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Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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ALL SUBMETRIZABLE NORMAL P-SPACES ARE PERFECT

A. V. ARHANGEL'SKII

ABSTRACT. When is X a normal P-space? Some new results and problems in this direction are presented.

All topological spaces considered below are assumed to be T_1-spaces. Notation and terminology follow [3]. A space X is *perfect* if every closed subset of X is a G_δ in X. K. Morita has introduced the important class of P-spaces. He has shown that X is a normal P-space if and only if the product $X \times M$ is normal for every metrizable space M(see[9]). In particular, each normal perfect space is a normal P-space [10]. A space X is *submetrizable*, if there exists a one-to-one continuous mapping of X onto a metrizable space Z. The following assertion is the main result of this article.

Theorem 1. *If X is a submetrizable normal P-space, then X is perfect.*

Proof. Consider the product space $Z = X \times I$, where I is the unit segment with the usual topology. Obviously, the space Z is also a submetrizable normal P-space. Let us fix a one-to-one continuous mapping f of Z onto a metrizable space M. The "graph" mapping $g : Z \to Z \times M$ is described by the rule: $g(z) = (z, f(z))$, for every $z \in Z$. It is well known that g maps homeomorphically Z onto the closed subspace $g(Z)$ of the space $Z \times M$. Take any subspace Y of Z. Then Y is homeomorphic to $g(Y)$ and $g(Y) = g(Z) \cap (Z \times f(Y))$, since g is one-to-one. Therefore $g(Y)$ is closed in $Z \times f(Y)$. The space $Z \times f(Y)$ is normal as the product of a normal P-space with a metrizable space. It follows that the spaces $g(Y)$ and Y are also normal. Thus we have established that the space Z is hereditarily normal. Now we can apply Katetov's Theorem on hereditarily normal products ([7], [11], p.811) to the space $Z = X \times I$. According to this theorem, since not all countable subsets of I are closed in I, the space X has to be perfectly normal.

Recall that $iw(X) \leq \omega$ if there exists a one-to-one continuous mapping of X onto a separable metrizable space Y. K. Morita has characterized topological spaces X such that the product $X \times M$ is normal for every separable metrizable space M : X satisfies this condition if and only if X is a normal $P(\omega)$-space [9], [6]. The proof of Theorem 1 demonstrates that the following assertion is valid.

Theorem 2. *If X is a normal $P(\omega)$-space and $iw(X) \leq \omega$, then X is perfect.*

Corollary 1. *Let X be a paracompact P-space with a G_δ-diagonal. Then X is perfect.*

Proof. Every paracompact space with a G_δ-diagonal is submetrizable (see [5]). It remains to apply Theorem 1.

Corollary 2. *Every Lindelöf $P(\omega)$-space with a G_δ-diagonal is perfect.*

Proof. If X is a Lindelöf space with a G_δ-diagonal, then $iw(X) \leq \omega$ (see[5]). Now we apply Theorem 2.

Observe that we can reformulate Theorem 1 as follows: a normal submetrizable space is a P-space if and only if it is perfect. In a similar way we can reformulate Theorem 2 as well. Michael's line is an example of a paracompact submetrizable space which is not perfect. We also know that Michael's line is not a P-space [8]. Now it is more clear, why it is so: Theorem 1 provides a reason for that.

Our next result is a generalization of Theorem 1.

Theorem 3. *Let $\mathcal{P}$ be a class of spaces such that for any $X \in \mathcal{P}$ and any metrizable space Y, the product $X \times Y$ belongs to $\mathcal{P}$, and let $\mathcal{P}$ be closed hereditary. If X is submetrizable and $X \in \mathcal{P}$, then every subspace of $\mathcal{P}$ belongs to $\mathcal{P}$.*

Proof. The proof of Theorem 1 given above can be trivially transformed into a proof of Theorem 3.

Corollary 3. *If X is a submetrizable p-space, then X is a hereditarily p-space.*

Not every locally compact normal submetrizable space is a P-space. Indeed, there is even a collectionwise normal locally compact submetrizable space which is not perfectly normal (E. van Douwen [4]; see also [11]). By Theorem 1, this space cannot be a P-space.

Let $\mathcal{P}$ be a class of spaces. We shall say that a mapping $f : X \to Y$ is $\mathcal{P}$-open if for each open subset U of X there is a subspace $B \subset U$ which is open and closed in U and satisfies the conditions: $B \in \mathcal{P}$ and the set $g(U \backslash B)$ is open in Y. A mapping $f : X \to Y$ is said to be $\mathcal{P}$-simple if for every open subset Y of X there is a subspace $P \subset U$ open and closed in U such that $P \in \mathcal{P}$ and $f(U \backslash P)$ is an F_σ in Y.

The next assertion is obvious.

