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FORCING AND TOPOLOGICAL PROPERTIES

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ABSTRACT. Adding a dominating real is a *ccc* forcing which transforms a non-normal space of van Douwen to a metrizable space. An Ostaszewski space remains countably compact after adding Cohen reals.

Watson[W] posed the following problems, which are numbered 146 and 147 in [vMR].

Problem 78. *Can countable chain condition forcing make a non-normal space metrizable?*

Problem 79. *Is there, in ZFC, a cardinal-preserving forcing which makes a non-normal space metrizable?*

We answer both of these questions affirmatively by combining the following two well known constructions. The space is from Remark 12.6 of van Douwen [vD]; the forcing is from Exercise 2.8 of Kunen's text [K].

Topologize $Y = {}^{\omega}\omega \cup \omega \cup {}^{\omega}\omega \times \omega \times \omega$ as follows: Points of ${}^{\omega}\omega \times \omega \times \omega$ are isolated, a basic neighborhood of $f \in {}^{\omega}\omega$ has the form $f \cup f \times (f \backslash F)$, with F finite, and a basic neighborhood of $k \in \omega$ has the form $k \cup {}^{\omega}\omega \times k \times (\omega \backslash n)$, with $n \in \omega$.

The space Y is not normal because of the two closed sets ${}^{\omega}\omega$ and ω. Moreover, it should be clear that for $\mathcal{F} \subset {}^{\omega}\omega$, the subspace $\mathcal{F} \cup \omega \cup {}^{\omega}\omega \times \omega \times \omega$ of Y is metrizable iff the closed sets $\mathcal{F}$ and ω can be separated iff $\mathcal{F}$ is $<^*$-bounded in ${}^{\omega}\omega$ (it means that there is $g \in {}^{\omega}\omega$ so that $\{n \in \omega : g(n) \leq f(n)\}$ is finite for all $f \in \mathcal{F}$). Therefore, a forcing which makes ${}^{\omega}\omega \cap M$ (the ground model reals) $<^*$-bounded in ${}^{\omega}\omega \cap M[G]$ (the reals of the extension) will make the non-normal space Y metrizable. We now give Kunen's description of "adding a dominating real" forcing.

$\mathbb{P}$ is the set of pairs $\langle p, F\rangle$ such that p is a finite partial function from ω to ω and F is a finite subset of ω^ω. $\langle p, F\rangle \leq \langle q, G\rangle$ iff $q \subset p$, $G \subset F$, and

$$\forall f \in G \, \forall n \in (\mathrm{dom}(p)\backslash\mathrm{dom}(q)) \, (p(n) > f(n)).$$

This forcing is countable chain condition, and hence cardinal preserving, because the set of finite partial functions from ω to ω is countable.

The following lemma might be folklore, at least in the case when X is an Ostaszewski space, but maybe a proof should be published. A space satisfying the hypothesis of the lemma must be a NAC(Normal Almost Compact) space [FKL].

Lemma. *If X is a T_1 countably compact space, and the uncountable closed subsets of X from a filter base, then X remains countably compact after adding Cohen reals.*

Proof. First we note that in the extension X is countably compact iff every countable subset has a cluster point. Every countable set in the extension appears in an intermediate model where countably many Cohen reals have been added. So it suffices to consider the case when $\mathbb{P}$ is countable. Towards a contradiction, let T and τ_n, $n \in \omega$ be $\mathbb{P}$-names so that $\mathbf{1} \Vdash T = \{\tau_n : n \in \omega\} \wedge T$ has no cluster point. For $p \in \mathbb{P}$, define

$$H_p = \{x \in X : \forall U, \text{open containing } x \exists q < p(q \Vdash |T \cap U| > 1)\}$$

It is easy to see that H_p is closed.

Case 1: H_p is uncountable for all $p \in \mathbb{P}$. By our hypothesis on X, there is a point $z \in \bigcap\{H_p : p \in \mathbb{P}\}$, and it is routine to see that $\mathbf{1}$ forces that z is a cluster point of T. Contradiction!

Case 2: $H_{p_0} = \{y_n : n \in \omega\}$ for some $p_0 \in \mathbb{P}$. By induction on $n \in \omega$, define $a_n \in X$, U_n an open subset of X, and $p_n \in \mathbb{P}$ so that

(1) $y_n \in U_n$

(2) $p_{n+1} \leq p_n$,
(3) $p_{n+1} \Vdash \tau_n = \check{a}_n$
(4) $p_{n+1} \Vdash |T \cap \check{U}_n| \leq 1$

Item (4) is possible because we started with the assumption that **1** forces that T has no cluster point. This induction was done in the ground model, where X is countably compact. So there is a, a cluster point of $\{a_n : n \in \omega\}$. Every open set containing a contains infinitely many a_n's, so a should be in H_p. However, for each $n \in \omega$, U_n is an open set containing y_n and at most one point of T. Hence a is not y_n for any $n \in \omega$. Contradiction!

References

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