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by

Jerry E. Vaughan

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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A COMPARISON OF TWO CONSTRUCTIONS IN TOPOLOGY

JERRY E. VAUGHAN*

ABSTRACT. We compare two constructions of families of sequentially compact Hausdorff spaces whose product is not countably compact, and show that every space that can be constructed by one construction is homeomorphic to a subspace of some space constructed by the other construction, but not vice versa.

1. INTRODUCTION

In [3], Peter Nyikos and I gave two constructions which give for every ultrafilter $u \in \omega^*$ a space X_u which is sequentially compact, Hausdorff, and not u–compact (such spaces provide a solution to the Scarborough-Stone problem [4] in the class of Hausdorff spaces). The two constructions in [3] produce spaces with similar properties. For instance, spaces constructed by either construction are always scattered, locally countable, weakly T_3 [3], weakly first countable [3], Hausdorff, non-regular spaces. The purpose of this paper is to answer the natural question: Is one of the two constructions more general that the other in some sense?

By abuse of notation, we let C denote both a construction in topology, and the class of spaces that can be constructed by the construction. In order to give a rigorous comparison of two constructions, we need a rigorous definition of the constructions. We take the two constructions as defined in [3], and recall their definitions in §2, §3 below. The construction defined in §2 is a modification using weak bases of the Osteszewski construction, and the construction defined in §3

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is an iteration of the well-known construction of the class of spaces called Ψ. We denoted these two constructions by C_1 and C_2.

Since C_1 and C_2 produce spaces which are in many ways similar, it should be noted that in fact, $C_1 \cap C_2 = \emptyset$ because C_1 produces spaces having c (= the cardinality of the continuum) isolated points, and C_2 produces spaces having only countably many isolated points.

The relation between the two constructions is described in §4 where we show that every space $X \in C_2$ is homeomorphic to a subspace of a space $Y \in C_1$. The converse, however, fails. In §5, we construct a space $X \in C_1$ with the property that every nonisolated point in X is the limit of a convergent sequence from a countable, discrete dense subset, and we show such a space X cannot be embedded into any space $Y \in C_2$.

The notion of a weak base is used in both constructions, and we now recall that definition.

Definition 1.1. (A.V. Arhangel'skiĭ[1]) *Let X be a space and for each $x \in X$ let $\mathcal{B}(x)$ be a family of subsets of X each of which contains x. We say that $\mathcal{B} = \cup\{\mathcal{B}(x) : x \in X\}$ is a* weak base *for X provided*

(1) for all $B_1, B_2 \in \mathcal{B}(x)$, there exists $B_3 \in \mathcal{B}(x)$ such that $B_3 \subset B_1 \cap B_2$, and

(2) For every $U \subset X$, U is open if and only if for every $x \in U$ there exists $B \in \mathcal{B}(x)$ such that $B \subset U$.

For an isolated point x, $\{x\} \in \mathcal{B}(x)$. For the spaces considered in this paper, the nonisolated points have a weak base of a special kind.

Definition 1.2. *Let X be a space with weak base $\mathcal{B} = \cup\{\mathcal{B}(x) : x \in X\}$, and for all nonisolated points $x \in X$ let K_x be a countable set that converges to x and $x \notin K_x$. If*

$$\mathcal{B}(x) = \{H \cup \{x\} : H \text{ is a cofinite subset of } K_x\},$$

we say that K_x generates the weak base for x, *and that X has a* weak base generated by convergent sequences.

Throughout the remainder of the paper, let $u \in \omega*$ be arbitrary but fixed.

2. Construction C_1: Modified Ostaszewski Construction

The underlying set for the space is the cardinal c. The topology is constructed by transfinite induction as follows.

Step 1. Well-order $[c]^\omega = \{K_\alpha : \omega \leq \alpha < c\}$, the set of all countably infinite subsets of c, so that $K_\alpha \subset \alpha$ for all $\omega \leq \alpha < c$.

Step 2. Begin the induction by starting with the discrete topology T_ω on ω.

