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ELEMENTARY SUBMODELS FOR A PARTITION OF $\beta\omega\backslash\omega$

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ABSTRACT. Elementary submodels give a partition of the Stone-Čech remainder $\omega^* = \beta\omega\backslash\omega$ into $2^{\mathbf{c}}$-many, countably compact, dense subspaces: $\omega^* = \oplus_{\alpha<2^{\mathbf{c}}} C_\alpha$, $|C_\alpha| = \mathbf{c}$. This partition is topologically very non-homogeneous; indeed, can be the most non-homogeneous partition.

0. INTRODUCTION

Recently A. Dow [6] and I. Bandlow [1] applied elementary submodels to general topology very successfully. We give another application of elementary submodels to get a topologically very non-homogeneous partition of ω^*

$$\omega^* = \oplus_{\alpha<2^{\mathbf{c}}} C_\alpha,\ |C_\alpha| = \mathbf{c}$$

into countably compact, dense subspaces. Here, $\omega^* = \beta\omega\backslash\omega$ is the remainder of the Stone-Čech compactification of the set of natural numbers ω. $\oplus$ denotes just "disjoint union", and $\mathbf{c}$ is the cardinality of the continuum. Our "non-homogeneous" means that each C_α differs topologically from other C_β's (precisely, see Theorems 1 & 2). Our partition has the property that each C_α is of the smallest possible cardinality, i.e. $\mathbf{c}$, as for a countably compact dense subset in ω^*. If we allow C_α to be bigger in size, we have more freedom in choosing countably compact subspaces, and consequently it becomes easier to get a non-homogeneous partition of ω^* (cf.[4]). To lower the

*Dedicated to Professor Yukihiro Kodama on his 60th birthday.

size of C_α, we essentially need elementary submodel. Though the topological abstract of Theorem 1 was given in [4] without elementary submodels, our method is simpler and powerful e-nough to give further the best possible "non-homogeneous" partition of ω^* (Theorem 2).

1. Preliminaries

Let us denote by $T = \omega^*/\sim$ the set of all types of points in ω^* where the equivalence relation $x \sim y$ for points $x,\ y \in \omega^*$ means that there exists a permutation p of ω such that its Stone extension βp maps x to y. The equivalence class containing $x \in \omega^*$ is denoted by $[x] \in T$. Note that $|T| = 2^{\mathbf{c}}$ since $|\,[x]\,| = \mathbf{c}$ for each $x \in \omega^*$. For spaces $X,\ Y$ we write $X \approx Y$ if X is homeomorphic with Y. For a subspace X of ω^*, $\mathbb{D}(X)$ denotes the set of all countably infinite, discrete subsets in X. Note that every $A \in \mathbb{D}(\omega^*)$ is C^*-embedded in ω^*, hence $cl\ A \approx \beta\omega$. We write $A^* = cl\ A \setminus A$. Let $h : \omega \approx A$ and let $\beta h : \beta\omega \approx \beta A = cl\ A \subseteq \omega^*$ be its Stone extension. For a point $x \in A^*$ the type of the point $y \in \omega^*$ such that $\beta h(y) = x$ is denoted by $\tau(x, A)$ and called the *relative type* of x with respect to A. We consider the Rudin-Frolik order $\sqsubset$ on T which is defined as follows: Let $t_1 = [x_1]$ and $t_2 = [x_2]$; then $t_1 \sqsubset t_2$ iff there exists an embedding $f : \beta\omega \to \omega^*$ such that $f(x_1) = x_2$. In terms of relative types this is equivalent to say $[x_1] = \tau(x_2, A)$ for some $A \in \mathbb{D}(\omega^*)$ with $x_2 \in A^*$. Define $t_1 \sqsubseteq t_2$ by $t_1 \sqsubset t_2$ or $t_1 = t_2$; then $\sqsubseteq$ is a partial ordering on T. When $t_1 \sqsubset t_2$, Frolik called "t_1 *produces* t_2". For basic properties of relative types and the partial order $\sqsubseteq$ see [9], [8] and [7].

