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METRIC DIMENSION OF BOUNDED SUBSPACES OF EUCLIDEAN SPACES

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1. Introduction

In 1928, P. Alexandroff[1] proved that a compact subspace X of Euclidean n-space R^n has dimension $\leq m$ iff for every compact polyhedron K in R^n of dimension $n-m-1$ and every $\epsilon > 0$ there exists an ϵ-translation f of X into R^n such that $f(X) \cap K = \emptyset$. In this note, we study the metric dimension μ dim of bounded subspaces in Euclidean spaces and we shall prove

Theorem 1. *Let X be a bounded subspace in R^n. Then μ dim $X \leq m$ iff for every compact polyhedron K with dim $K = n - m - 1$ and every $\epsilon > 0$ there exists an ϵ-translation $f : X \to R^n$ such that $f(X) \cap K = \emptyset$.*

In [4], K. Sitnikov constructed a two dimensional subspace S in R^3 whose metric dimension is equal to one. We shall modify his construction to obtain a subspace $S_{n,m}$ in R^n for all integers m, n with $0 \leq m < n$ such that

$$\dim S_{n,m} = \min\{2m, n-1\} \text{ and } \mu \dim S_{n,m} = m.$$

Moreover we shall prove the following

Theorem 2. *A bounded subspace X in R^n has μ dim $X \leq m$ iff for every $\epsilon > 0$ there exist an ϵ-translation $f : X \to R^n$ such that $f(X) \subset S_{n,m}$.*

Throughout this paper we denote by R,Z and N the set of real numbers, integers and positive integers, respectively.

2. Lemmas.

Let X be a subspace of R^n. For $\epsilon > 0$ a continuous map $f : X \to R^n$ is called an *ϵ-translation* if $d(x, f(x)) < \epsilon$ for each $x \in X$ where d is the usual metric. The *metric dimension* μ dim X was defined originally by Alexandroff to be the least integer m such that for every $\epsilon > 0$, X admits an ϵ-translation into a polyhedron which is an underlying space of a locally finite simplicial complex of dimension $= m$. For a general metric space (X, ρ) the metric dimension μ dim (X, ρ) is defined to be the least integer m such that for each $\epsilon > 0$ there exists an open cover $\mathcal{U}$ of X of order $\leq m + 1$ with mesh $\mathcal{U} < \epsilon$. As was proved by Y. Smirnov[5], these two definitions of metric dimension are equivalent in the class of subspaces of Euclidean spaces.

Lemma 1. (Katětov[3]). *Every metric space (X, ρ) satisfies the relation dim $X \leq 2$ μ dim (X, ρ).*

Lemma 2. (Wilkinson [6]). *If $A_1, A_2, \ldots$ are closed proper subsets of R^n such that dim $(A_i \cap A_j) \leq k$ whenever $i \neq j$, then dim $(R^n - \cup A_i) \geq n - k - 2$.*

Lemma 3. *Let $B_1, B_2, \ldots$ be a sequence of $(n-m-1)$-planes in $R^n, 0 \leq m < n$, and let $p_1, \ldots, p_s$ be points in R^n. Then for any $\epsilon > 0$ there exist points $q_1, \ldots, q_s$ in R^n satisfying the conditions*

$(1)_s$ $d(p_i, q_i) < \epsilon, 1 \leq i \leq s$,

$(2)_s$ *$q_1, \ldots, q_s$ are in a general position, and*

$(3)_s$ *if H is a k-plane spanned by $k+1$ points from $q_1, \ldots, q_s$ with $0 \leq k \leq m$, then $H \cap B_i = \emptyset$ for every $i \in N$.*

Proof. Let us take a point $q_1 \in R^n - \cup B_i$ with $d(p_1, q_1) < \epsilon$, and suppose that for some $t < s$ the points $q_1, \ldots q_t$ are chosen in such a way that the conditions $(1)_t \sim (3)_t$ are satisfied. Every family of k points from $q_1, \ldots q_t$ with $k \leq n$ spans a $(k-1)$-plane by $(2)_t$, and we denote by $\mathcal{H}_1$ the totality of these planes. Also every B_i and every family of k points from $q_1, \ldots, q_t$ with $k \leq m$ determine a plane of dimension $\leq n-m+k-1 \leq n-1$, and by $\mathcal{H}_2$ we denote the set of these planes. Then $H = \bigcup(\mathcal{H}_1 \cup \mathcal{H}_2)$ is the countable union of nowhere dense subspace of R^n. Hence we can take a point q_{t+1} such that $d(p_{t+1}, q_{t+1}) < \epsilon$ and $q_{t+1} \in R^n - H$. Then the points $q_1, \ldots, q_{t+1}$ satisfy the conditions $(1)_{t+1} \sim (3)_{t+1}$ which completes the proof of the lemma by induction on t.

3. Construction of $S_{n,m}$

Let m, n be integers with $0 \leq m < n$. It is easy to define a subset $T_i = \{t_{i,k} \mid k \in Z\}$ of $R, i \in N$, satisfying the conditions

(1) $t_{i,k+1} - t_{i,k} = 1/i$ for every i, k and

(2) $T_i \cap T_j = \emptyset$ whenever $i \neq j$.