Step 3. (Inductive step). Assume we have constructed for all $\omega \leq \alpha < \gamma$, where $\gamma < c$, Hausdorff topologies T_α on α and $\mathcal{B}(\alpha)$ countable families of subsets of $\alpha + 1$ each containing the ordinal α, such that $\beta < \alpha < \gamma$ implies

(1) (β, T_β) is an open subspace of (α, T_α),

(2) $\bigcup\{\mathcal{B}(\beta) : \beta < \alpha\}$ is a weak base for (α, T_α),

(3) if β is not isolated in $(\beta + 1, T_{\beta+1})$, then there exists $H_\beta \in [K_\beta]^\omega$ such that H_β converges to β and generates the weak base $\mathcal{B}(\beta)$ for β in $(\beta + 1, T_{\beta+1})$,

(4) β is isolated in $(\beta + 1, T_{\beta+1})$ if and only if K_β has a limit point in (β, T_β),

(5) u does not converge to any point in (α, T_α),

(6) K_β has a cluster point in $(\beta + 1, T_{\beta+1})$.

We now proceed to the construction of the space (γ, T_γ).

Step 3A. γ is a limit ordinal. Take T_γ to be the topology on γ having $\bigcup\{T_\alpha : \alpha < \gamma\}$ as a base. Then (γ, T_γ) works by [3, Lemma 2.10].

Step 3B. γ is a successor ordinal, say $\gamma = \alpha + 1$.

Step 3B(i). If K_α has a limit point in (α, T_α), we define $\mathcal{B}(\alpha) = \{\{\alpha\}\}$, and let T_α be the topology on $\alpha + 1$ having $T_\alpha \cup \{\{\alpha\}\}$ as a base. Thus the point α is isolated.

Step 3B(ii). If K_α does not have a limit point in (α, T_α), then choose an infinite set $H_\alpha \subset K_\alpha$ for which there exists $W \in T_\alpha$ such that $H_\alpha \subset W$, and $W \cap \omega \notin u$ (this is possible by [3, Theorem 3.2]). Take $\mathcal{B}(\alpha)$ to be the set of all sets of the form $L \cup \{\alpha\}$ where L is a cofinite subset of H_α. Take $T_{\alpha+1}$ to be the topology on $\alpha+1$ having as a base all sets $U \subset \alpha+1$ such that $U \cap \alpha \in T_\alpha$ and if $\alpha \in U$ then there exists $B \in \mathcal{B}(\alpha)$ such that $B \subset U$. Thus $T_\alpha \subset T_{\alpha+1}$.

Note that Construction C_1 can produce non-homeomorphic spaces because of the freedom in Step 1 and Step 3B(ii).

Definition 2.1. The natural convention at Step 3B(ii): *if there exists $W \in T_\alpha$ such that $K_\alpha \subset W$ and $W \cap \omega \notin u$, then take $H_\alpha = K_\alpha$ (i.e., if it is possible to use the entire set K_α to determine the topology at α then use it). If no such $W \in T_\alpha$ exists, then take any $H_\alpha \in [K_\alpha]^\omega$ for which there exists $W \in T_\alpha$ such that $H_\alpha \subset W$, and $W \cap \omega \notin u$.*

3. Construction C_2: Iterations of Ψ

Definition 3.1. *A family $\mathcal{A} \subset [X]^\omega$ is called* almost disjoint *provided if A, A' are distinct elements of $\mathcal{A}$, then $A \cap A'$ is finite.*

Definition 3.2. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A} \subset [X]^\omega$ an almost disjoint family of closed discrete subsets of X. We define a space, denoted $\Psi(X, \mathcal{A})$ as follows: The underlying set of the space is $X \cup \mathcal{A}$, where we write x_A instead of A for all $A \in \mathcal{A}$, and the topology is that having as a base*

$$\mathcal{T} \cup \bigcup_{A \in \mathcal{A}} \{\{x_A\} \cup U : U \in \mathcal{T} \text{ and } A - U \text{ is finite}\}$$

Thus A converges to x_A and generates the weak base at x_A in $\Psi(X, \mathcal{A})$.

Definition 3.3. *Let $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \gamma\}$ be a family of pairs with each X_α a topological space, such that for all $\alpha < \gamma$*

(1) $\mathcal{A}_\alpha \subset [X_\alpha]^\omega$ is an almost disjoint family of closed discrete subsets of X_α,

(2) if $\alpha + 1 < \gamma$, then $X_{\alpha+1} = \Psi(X_\alpha, \mathcal{A}_\alpha)$,

(3) if α is a limit ordinal, then $X_\alpha = \cup\{X_\beta : \beta < \alpha\}$.