For each subspace X of ω^* we associate the quantities $E(X) \subseteq \hat{E}(X) \subseteq T$ defined by

$$E(X) = \{\tau(x, A) \mid A \in \mathbb{D}(X),\quad x \in X \cap A^*\},$$

$$\hat{E}(X) = \{\tau(x, A) \mid A \in \mathbb{D}(\omega^*),\quad x \in X \cap A^*\}.$$

$E(X)$ is identical with the set of all spaces, up to homeomorphisms, of the form $\omega \cup \{x\}(\subseteq \beta\omega$ where $x \in \omega^*)$ that is embeddable into X. So, we call $E(X)$ the *embedding type* of the space X. Note that $E(X)$ consists of the relative types made within X, while $\hat{E}(X)$ may contain some other types made from the outside of X. The embedding type $E(X)$ is a kind of "degree of compactness". For example, if X consists of only weak P-points (cf.[8]), then $E(X)$ is empty. If X is infinite and countably compact, then $E(X)$ is non-empty. If X contains a copy of ω^* or $\beta\omega$, then $E(X) = T$.

Fact 1. ω^* contains $2^{\mathbf{c}}$-many disjoint copies of ω^*; therefore each subspace in ω^* of cardinality less than $2^{\mathbf{c}}$ misses a copy of ω^*. Especially, if $|x| < 2^{\mathbf{c}}$, then $E(\omega^*\backslash X) = T$.

The notion of the embedding type of a space becomes effective once we note

Fact 2. Let $X, Y \subseteq \omega^*$. If X is topologically embeddable into Y, then $E(X) \subseteq E(Y)$. Hence, $E(X)\backslash E(Y) \neq \emptyset$ implies that X is *not* embeddable into Y.

Define $\tau[x, X] = \{\tau(x, A) \mid A \in \mathbb{D}(X),\ x \in A^*\}$ (cf.[9]). Then $E(X) = \cup\{\tau[x, X] \mid x \in X\}$, $\hat{E}(X) = \cup\{\tau[x, \omega^*] \mid x \in X\}$. The reason we introduced $\hat{E}(\ )$ besides $E(\ )$ is that $\hat{E}(\ )$ is often much easier to handle. For example we have the equality $\hat{E}(\cup_\alpha X_\alpha) = \cup_\alpha \hat{E}(X_\alpha)$ for arbitrary many subspaces $X_\alpha \subseteq \omega^*$, but the corresponding equality for $E(\ )$ is generally false even for two subspaces. Frolik [7] showed that every type is produced by at most $\mathbf{c}$ many types, that is, the cardinality of $\tau[x, \omega^*] = \{t \in T \mid t \sqsubset [x]\}$ is at most $\mathbf{c}$. So, we get the following estimation.

Fact 3. If $|X| < 2^{\mathbf{c}}$, then $|E(X)| \leq |\hat{E}(X)| < 2^{\mathbf{c}}$.

Next, we present some preliminaries about elementary submodels. Consult [2], [6] or [1] for more precise descriptions.

Let M be a non-empty subset of H. M is an *elementary submodel* of H, denoted $M \prec H$, provided that for every formula $\varphi(x_1, \ldots x_n)$ of the language of set theory and for every $a_1, \ldots a_n \in M$ we have

$$M \models \varphi(a_1, \ldots a_n) \text{ iff } H \models \varphi(a_1, \ldots a_n).$$

We frequently use the next criterion for elementary submodels, which refers semantically only to the larger structure H.