Denote by $\tau_i(k)$ the closed interval $[t_{i,k}, t_{i,k+1}]$ and define

$$\mathcal{T}_i = \{\tau_i(k_1, \ldots, k_n) \mid k_1, \ldots, k_n \in Z\}$$

$$\text{where } \tau_i(k_1, \ldots, k_n) = \tau_i(k_1) \times \ldots \times \tau_i(k_n).$$

Then $\mathcal{T}_i$ is a closed cover of R^n consisting of congruent n-cubes whose edges have length $1/i$ and whose interiors are pairwise disjoint. For $\tau \in \mathcal{T}_i$ we denote by $B_j(\tau)$ the union of all j-dimensional faces of τ and define

(3) $B_{i,j} = \bigcup\{B_j(\tau) \mid \tau \in \mathcal{T}_i\}$ and

(4) $S_{n.m} = R^n - \bigcup\{B_{i,n-m-1} \mid i \in N\}$.

Clearly $B_{i,n-m-1}$ is a countable union of $(n-m-1)$-planes and hence $S_{n,m}$ is a dense G_δ subspace of R^n.

Lemma 4. *The space $S_{n,m}$ satisfies the equalities*

(5) μ *dim* $S_{n,m} = m$ *and*

(6) $\dim S_{n,m} =$ *min* $\{n-1, 2m\}$.

Proof. Let $\epsilon > 0$ be given. Take an $i \in N$ such that $\sqrt{n}/2i < \epsilon$ and let $g : R^n \to R^n$ be the mapping defined by $g(x_1, \ldots, x_n) = (x_1 + 1/2i, \ldots, x_n + 1/2i)$. Then $\mathcal{T}_i^* = \{g(\tau) \mid \tau \in \mathcal{T}_i\}$ consists of n-cubes whose vertices are the centers of n-cubes in $\mathcal{T}_i$ and forms a decomposition of R^n dual to $\mathcal{T}_i$. Now we put $B_j^* = \cup\{B_j(\sigma) \mid \sigma \in \mathcal{T}_i^*\}$ where $B_j(\sigma)$ is the union of all j-dimensional faces of σ. For each $\sigma \in T_i^*, \sigma \cap B_{i,n-m-1}$ contains the center c and the projection of $\sigma - \{c\}$ from c onto $B_{n-1}(\sigma)$ defines a map

$$p_{\sigma,n} : \sigma - B_{i,n-m-1} \to B_{n-1}(\sigma) - B_{i,n-m-1}.$$

Similarly for each j-dimensional face ν of σ with $m+1 \leq j \leq n$, we can project $\nu - B_{i,n-m-1}$ onto $B_{j-1}(\nu) - B_{i,n-m-1}$ from the center of ν, and these projections determine a continuous map

$$p_{\sigma,j} : B_j(\sigma) - B_{i,n-m-1} \to B_{j-1}(\sigma) - B_{i,n-m-1}.$$

Since $B_m(\sigma) \cap B_{i,n-m-1} = \emptyset$ the composed map $p_\sigma = p_{\sigma,m+1} \circ \ldots \circ p_{\sigma,n}$ defines a map

$$p_\sigma : \sigma - B_{i,n-m-1} \to B_m(\sigma).$$

The family $\{p_\sigma \mid \sigma \in \mathcal{T}_i^*\}$ determines uniquely a continuous map $f : S_{n,m} \to B_m^* = \cup B_m(\sigma)$ because $\cup\{\sigma - B_{i,n-m-1} \mid \sigma \in \mathcal{T}_i^*\} = S_{n,m}$. Moreover for each $x \in S_{n,m}$ we have $d(x, f(x)) < \sqrt{n}/2i < \epsilon$, which implies that μ dim $S_{n,m} \leq m$ because B_m* is an m-dimensional polyhedron. On the other hand, in view of Lemma 3, $S_{n,m}$ contains m-dimensional planes and hence μ dim $S_{n,m} \geq m$ which proves (5).

Now we proceed to the proof of (6). In case $n \leq 2m+1$, it follows from (2) that

$$B_{i,n-m-1} \cap B_{j,n-m-1} = \emptyset \text{ if } i \neq j.$$

By Lemma 2 we have dim $S_{n,m} =$ dim $(R^n - \cup B_{i,n-m-1}) \geq n-1$. Since $S_{n,m}$ has no interior points in R^n we have dim $S_{n,m} = n-1 = \min\{n-1, 2n\}$. In case $n > 2m+1$, it follows from (2) that

$$\dim (B_{i,n-m-1} \cap B_{j,n-m-1}) = n - 2m - 2 \text{ whenever } i \neq j,$$

which implies that dim $S_{n,m} \geq n - (n-2m-2) - 2 = 2m$ by Lemma 2. In view of (5) and Lemma 1, we have dim $S_{n,m} \leq 2m$. Thus we obtain dim $S_{n,m} = 2m$, which proves (6). This completes the proof of the lemma.