Such a family is called a Ψ-system of length γ, *and the space $\Psi_\gamma = \cup\{X_\beta : \beta < \gamma\}$ is called the* iterate (or limit) of the Ψ-system, *and we write* $\Psi_\gamma = \lim\{X_\beta : \beta < \gamma\}$.

Construction C_2:

Step 1. Let X_0 denote ω with the discrete topology.

Step 2. Pick any maximal almost disjoint family $\mathcal{A}_0 \subset [\omega]^\omega$, such that $\mathcal{A}_0 \cap u = \emptyset$.

Step 3. (Inductive step) Assume we have constructed for all $\alpha < \gamma$, where $\gamma < \omega_1$, spaces X_α and maximal almost disjoint families

$$\mathcal{A}_\alpha \subset \{H \in [X_\alpha]^\omega : \text{H is closed discrete in } X_\alpha\}$$

such that $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \gamma\}$ is a Ψ-system of length γ satisfying

(1) X_α is T_2 and has a weak base generated by convergent sequences, hence consisting of countable, compact sets, and

(2) u has no limit point in X_α.

Step 3A. γ is a limit ordinal: Take T_γ to be the topology on

$$X_\gamma = \bigcup\{X_\alpha : \alpha < \gamma\}$$

having $\bigcup\{T_\alpha : \alpha < \gamma\}$ as a base. Pick a maximal almost disjoint family A_γ of countable closed discrete subsets of X_γ so that u has no limit point in $\Psi(X_\alpha, \mathcal{A}_\alpha)$, (this is possible by [3, Lemma 4.6]).

Step 3B. $\gamma = \alpha + 1$: We are given X_α, and $\mathcal{A}_\alpha$ satisfying (1) and (2); so we put $X_{\alpha+1} = \Psi(X_\alpha, \mathcal{A}_\alpha)$. We define $A_{\alpha+1}$ just as we did A_γ in Step 3A.

This completes the induction and give us a Ψ-system of T_2-spaces of length ω_1. We take X_u to be the iterate of this Ψ-system.

Construction C_2 is capable of producing non-homeomorphic spaces because of the freedom in the choice of the maximal almost disjoint families at each step. Note that C_1 is an induction on c, and C_2 is an induction on ω_1. Also note that since we start with $X_0 = \omega$, we have $|\Psi_{\omega_1}| \leq c$.

4. Every $X \in C_2$ can be embedded into a space $Y \in C_1$

We will prove

Theorem 4.1 *Every space constructed by C_2 (iterations of Ψ) is homeomorphic to a subspace of some space constructed by C_1 (modification of Ostaszewski technique).*

We begin with some lemmas.

Lemma 4.2. *Let X and Y be spaces with weak bases generated by convergent sequences. For each nonisolated point x in X, let K_x be a countable subset of X which converges to x and generates the weak base at x in X. If $h : X \to Y$ is a one-to-one map such that h maps isolated points to isolated points, and $h(K_x)$ generates the weak base at $h(x)$ for all nonisolated $x \in X$, then h is a homeomorphism onto an open subset of Y.*

Lemma 4.3. *Let X be a Hausdorff space with a weak base generated by convergent sequences. A sequence S converges to a point $x \in X$ if and only if $S - K_x$ is finite, where K_x is the sequence that converges to x and generates the weak base at x.*

Proof. Suppose S converges to x, and $S' = S - K_x$ is infinite. Since X is Hausdorff, no point of K_x is a limit point of S'; so there exists an open set $U \supset K_x$ such that $U \cap S' = \emptyset$. Thus $W = \{x\} \cup U$ is an open set which contains x and misses S', and this contradicts the hypothesis that S converges to x. The other half of the lemma is trivial.