Tarski Criterion. $M \prec H$ iff for every formula $\psi(x_0, x_1, \ldots x_n)$ and every $a_1, \ldots a_n \in M$ if $H \models \exists x\ \psi(x, a_1, \ldots a_n)$, then $\exists a \in M$, $H \models \psi(a, a_1, \ldots a_n)$. a is said to be *definable from* $a_1 \ldots . a_n$ if there exists a formula φ which characterizes a, that is $\varphi(a, a_1, \ldots a_n)$ and $\forall x(\varphi(x, a_1, \ldots a_n) \to x = a)$. If $M \prec H$, and $a \in H$ is definable in H from elements of M, then a in fact belongs to M. In this paper the role of H is played by the set $H(\theta)$ of all sets hereditarily of cardinality $< \theta$. We take θ to be a sufficiently large regular cardinal. For our purposes it suffices to take θ bigger than $\kappa_\omega = sup\{\kappa_n \mid n \in \omega\}$ where $\kappa_0 = \omega, \kappa_{n+1} = 2^{\kappa_n}$; then all the power sets such as $\mathcal{P}(\omega), \mathcal{P}(\mathcal{P}(\omega)), \ldots$ belong to $H(\theta)$, and $H(\theta)$ becomes essentially a model of ZFC.

In the sequel of this paper M always denotes an elementary submodel of the sufficiently large $H(\theta)$ such that $M^\omega \subseteq M$ and $\mid M \mid = \mathbf{c}$. Here, $M^\omega \subseteq M$ means that M is closed under countable sequences. From the definability in ZFC we know for example ω, $\mathcal{P}(\omega)$, $\mathbf{c}$, $\beta\omega$, ω^*, T, (ω^*, τ), where $\tau = \{a^* \mid a \in \mathcal{P}(\omega)\}$ is a clopen base for ω^*, are all elements of M. The condition $M^\omega \subseteq M$ implies moreover that $\mathcal{P}(\omega)$, $\mathbf{c}$, $\tau \subseteq M$; but neither $\omega^* \subseteq M$ nor $T \subseteq M$ is true because $|M| = \mathbf{c} < 2^{\mathbf{c}} = |\omega^*| = |T|$. Note that, since $\tau \subseteq M$, the two kinds of topologies on $\omega^* \cap M$ coincide; that is, the subspace topology generated by $\{a^* \cap M \mid a^* \in \tau\}$ is the same as the topology generated by $\{a^* \cap M \mid a^* \in \tau \cap M\}$.

Fact 4. For every $x \in M$,
(1) $|x| \leq \mathbf{c}$ implies $x \subseteq M$;

(2) $|x| \geq \mathbf{c}$ implies $|x \cap M| = \mathbf{c}$.

Proof. (1) $|x| \leq \mathbf{c}$ implies that $H(\theta) \models$"$\exists f : \mathbf{c} \to x$ onto ". (We often omit the mention about $H(\theta)$.) By the elementarity of M, that is, by the Tarski Criterion, we can assume that $f \in M$. Since $\mathbf{c} \subseteq M$, we have $f(\alpha) \in M$ for every $\alpha \in \mathbf{c}$ (note that $f(\alpha)$ is definable from $f, \alpha \in M$). Hence $x = f[\mathbf{c}] \subseteq M$.

(2) $|x| \geq \mathbf{c}$ implies " $\exists y \subseteq x \ |y| = \mathbf{c}$". By the elementarity of M we can assume $y \in M$. Then by the above (1) we know $y \subseteq M$, i.e. $y \subseteq x \cap M$. Hence $\mathbf{c} = |y| \leq |x \cap M|$. Since $|M| = \mathbf{c}$ we have $|x \cap M| = \mathbf{c}$.

From Fact 4 we know that $|\omega^* \cap M| = |T \cap M| = \mathbf{c}$ and $\tau[x, \omega^*] \subseteq M$ for every $x \in \omega^* \cap M$.

Proposition 1. *For every $X \subseteq \omega^*$ with $X \in M$, we have $E(X \cap M) = E(X) \cap M$ and $\hat{E}(X \cap M) = \hat{E}(X) \cap M$.*

Proof. By the elementarity of M and the definability of $\tau(x, A)$, we have

$t \in E(X) \cap M$

iff $t \in M$ and $\exists A \in \mathbb{D}(X)$, $\exists x \in X \ \ t = \tau(x, A)$

iff $t \in M$ and $\exists A \in \mathbb{D}(X \cap M)$, $\exists x \in X \cap M \ \ t = \tau(x, A)$

iff $t \in E(X \cap M)$.

