{H}_2$, the quasi-uniformity $\mathcal{V}$ is totally bounded and both $\mathcal{H}_1$ and $\mathcal{H}_2$ are ultrafilters, we see that there are $a \in A$ and $b \in B$ such that $V^{-1}(a) \in \mathcal{H}_1$ and $V(b) \in \mathcal{H}_2$. Therefore $(b, a) \in (X \times X)\backslash(B \times A)$ - a contradiction. We conclude that α is a proximity on X. Since $\mathcal{V}$ is totally bounded, it is a uniformity on X. To see this either use [5, Theorem 1 or 3, Theorem 1.33] or argue as follows: Let $W, V \in \mathcal{V}$ with $W^2 \subseteq V$. There is a finite cover $\{B_i : i \in \{1, \dots, n\}\}$ of X such that $B_i \times B_i \subseteq W$ whenever $i \in \{1, \dots, n\}$. Since α is a proximity, for each $i \in \{1, \dots, n\}$ there exists $H_i \in \mathcal{V}$ with $H_i^{-1}(B_i) \subseteq W(B_i)$. Let $H = \cap_{i=1}^n H_i$ and consider an arbitrary $x \in X$. There is $j \in \{1, \dots, n\}$ such that $x \in B_j$. Then $H^{-1}(x) \subseteq H_j^{-1}(B_j) \subseteq W(B_j) \subseteq W^2(x) \subseteq V(x)$. We conclude that $H^{-1} \subseteq V$ and that $\mathcal{V}$ is a uniformity.

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