Lemma 4.4. *Let Ψ_{ω_1} be an iteration of the Ψ-system $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \omega_1\}$ given by Construction C_2, let $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha < \omega_1\}$, and let $\kappa = |\Psi_{\omega_1}|$. Then $\kappa = |\mathcal{A}|$ and κ has uncountable cofinality.*

Proof. Note that $\mathcal{A}$ is an almost disjoint family of countable subsets of Ψ_{ω_1}. Further, it follows from Lemma 4.3 that $\mathcal{A}$ is a maximal such family because Ψ_{ω_1} is sequentially compact (in other words, because each $\mathcal{A}_\alpha$ is maximal, and the iteration is of length ω_1). Thus $\mathcal{A}$ is a maximal almost disjoint family of countable subsets of Ψ_{ω_1}. In addition, $|\mathcal{A}| = |\Psi_{\omega_1}|$ since the correspondence $A \to x_A$ from $\mathcal{A}$ onto $\Psi_{\omega_1} - \omega$ is a bijection. The lemma now follows from the well-known fact that if λ is a cardinal of countable cofinality, then every maximal almost disjoint family of countable subsets of λ has cardinality greater than λ (this fact is easy to prove using a standard diagonal argument).

Lemma 4.5. *Let Ψ_{ω_1} be the iteration of the Ψ-system $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \omega_1\}$. Then Ψ_{ω_1} can be well-ordered in order type $\kappa = |\Psi_{\omega_1}|$ in such a way that $\Psi_{\omega_1} = \{p_\alpha : \alpha < \kappa\}$, $p_n = n$ for all $n \in \omega$, and for all $\omega \leq \alpha < \kappa$ (*) if $A \in \mathcal{A}$ such that $p_\alpha = x_A$ then $A \subset \bigcup\{p_\beta : \beta < \alpha\}$.*

Proof. Well-order Ψ_{ω_1} in order type κ, put $p_n = n$ for all $n \in \omega$, and for $\omega \leq \alpha < \kappa$ define p_α by induction by selecting p_α to be the first point in

$$\Psi_{\omega_1} - \{p_\beta : \beta < \alpha\}$$

that satisfies (*). This selection is possible since any point in X_σ, where $\sigma < \omega_1$ is the first ordinal such that $X_\sigma - \{p_\beta : \beta < \alpha\} \neq \emptyset$, satisfies (*). Claim: every point $p \in \Psi_{\omega_1}$ is listed as some p_α for $\alpha < \kappa$. If this is not the case, then let $\omega \leq \sigma < \omega_1$ be the first ordinal such that X_σ contains a point p that is never listed. Say $p = x_A$. By uncountable cofinality (Lemma 4.4), there exists $\alpha < \kappa$ such that $A \subset \bigcup\{p_\beta : \beta < \alpha\}$. Since p is never listed, for every $\alpha < \beta < \kappa$ we choose for p_β a point that precedes p in the order on Ψ_{ω_1}, but this contradicts the fact that this order has order type κ.

Proof of Theorem 4.1. Let Ψ_{ω_1} be an iteration of the Ψ-system $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \omega_1\}$ given by Construction C_1, and let $\kappa = |\Psi_{\omega_1}| = |\mathcal{A}|$. Recall that $\kappa \leq c$. Let $Y \subset c$ such that $\omega + 1 \subset Y$, $|Y| = \kappa$, and $|c - Y| = c$. Let $\{y_\alpha : \alpha < \kappa\}$ be an order preserving enumeration of Y (thus $y_n = n$ for all $n \in \omega$, and $y_\omega = \omega$). Assume we have the order given by Lemma 4.5: $\Psi_{\omega_1} = \{p_\alpha : \alpha < \kappa\}$. Since $\Psi_{\omega_1} - \omega$ is in one-one correspondence with $\mathcal{A}$ we can order $\mathcal{A} = \{A_\alpha : \omega \leq \alpha < \kappa\}$ so that $p_\alpha = x_{A_\alpha}$. Define a mapping $h : \Psi_{\omega_1} \to Y$ by $h(p_\alpha) = y_\alpha$ for all $\alpha < \kappa$ (thus $h(p_n) = n = y_n$ for all $n \in \omega$, and $h(p_\omega) = \omega = y_\omega$). Since h is a bijection onto Y, $h(\mathcal{A})$ is a maximal almost disjoint family in $[Y]^\omega$.

In accordance with Step 1 of C_1 we must order $[c]^\omega = \{K_\alpha : \omega \leq \alpha < c\}$ so that $K_\alpha \subset \alpha$ for all α. This order will be constructed by transfinite induction, and will satisfy several additional properties (one purpose of these properties is to make $h(A_\beta)$ be the first set in the order from $[h(A_\beta)]^\omega$).