Hence $E(X) \cap M = E(X \cap M)$. The proof of $\hat{E}(X) \cap M = \hat{E}(X \cap M)$ is similar.

Since $E(\omega^*) = \hat{E}(\omega^*) = T$, Proposition 1 implies that $E(\omega^* \cap M) = \hat{E}(\omega^* \cap M) = T \cap M$.

Let $X \subseteq Y \subseteq \omega^*$ and $S \subseteq T$. X is called *S-countably compact* (briefly, $S -$ c.c.) if $\forall s \in S \ \ \forall A \in \mathbb{D}(X) \ \exists x \in A^* \cap X \ \ s = \tau(x, A)$. X is called *extra S-countably compact in Y* if $\forall s \in S \ \ \forall A \in \mathbb{D}(Y) \ \exists x \in A^* \cap X \ s = \tau(x, A)$. X is called *extra countably compact in Y* if $\forall A \in \mathbb{D}(Y) \ \exists x \in A^* \cap X$. (The term "extra c.c." is due to Comfort [5].) If S is non-empty, we have the implications:

$$\begin{array}{ccc} \text{extra } S - c.c. \text{ in } \omega^* & \rightarrow & S - c.c. \\ \downarrow & & \downarrow \\ \text{extra } c.c. \text{ in } \omega^* & \rightarrow & \text{countably compact.} \end{array}$$

Note that if X is $S - c.c.$, we have $S \subseteq E(X)$. The next proposition is immediate from the elementarity of M.

Proposition 2. *Let $X \subseteq \omega^*$, $S \subseteq T$ and $X, S \in M$.*
(1) If X is $S - c.c.$, then $X \cap M$ is $S \cap M - c.c.$:
(2) If X is extra $S - c.c.$ in ω^*, then $X \cap M$ is extra $S \cap M - c.c.$ in $\omega^* \cap M$.

Lemma 1. *For every $X \subseteq \omega^*$, the subspace $\omega^* \backslash X$ is extra $(T \backslash \hat{E}(X)) - c.c$ in ω^*.*

Proof. Let $A \in \mathbb{D}(\omega^*)$ and $s \in T \backslash \hat{E}(X)$. Then obviously there exists a point $x \in A^*$ with $\tau(x, A) = s$. We need to show $x \in \omega^* \backslash X$. If it were $x \in X$, then the definition of $\hat{E}(X)$ implies $s \in \hat{E}(X)$, which contradicts the choice of s.

ω^* is obviously $T - c.c$, hence Proposition 2(1) implies that $\omega^* \cap M$ is $T \cap M - c.c.$ By the above lemma $\omega^* \backslash M$ is extra $(T \backslash M) - c.c$ in ω^*. Note that $E(\omega^* \backslash M)$ is not equal to $T \backslash M$, but that $E(\omega^* \backslash M) = T$ by Fact 1. When a subspace $X \subseteq \omega^*$ satisfies $[x] \subseteq X$ for every $x \in X$, we call such a space T-*saturated.* Any T-saturated space is dense in ω^*. $\omega^* \cap M$ is T-saturated by Fact 4(1); consequently, $\omega^* \backslash M$ is also T-saturated.

2. The non-homogeneous partition of ω^*

To construct a partition of ω^* as in the Abstract, we need

Lemma 2. *Let $X \subseteq \omega^*$, $|X| < 2^{\mathbf{c}}$ and $X \in M$. Define $C = \omega^* \cap M \backslash X$. Then this C has the following properties:*
$E(C) = T \cap M$, $|(T \backslash \hat{E}(X)) \cap M| = \mathbf{c}$ and
C is extra $(T \backslash \hat{E}(X)) \cap M - c.c.$ in $\omega^* \cap M$.