Proposition 1. *Every $\mathcal{P}$-open mapping into a perfect space is $\mathcal{P}$-simple.*

A mapping is a $\mathcal{P}$-condensation if it is one-to-one, continuous and $\mathcal{P}$-simple. A space X is called a Michael $\mathcal{P}$-space if there exists a $\mathcal{P}$-condensation of this space onto a regular σ-compact space. The natural identity mapping of the Michael line onto R (see [8]) is $\mathcal{P}$-open and $\mathcal{P}$-simple, where $\mathcal{P}$ is the class of all discrete spaces.

Proposition 2. *Let $\mathcal{P}$ be a class of spaces satisfying the following conditions: 1) $\mathcal{P}$ is closed hereditary; 2) for each $X \in \mathcal{P}$ and each regular σ-compact space Y the product space $X \times Y$ belongs to $\mathcal{P}$; 3) If $X \in \mathcal{P}$ and $Y \in \mathcal{P}$, then the free* *topological sum $X \oplus Y$ belongs to $\mathcal{P}$. Then for each Michael $\mathcal{P}$-space $X \in \mathcal{P}$ every open subspace of X is in $\mathcal{P}$.*

Proof. Let U be an open subspace of X. Let us fix a $\mathcal{P}$-condensation $f : X \to Y$, where Y is a regular σ-compact space. We can also choose a subspace P of U open and closed in U such that $P \in \mathcal{P}$ and $f(U \backslash P)$ is an F_σ in Y. By the argument in the proof of Theorem 1, the space $V = U \backslash P$ is homeomorphic to a closed subspace of the product space $X \times Z$, where Z is the subspace $f(U \backslash P)$ of Y. Since Y is regular and σ-compact, and Z is an F_σ in Y, the space Z is also σ-compact and regular. By conditions 1) and 2), it follows that $V \in \mathcal{P}$. It remains to apply condition 3).

Very often it happens that if every open subspace of a space X has a property $\mathcal{P}$, then every subspace of X has $\mathcal{P}$. In particular, among such properties we find the following ones: paracompactness, Lindelöf property, metacompactness, subparacompactness, submetacompactness, screenability and weak θ-refinability; this list is by no means complete. Each of the properties mentioned above is closed hereditary and is preserved by the products with σ-compact spaces in the class of regular spaces (see, for example, [3], Theorem 6.1). Except for Lindelöf property, they are also preserved by the free topological sums. Therefore the following assertion is valid.

Theorem 4. *Let X be a regular Michael $\mathcal{P}$-space belonging to $\mathcal{P}$, where $\mathcal{P}$ is one of the following: paracompact, screenable, metacompact, subparacompact, submetacompact, weakly θ-refinable. Then X has this property hereditarily, i.e. every subspace of X is in $\mathcal{P}$.*

This nicely agrees with our knowledge that the Michael line is hereditarily paracompact [8].

Theorem 5 *Let $f : X \to Y$ be a $\mathcal{P}$-open mapping, where $\mathcal{P}$ is the class of all normal countably paracompact spaces, $X \in \mathcal{P}$ and Y is locally compact, separable and metrizable. Then every subspace of X is normal and countably paracompact.*

Proof. The proof of this assertion is similar to the proof of Theorem 4: we just use the fact that the product of a normal countably paracompact space with a locally compact, separable, metrizable space is normal and countably paracompact. Recall also that if every open subspace of a space X is normal, then every subspace of X is normal, and that the same is true for countable paracompactness as well.

We close the article with a few open questions. The first three of them are closely related to each other.

Problem 1. *Is every submetrizable P-space perfect? Is this true for Tychonoff submetrizable P-spaces?*

Problem 2. *Is every normal P-space with a G_δ-diagonal perfect?*

Problem 3. *Let X be a $P(\omega)$-space such that $iw(X) \leq \omega$. Is then X perfect?*

Problem 4. *Is every locally compact normal P-space with a G_δ-diagonal submetrizable? Is it perfect?*

We would like to have an answer in ZFC, but a consistency answer is also welcome.

Problem 5. *Is each locally compact normal P-space submetacompact? Weakly θ-refinable? What if we add to our assumptions on X that it is collectionwise normal?*

Problem 6. *Is every locally compact metacompact normal P-space paracompact?*

We are looking for an answer to the last question in ZFC, since W. S. Watson has demonstrated that under $V = L$ every normal locally compact metacompact space is paracompact [12]. A part of motivation for problem 6 comes from the following result of Arhangel'skii: every perfectly normal metacompact locally compact space is paracompact (see [3] for a discussion of related matters).

Problem 7. *Is every normal (collectionwise normal) locally compact submetacompact (subparacompact) space a P-space? In other words is the product of any such space with every metrizable space normal?*

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Department of Mathematics
Moscow State University
Moscow, 119899 Russia