Let $[c]^\omega$ have a well-order in order type c, and assume we have defined $K_\alpha \in [c]^\omega$ for $\omega \leq \alpha < \gamma$, where $\gamma < c$ such that

(1) $K_\alpha \subset \alpha$,

(2) if $\alpha = y_\beta \in Y$, then $K_\alpha = h(A_\beta)$,

(3) if $K_\alpha \cap Y$ is infinite, and $\omega \leq \tau < \kappa$ is the smallest ordinal such that $K_\alpha \cap h(A_\tau)$ is infinite, then $y_\tau \leq \alpha$.

We define K_γ as follows: If $\gamma = y_\beta \in Y$, then to satisfy (2) we put $K_\gamma = h(A_\beta)$. Since $h(\mathcal{A})$ is an almost disjoint family and $K_\gamma \in h(\mathcal{A})$, the only τ for which $K_\gamma \cap h(A_\tau)$ is infinite is $\tau = \beta$; so $y_\tau = y_\beta = \gamma$, and therefore (3) holds. (1) holds because $A_\beta \subset \{p_\tau : \tau < \beta\}$; so

$$h(A_\beta) \subset h(\{p_\tau : \tau < \beta\}) = \{y_\tau : \tau < \beta\} \subset y_\beta = \gamma.$$

If $\gamma \notin Y$, define

$$\mathcal{H}_\gamma = \{H \in [c]^\omega - \{K_\alpha : \alpha < \gamma\} : H \subset \gamma\},$$

and let K be the first set in $\mathcal{H}_\gamma$. We define K_γ by an induction which starts by inspecting the set K:

(i) if $K \cap Y$ is finite, define $K_\gamma = K$,

(ii) if $K \cap Y$ is infinite and $\tau < \kappa$ is the first ordinal such that $K \cap h(A_\tau)$ is infinite, and $y_\tau \leq \gamma$, then define $K_\gamma = K$,

(iii) otherwise, discard K, and inspect the next set in $\mathcal{H}_\gamma$, and repeat (i), (ii), and (iii). Continue in this manner until some set in $\mathcal{H}_\gamma$ is defined as K_γ (there exists $K \in \mathcal{H}_\gamma$ that satisfies (ii): any $K \in [h(A_\omega)]^\omega - \{K_\alpha : \alpha < \gamma\}$; so the process eventually will define K_γ).

Claim 4.6. *For every $K \in [c]^\omega$, there exists $\alpha < c$ such that $K = K_\alpha$.*

Proof. Suppose some K is not listed. There exists $\alpha < c$ such that $K \subset \alpha$, and if $K \cap Y$ is infinite, there exists $\tau < \kappa$ such that $K \cap h(A_\tau)$ is infinite. If $K \cap Y$ is finite put $\gamma = \alpha$, and if $K \cap Y$ is infinite put $\gamma = \max\{\alpha, y_\tau\}$. At each step $\gamma < \sigma < c$ with $\sigma \notin Y$, we have $K \in \mathcal{H}_\sigma$, but K was not inspected at step σ since if it had been, it would have been put equal to K_σ by

(i) (if $K \cap Y$ is finite) or by (ii) (if $K \cap Y$ is infinite). Thus for each $\gamma < \sigma < c$, with $\sigma \notin Y$, a set which precedes K in the order on $[c]^\omega$ was defined to be K_σ, but this is impossible since $|c - Y| = c$, and the order on $[c]^\omega$ has order type c.

We now apply construction C_1 to $\{K_\alpha : \omega \leq \alpha < c\}$, and use the natural convention at step 3B(ii). Denote the resulting spaces by (α, T_α) for all $\alpha \leq c$, and let $X = (c, T_c)$.

