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METRIC DIMENSION OF BOUNDED SUBSPACES OF EUCLIDEAN SPACES II

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1. INTRODUCTION.

The present paper is a continuation of [3]. We consider the following conditions for a subspace X in Euclidean space R^n:
(a) For every (n-m-1)-dimensional plane H in R^n and every $\epsilon > 0$ there exists an ϵ-translation $f : X \to R^n$ such that $f(X) \cap H = \emptyset$.
(b) For every sequence $\{H_i\}$ of (n-m-1)-dimensional planes in R^n and every $\epsilon > 0$ there exists an ϵ-translation $f : X \to R^n$ such that $f(X) \cap (\cup H_i) = \emptyset$.
(c) For every (n-m-1)-dimensional compact polyhedron K in R^n and every $\epsilon > 0$ there exists and ϵ-translation $f : X \to R^n$ such that $f(X) \cap K = \emptyset$.

In [3] we have proved that the implications (b) $\leftrightarrow$ (c) $\leftrightarrow$ μ dim $X \leq m$ are valid if X is bounded in R^n. This extend the classical Alexandroff's theorem which proves the equivalence (c) $\leftrightarrow$ dim $X \leq m$ for every compact space X in R^n because the metric dimension μ dim$X$ and dim X coincide if X is compact.

On the other hand, Cogosvili stated a theorem in [1] asserting the equivalence (a) $\leftrightarrow$ dim $X \leq m$. But his proof contains a gap, as was pointed out by [2].

The purpose of this paper is to give the following example which exhibits the equivalence (a) $\leftrightarrow$ μ dim $X \leq m$ does not hold in general.

Example There exists a bounded subspace X in R^3 which satisfies the conditions that for every line h in R^3 and every $\epsilon > 0$ there exists an ϵ-translation f of X into R^3 such that $f(X) \cap h = \emptyset$ but μ dim $X = 2$.

The space X given in the example is not compact, and the problem whether the above Cogosvili's theorem holds for compact spaces remains open.

2. Proof of Example.

In the following, we denote by N the set of positive integers and set

$$B^3 = \{x \in R^3 : \| x \| \leq 1\}, S^2 = \{x \in R^3 : \| x \| = 1\} \text{ and}$$
$$S_+ = \{x \in S^2 : x_3 > 0\}.$$

Lemma *There exists a sequence $\{g_i\}$ of lines in R^3 such that*
(1) $g_i \cap$ Int $B^3 = \emptyset$ *for every* $i \in N$,
(2) $g_i \cap g_j = \emptyset$ *if* $i \neq j$ *and*
(3) *For every line h in R^3 with $h \cap$* Int $B^3 \neq \emptyset$ and every $\epsilon > 0$ there exists $i \in N$ such that $g_i \cap B^3 \subset N_\epsilon(h)$ where $N_\epsilon(h)$ denotes the ϵ-neighborhood of h in R^3.

Proof. Let $A = \{t_i : i \in N\}$ be a dense subspace of S_+. We denote by T_i the tangent plane of S^2 at t_i and let $p_i : R^3 \to T_i$ be the orthogonal projection for every $i \in N$. It is easy to construct lines $g_{i,j}$ in R^3 for every $i, j \in N$ satisfying the conditions;
(4) $g_{i,j} \cap$ Int $B^3 \neq \emptyset$,
(5) $g_{i,j}$ is orthogonal to T_i,
(6) $g_{i,j} \cap g_{k,l} = \emptyset$ if $(i,j) \neq (k,l)$, and
(7) $\{x_{i,j} : x_{i,j} = p_i(g_{i,j}), j \in N\}$ is a dense subspace of $p_i(B^3)$ for every $i \in N$.

If we rewrite $\{g_{i,j} : i,j \in N\}$ as $\{g_i : i \in N\}$, the conditions (1) and (2) of Lemma are satisfied. To prove (3), let h be a line with $h \cap$ Int $B^3 \neq \emptyset$ and ϵ a positive number. Let T be the tangent plane of S_+ orthogonal to h and $p : R^3 \to T$ the orthogonal projection. Since A is dense in S_+, for each $k \in N$ there exists $i(k) \in N$ such that

$$\| t_{i(k)} - t \| < 1/k \text{ where } \{t\} = T \cap S_+.$$

Then for every $j \in N$ we have

$$\delta[p((g_{i(k),j} \cap B^3) \cup \{x_{i(k),j}\}] \to 0 \text{ as } k \to \infty$$

where $\delta(Z)$ denotes the diameter of Z.

From (7) it follows that $p(\{x_{i(k),j} : j \in N\})$ is dense in $pp_{i(k)}(B^3)$ which converges to $p^2(B^3) = p(B^3) \ni p(h)$ as $k \to \infty$. Therefore there exist $k, j \in N$ such that

$$\delta[p((g_{i(k),j} \cap B^3) \cup \{x_{i(k),j}\}] < \frac{1}{2}\epsilon \text{ and}$$

$$\| p(x_{i(k),j}) - p(h) \| < \frac{1}{2}\epsilon.$$

Then we have $p(g_{i(k),j} \cap B^3) \subset N_\epsilon(p(h))$, which implies that $g_{i(k),j} \cap B^3 \subset N_\epsilon(h)$. This completes the proof of the lemma.

Proof of Example. Let $\{g_i\}$ be the sequence of lines in Lemma and we set $X = \text{Int } B^3 - \cup\{g_i\}$. We prove that X satisfies the conditions of the example. Suppose that h is a line in R^3 and $\epsilon > 0$. We must prove

(1) there exists and ϵ-translation $f : X \to R^3$ such that $f(X) \cap h = \emptyset$. We may assume that $h \cap$ Int $B^3 \neq \emptyset$. By Lemma, there exists i such that $g_i \cap B^3 \subset N_\epsilon(h)$. It is easy to construct a homeomorphism $\psi : R^3 \to R^3$ such that

(2) $\psi(x) = x$ if $x \in R^3 - N_\epsilon(h)$,

(3) $\| \psi(x) - x \| < \epsilon$ if $x \in N_\epsilon(h)$, and

(4) $\psi^{-1}(h) \cap B^3 \subset g_i$.

Then the mapping $f = \psi \mid X : X \to R^3$ is an ϵ-translation and $\psi(X) \cap h \subset \psi(B^3 - g_i) \cap h = \emptyset$. This proves (1) and it remains to prove

(5) μ dim $X = 2$.

We can take a parallelotope P in B^3 with all vertices contained in X. Let $\{C_i, C_i'\}, i = 1, 2, 3,$ be pairs of opposite faces of P.

Then no line g_i meets all of C_i and C_i', $i = 1, 2, 3$. Hence by [4, Theorem 5.5] we obtain dim $X \geq \mu$ dim $(X \cap P) \geq 2$, which implies (5). This completes the proof.

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