4. Proof of theorems.

Proof of Theorem 1. Necessity. Suppose that a bounded subspace X of R^n has μ dim $X \leq m$, and let K be a compact polyhedron of dimension $n-m-1$ and ϵ a positive number. Let $B_1, \dots, B_r$ be a finite family of planes in R^n of dimension $n-m-1$ such that K is contained in $\cup B_i$. Since μ dim $X \leq m$, there exists a finite open cover $\mathcal{U} = \{U_1, \dots, U_s\}$ of X such that ord $\mathcal{U} \leq m+1$ and mesh $\mathcal{U} < \epsilon/2$(cf.[2,1.6.C]). By lemma 3 we can choose points $q_1, \dots, q_s$ in R^n such that diam$(U_i \cup \{q_i\}) < \epsilon$ and every m-plane spanned by $m+1$ points from $q_1, \dots, q_s$ is disjoint from $\cup B_i$. Then the κ-mapping $f : X \to R^n$ determined by $\mathcal{U}$ and $q_1, \dots, q_s$ satisfies the conditions that $f(X) \cap K \subset f(X) \cap (\cup B_i) = \emptyset$ and $\mathrm{d}(x, f(x)) < \epsilon$ for every $x \in X$(cf.[2, Theorem 1.10.6]).

Sufficiency. We use the same notations as in the proof of the Lemma 4. Suppose that a bounded subspace X in R^n satisfies the condition of the theorem and let $\epsilon > 0$ be given. Choose an $i \in N$ such that $\sqrt{n}/i < \epsilon$. Since X is bounded there exists a finite subcollection $\mathcal{S}$ of $\mathcal{T}_i^*$ such that $S = \cup \mathcal{S}$ contains $\epsilon/2$-neighborhood of X. Then $K = S \cap B_{i,n-m-1}$ is an $(n-m-1)$-dimensional compact polyhedron and hence by the assumption there exists a $\frac{1}{2}\epsilon$-translation $h : X \to R^n$ such that $h(X) \cap K = \emptyset$. Define a map $k : h(X) \to R^n$ by $k \mid (\sigma - B_{i,n-m-1}) = p_\sigma$ for every $\sigma \in \mathcal{S}$. Then $k(h(X)) \subset B_m^*$ and $d(x, k(h(x))) < \epsilon/2 + \sqrt{n}/2i < \epsilon$ for every x in X, so that μ dim $X \leq m$ since B_m^* is an m-dimensional polyhedron. This completes the proof of Theorem 1.

Proof of Theorem 2. Necessity. Suppose that a bounded subspace X of R^n has μ dim $X \leq m$ and let $\epsilon > 0$ be given. Then there exists a finite open cover $\mathcal{U} = \{U_1, \dots, U_s\}$ of X such that ord $\mathcal{U} \leq m+1$ and mesh $\mathcal{U} < \frac{1}{2}\epsilon$. By Lemma 3 we can choose points $q_1, \dots, q_s \in R^n$ such that diam$(U_i \cup \{q_i\}) < \epsilon$ and every m-plane spanned by $m+1$ points from $q_1, \dots, q_s$ is disjoint from $\cup\{B_{i,n-m-1} \mid i \in N\}$ which is a countable union of $(n-m-1)$-planes. Then the κ-mapping $f : X \to R^n$

determined by $\mathcal{U}$ and $q_1, \dots, q_s$ is the desired ϵ-translation such that $f(X) \subset S_{n,m}$.

Sufficiency. By the assumption for a given $\epsilon > 0$ there exists an $\frac{1}{3}\epsilon$-translation f of X into $S_{n,m}$ and an open cover $\mathcal{U}$ of $S_{n,m}$ such that ord $\mathcal{U} \leq m+1$ and mesh $\mathcal{U} < \epsilon/3$. Clearly $\mathcal{V} = f^{-1}\mathcal{U}$ is an open cover of X such that ord $\mathcal{V} \leq m+1$ and mesh $\mathcal{V} < \epsilon$ and hence μ dim $X \leq m$. This completes the proof.

Corollary 1. *For a bounded subspace X in R^n the following conditions are equivalent.*

(1) μ dim $X \leq m$.

(2) For each $\epsilon > 0$ there exist an ϵ-translation $f : X \to R^n$ and a compact polyhedron K of dimension m such that $f(X) \subset K \subset S_{n,m}$.

(3) For each $\epsilon > 0$ there exist an ϵ-translation $f : X \to R^n$ such that $Cl(f(X)) \subset S_{n,m}$.

(4) For each $\epsilon > 0$ there exist an ϵ-translation $f : X \to R^n$ such that $f(X) \subset S_{n,m}$.

Corollary 2. *A bounded subspace X in R^n has μ dim $X \leq m$ iff for every sequence $A_1, A_2, \dots$ of planes in R^n of dimension $\leq n - m - 1$ and every $\epsilon > 0$, there exist an ϵ-translation $f : X \to R^n$ such that $f(X) \cap (\cup A_i) = \emptyset$.*

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