Claim 4.7. *For every $\omega \leq \alpha < c$, if $\alpha \in Y$, then K_α converges to α and generates the weak base at α, and if $\alpha \notin Y$ and $K_\alpha \cap Y$ is infinite, then α is isolated in X.*

Proof. Assume true for $\omega \leq \alpha < \gamma$, where $\gamma < c$. Case 1. $\gamma \in Y$. Then by (2) $\gamma = y_\beta$, and $K_\gamma = h(A_\beta)$ for some $\beta < \kappa$. We claim that K_γ has no limit points in (γ, T_γ): otherwise, let $\alpha < \gamma$ be the first limit point of K_γ. First we note that $K_\alpha \cap K_\gamma$ is finite: since $K_\gamma = h(A_\beta) \subset Y$, this is clear if $K_\alpha \cap Y$ is finite; so we assume that $K_\alpha \cap Y$ is infinite. Thus, since α is not isolated, it follows from the induction hypothesis that $\alpha \in Y$, therefore by (2) $K_\alpha \in h(\mathcal{A})$; so by almost disjointness, $K_\alpha \cap K_\gamma$ is finite. By virtue of the construction C_1, there exists $H_\alpha \subset K_\alpha$ such that H_α converges to α and generates the weak base at α in X. Let H'_α be a cofinite subset of H_α such that $H'_\alpha \cap K_\gamma$ is empty. For each $\sigma \in H'_\alpha$, $\sigma < \alpha$. Since α is minimal, there is a neighborhood U_σ of σ such that $U_\sigma \cap K_\gamma = \emptyset$. Thus

$$W = \{\alpha\} \cup \bigcup\{U_\sigma : \sigma \in H'_\alpha\}$$

is a neighborhood of α containing at most one point of K_γ. This contradicts our assumption that α is a limit point of K_γ, and hence K_γ has no limit points in (γ, T_γ). Now it follows by Construction C_1 Step 3B(ii) and 2.1, that K_γ converges to γ and generates the weak base at γ.

Case 2. $\gamma \notin Y$ and $K_\gamma \cap Y$ is infinite. Let $\tau < \kappa$ be the smallest ordinal such that $K_\gamma \cap h(A_\tau)$ is infinite. By (3) above, we have $y_\tau \leq \gamma$, and since $\gamma \notin Y$ we have $y_\tau < \gamma$. By the induction hypothesis, $K_{y_\tau} = h(A_\tau)$ converges to y_τ; so K_γ has y_τ as a limit point in X. Since $y_\tau < \gamma$, K_γ has y_τ as a limit point in (γ, T_γ). Thus by Step 3B(i), γ is isolated in X.

Claim 4.8. *h is a homeomorphism from Ψ_{ω_1} onto Y.*

Proof. We have that $h(n) = n$ for all $n \in \omega$, and A_α converges to p_α and generates the weak base at p_α for all $\omega \leq \alpha < \kappa$. Moreover, K_{y_α} converges to y_α and generates the weak base at y_α. Since $h(p_\alpha) = y_\alpha$, and $h(A_\alpha) = K_{y_\alpha}$, it follows from Lemma 4.2 that h is a homeomorphism.

Claim 4.9 *$h(\Psi_{\omega_1}) = Y$ is clopen in X. Thus, the ultrafilter u does not converge in X.*

Proof. By Lemma 4.2, $h(\Psi_{\omega_1}) = Y$ is open in X. Indeed, Y satisfies (2) of Definition 1.1. A similar proof shows that its complement is also open, or we can use the known results that X is a sequential, Hausdorff space [3, Lemma 2.8], and that in such a space, countably compact subspaces are closed.

This completes the proof of 4.1.

5. A SPACE $X \in C_1$ WHICH CANNOT BE EMBEDDED INTO ANY SPACE $Y \in C_2$

We will construct a space by C_1 in which every non-isolated point is the limit of a convergent sequence in ω, and show that such spaces cannot be embedded into any Ψ_{ω_1}.

Definition 5.1 *We say $Y \subset \Psi_{\omega_1}$ is* bounded *provided there exists $\alpha < \omega_1$ such that $Y \subset X_\alpha$.*

Lemma 5.2. *Every bounded, countably compact subset of Ψ_{ω_1} is countable.*

Proof. If the result is not true, take $\alpha < \omega_1$ to be the smallest ordinal such that there exists an uncountable, countably compact $Y \subset X_\alpha$. Thus, for all $\beta < \alpha$, $Y \not\subset X_\beta$ so by countable compactness, α is a successor ordinal; say $\alpha = \gamma + 1$. Again by countable compactness, $Y \cap (X_\alpha - X_\gamma)$ is finite. By local countability [3, Lemma 2.3] there is a countable open set $W \supset Y \cap (X_\alpha - X_\gamma)$. Thus $Y - W$ is an uncountable, countably compact subset of X_γ, and this contradicts the definition of α.