Proof. Since $|X| < 2^{\mathbf{c}}$, Fact 1 implies $E(\omega^*\backslash X) = T$. So, by Proposition 1 we have $E(C) = T \cap M$. Since by Lemma 1 $\omega^*\backslash X$ is extra $(T\backslash\hat{E}(X))-c.c.$ in ω^*, Proposition 2(2) implies that C is extra $(T\backslash\hat{E}(X))\cap M - c.c.$ in $\omega^* \cap M$. Fact 3 shows $|\hat{E}(X)| < 2^{\mathbf{c}}, hence |T\backslash\hat{E}(X)| = 2^{\mathbf{c}}$. Therefore Fact 2(2) implies that $|(T\backslash\hat{E}(X)) \cap M| = \mathbf{c}$.

Theorem 1. *Let* $\omega^* = \{x_\alpha \mid \alpha < 2^{\mathbf{c}}\}$. *Choose a sequence* $\langle M_\alpha \mid \alpha < 2^{\mathbf{c}}\rangle$ *of elementary submodels of the sufficiently large* $H(\theta)$ *such that for each* $\alpha < 2^{\mathbf{c}}$

$$M_\alpha^\omega \subseteq M_\alpha,\ |M_\alpha| = \mathbf{c},\ x_\alpha \in M_\alpha \text{ and } \langle M_\beta \mid \beta < \alpha\rangle \in M_\alpha.$$

Define $C_\alpha = \omega^* \cap (M_\alpha \backslash \bigcup_{\beta<\alpha} M_\beta)$.
Then $\omega^* = \oplus_{\alpha<2^{\mathbf{c}}} C_\alpha$ *is a partition of* ω^* *into pairwise non-homeomorphic, countably compact subspaces each of which is* T*-saturated (hence, dense in* ω^**) and of cardinality* $\mathbf{c}$. *Precisely,*

$$E(C_\alpha) = T \cap M_\alpha,\quad |T \cap (M_\alpha \backslash \bigcup_{\beta<\alpha} M_\beta)| = \mathbf{c} \text{ and}$$

$$C_\alpha \text{ is extra } T \cap (M_\alpha \backslash \bigcup_{\beta<\alpha} M_\beta) - c.c. \text{ in } \omega^* \cap M_\alpha.$$

This partition is quite "non-homogeneous" in the sense that each C_α *is* not *embeddable into* $\oplus_{\beta<\alpha} C_\beta$.

Proof. Choose $\langle M_\alpha \mid \alpha < 2^{\mathbf{c}}\rangle$ and define C_α as above. The condition $x_\alpha \in M_\alpha$ implies that $\{M_\alpha \mid \alpha < 2^{\mathbf{c}}\}$ covers ω^*; hence $\{C_\alpha \mid \alpha < 2^{\mathbf{c}}\}$ is a partition of ω^*. Obviously each C_α is T-saturated. Put $\tilde{M}_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Note that the condition $\langle M_\beta \mid \beta < \alpha\rangle \in M_\alpha$ implies $\alpha, \tilde{M}_\alpha \in M_\alpha$. (It is generally false that $\tilde{M}_\alpha \subseteq M_\alpha$. Since $|\tilde{M}_\alpha| = |\alpha| \cdot \mathbf{c}$, the inclusion $\tilde{M}_\alpha \subseteq M_\alpha$ is true iff $|\alpha| \leq \mathbf{c}$. See Remark 1 below.) Since $|\tilde{M}_\alpha| < 2^{\mathbf{c}}$, we can apply Lemma 2 for $X = \omega^* \cap \tilde{M}_\alpha$. Therefore, $E(C_\alpha) = T \cap M_\alpha, |T \cap M_\alpha \backslash \hat{E}(\omega^* \cap \tilde{M}_\alpha)| = \mathbf{c}$, and C_α is extra $(T \cap M_\alpha \backslash \hat{E}(\omega^* \cap \tilde{M}_\alpha)) -$ c.c. in $\omega^* \cap M_\alpha$. Since $\hat{E}(\omega^* \cap \tilde{M}_\alpha) = \bigcup_{\beta<\alpha} \hat{E}(\omega^* \cap M_\beta) = \bigcup_{\beta<\alpha}(T \cap M_\beta) = T \cap \tilde{M}_\alpha$, we get $T \cap M_\alpha \backslash \hat{E}(\omega^* \cap \tilde{M}_\alpha) = T \cap (M_\alpha \backslash \tilde{M}_\alpha)$. By Fact 2 we can conclude that C_α is not embeddable into $\oplus_{\beta<\alpha} C_\beta = \omega^* \cap \tilde{M}_\alpha$.