Recall that the *sequential closure* of a set A in a space X is defined to be

$$A \cup \{x \in X : \text{there is a sequence in } A \text{ converging to } x\}.$$

Lemma 5.3. *Let Ψ_{ω_1} be an iteration of the Ψ-system $\{(X_\alpha, \mathcal{A}_\alpha) : \alpha < \omega_1\}$ as given by Construction C_2. For all $\alpha < \omega_1$, the sequential closure of X_α is $X_{\alpha+1}$. In particular, the sequential closure of any countable subset of Ψ_{ω_1} is bounded.*

Proof. Let $\beta \geq \alpha + 2$ and $x \in X_\beta - X_{\alpha+1}$. For A such that $x = x_A$, we have by definition of Ψ_{ω_1}, that $A \cap X_\alpha$ is finite. By the Lemma 4.3, if S is a sequence that converges to x, then $S - A$ is finite; so $S \not\subset X_\alpha$. Thus, x is not in the sequential closure of X_α. For the other inclusion, we note that every point in $X_{\alpha+1}$ is the limit of a sequence from X_α by definition, and this completes the proof.

Theorem 5.4. *There exists a space constructed by C_1 in which every non-isolated point is in the sequential closure of ω.*

Proof. By Zorn's Lemma, let $\mathcal{A} \subset [\omega]^\omega$ be a maximal almost disjoint family of size c such that $\mathcal{A} \cap u = \emptyset$, and give $[c]^\omega$ any well-order of order type c. We proceed by induction to construct a topology on c, and a listing of $[c]^\omega$. Let S_ω denote the discrete topology on ω, and assume we have defined Hausdorff topologies S_α on α, and countable sets $K_\alpha \subset \alpha$ for all $\omega \leq \alpha < \gamma$, where $\gamma < c$, such that for $\omega \leq \beta < \alpha$,

(1) (β, S_β) is an open subspace of $(\beta + 1, S_{\beta+1})$,
(2) if β is not isolated in $(\beta + 1, S_{\beta+1})$, then
(a) $K_\beta \cap \omega$ is infinite, and $K_\beta \cap \omega \subset A$ for some $A \in \mathcal{A}$,
(b) There exists an open set $W \in S_\beta$ such that $K_\beta \subset W$ and $W \cap \omega \notin u$,
(c) K_β converges to β, and generates the weak base at β in $(\beta + 1, S_{\beta+1})$,
(3) β is isolated in $(\beta + 1, S_{\beta+1})$ if and only if K_β has a limit point in (β, S_β),
(4) u does not converge to any point in (α, S_α).

We construct S_γ and K_γ. If γ is a limit ordinal take S_γ to be the topology on γ having $\cup\{S_\alpha : \alpha < \gamma\}$ as a base. Then (γ, S_γ) has a weak base generated by convergent sequences, and u does not converge in (γ, S_γ). Define K_γ as follows: Put

$$\mathcal{H}_\gamma = \{H \in [c]^\omega - \{K_\alpha : \omega \leq \alpha < \gamma\} : H \subset \gamma\}$$

and let B_γ be the first element of $\mathcal{H}_\gamma$ in the given order of $[c]^\omega$.

Case 1. If B_γ has a limit point in (γ, S_γ), put $K_\gamma = B_\gamma$. Case 2. If B_γ has no limit point in (γ, S_γ), then since u has no limit point in (γ, S_γ), by ω-collectionwise Hausdorff (see [3, Lemma 2.6]) there exists an infinite set $B' \subset B_\gamma$ and an open set $W' \supset B'$ such that $W' \cap \omega \notin u$. If $B' \cap \omega$ is infinite, then there exists $A \in \mathcal{A}$ such that $B' \cap A$ is infinite; put $K_\gamma = (B' - \omega) \cup (A \cap B')$ and $W = W' \cup A$. If $B' \cap \omega$ is finite, then pick an $A \in \mathcal{A}$ such that A has no limit points in (γ, S_γ), and put $K_\gamma = (B' - \omega) \cup A$ (since $|\mathcal{A}| = c$ we can pick $A \in \mathcal{A}$ such that $K_\beta \cap \omega \not\subset A$ for all $\beta < \gamma$; thus A has no limit points in (γ, S_γ) by 4.3 and 2(a),(c)) and $W = W' \cup A$. Note that in Case 2, $K_\gamma \cap B_\gamma$ is infinite, and K_γ has no limit point in (γ, S_γ).

If γ is a successor ordinal, say $\gamma = \alpha+1$, we are given S_α and K_α. If K_α has a limit point in (α, S_α), them make α isolated in $(\alpha+1, S_{\alpha+1})$ as in Step 3B(i) of C_1. If K_α does not have a limit point in (α, S_α), then take $S_{\alpha+1}$ to be the topology on $\alpha + 1$ defined as in Step 3B(ii) of C_1 using 2.1 so that K_α converges to α, and generates the weak base at α. By 2(b), u does not converge in $(\alpha + 1, S_{\alpha+1})$. Define $K_{\alpha+1}$ in the same manner as K_γ was defined in the limit ordinal case.

This completes the induction.

Claim 5.5. *If $K \in [c]^\omega$, and α and β are ordinals such that $\omega \leq \alpha < \beta < c$ and $K = B_\alpha = B_\beta$, then $K = K_\alpha$ or $K = K_\beta$.*

Proof. If $B_\alpha(= K)$ has a limit point in (α, T_α), then $K_\alpha = K$. Otherwise, K_α was defined so that $K_\alpha \cap K$ is infinite, and the topology at α was defined so that K_α converges to α. Thus K has α as a limit point in $(\alpha + 1, T_{\alpha+1})$, hence in (β, T_β). Since $B_\beta(= K)$ has a limit point in (β, T_β), $K_\beta = B_\beta$; so $K = K_\beta$.

Claim 5.6. *Every set $K \in [c]^\omega$ is listed as K_α for some $\omega \leq \alpha < c$.*

Proof. If this is not true, let K be a set in $[c]^\omega$ that is not listed as any K_α. By 5.5 there is at most one α such that $K = B_\alpha$, and since K is countable, there exists $\beta < c$ such that $K \subset \beta$, and $K \neq B_\gamma$ for all $\beta < \gamma < c$. Thus $K \in \mathcal{H}_\gamma$, and $K \neq B_\gamma$ for all $\beta < \gamma < c$; so the B_γ are sets that precede K in the order on $[c]^\omega$. Since that order has order type c, there exists at least one set B that precedes K and such that $B = B_\sigma$ for infinitely many σ. Pick $\sigma_0 < \sigma_1 < \sigma_2$ such that $B = B_{\sigma_i}$ for $i \in \{0, 1, 2\}$. By 5.5, $B = K_{\sigma_0}$ or $B = K_{\sigma_1}$; so $B \notin \mathcal{H}_{\sigma_2}$. But $B = B_{\sigma_2} \in \mathcal{H}_{\sigma_2}$. This is a contradiction.

Claim 5.7 *Every non-isolated point in the space (c, S_c) is in the sequential closure of ω.*

Proof. If β is a non-isolated point, then by 2(c), K_β converges to β, and by 2(a), $K_\beta \cap \omega$ is infinite. Thus, there is a sequence in ω which converges to β.

Now apply construction C_1 using the above order on $[c]^\omega = \{K_\alpha : \alpha < c\}$, and using the natural convention at Step 3B(ii). Denote the resulting topology on α by T_α for $\omega \leq \alpha \leq c$.

Claim 5.8. *For every $\omega \leq \alpha \leq c$ we have $S_\alpha = T_\alpha$.*

Proof. This follows easily by induction.

Claim 5.9. *In the space (c, T_c) constructed above by C_1, a point is non-isolated if and only if it is in the sequential closure of $\omega \subset X$.*

Proof. This follows from 5.7 and 5.8.

Corollary 5.10. *The space (c, T_c) constructed above cannot be embedded into any space constructed by C_2.*

Proof. Suppose $h : (c, T_c) \to \Psi_{\omega_1}$ is an embedding. By 5.2, the countably compact subspace $h(c)$ is unbounded, but by 5.3 the sequential closure of $h(\omega)$ is bounded. By 5.9 the set of non-isolated points of $h(c)$ is bounded. This contradicts that $h(c)$ is countably compact.

Thus, (c, T_c) is a space constructed by C_1 which cannot be embedded into any space construced by C_2.

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University of North Carolina at Greensboro
Greensboro, NC 27412