Remark 1. Note that, since M_α is of cardinality $\mathfrak{c}$, $\alpha \subseteq M_\alpha$ is true iff $|\alpha| \leq \mathfrak{c}$. So, the above sequence $\langle M_\alpha \mid \alpha < 2^{\mathfrak{c}} \rangle$ is not necessarily a chain under inclusion. It becomes so iff $\mathfrak{c}^+ = 2^{\mathfrak{c}}$. Consequently, $\langle \tilde{M}_\alpha \mid \alpha < 2^{\mathfrak{c}} \rangle$, where $\tilde{M}_\alpha = \cup_{\beta<\alpha} M_\beta$ as above, becomes what we call an elementary chain if $\mathfrak{c}^+ = 2^{\mathfrak{c}}$.

Remark 2. In Theorem 1, C_β *is* embeddable into C_α if $\beta \in \alpha \cap M_\alpha$. Indeed, since $\omega^* \setminus \cup_{\gamma<\alpha} M_\gamma$ contains a copy of ω^* by Fact 1, C_α contains a copy of $\omega^* \cap M_\alpha$. If $\beta \in \alpha \cap M_\alpha$, then $C_\beta \subseteq \omega^* \cap M_\beta \subseteq \omega^* \cap M_\alpha$; hence C_α has a copy of C_β.

3. The "most" non-homogeneous partition of ω^*

Recall that the Rudin-Frolik order $(T, \sqsubseteq)$ is a "pseudo-tree", that is, for every $t \in T$ the set $\{s \in T \mid s \sqsubset t\}$ is linearly ordered. Using this pseudo-tree structure and taking more care of elementary submodels, we will refine the construction in Theorem 1. The resulted partition in Theorem 2 turns out to be the best possible non-homogeneous one. K. Eda asked in view of Remark 2 whether Theorem 1 can be refined so that each C_α is not embeddable into other C_β even in case $\beta > \alpha$. Our partition surely answers in the affirmative.

Let π denote the canonical quotient map $\pi : \omega^* \to T = \omega^*/\sim$. Once we get a partition of T, we can pull it back to ω^* by π. So, we will partition T rather than ω^*. For each $t \in T$ put

$$L(t) = \{s \in T \mid s \sqsubset t\}, \quad \tilde{L}(t) = L(t) \cup \{t\},$$
$$U(t) = \{s \in T \mid t \sqsubset s\}, \quad \tilde{U}(t) = U(t) \cup \{t\},$$
$$C(t) = L(t) \cup \{t\} \cup U(t) = \{s \in T \mid s \text{ is comparable with } t\}.$$

It is well known that for every $t \in T$ the lower part $L(t)$ forms a linearly ordered set of cardinal $\leq \mathfrak{c}$, while the upper part $U(t)$ is of cardinality $2^{\mathfrak{c}}$. Kunen observed that $(U(t), \sqsubseteq)$ is order-isomorphic to the ultrapower $(T, \sqsubseteq)^\omega/x$ where $[x] = t$; hence $U(t)$ contains $2^{\mathfrak{c}}$-many incomparable elements (cf.[3]). So, we can say that $(T, \sqsubseteq)$ is a "thick" pseudo-tree.

Lemma 3. *For every $t \in T$, $\pi^{-1}(U(t))$ is extra $\tilde{U}(t)$ – c.c. in ω^*.*

Proof. Let $t \sqsubseteq s$ and $A \in \mathbb{D}(\omega^*)$. Obviously there exists a point $x \in A^*$ such that $s = \tau(x, A)$. Then $s \sqsubset [x]$; hence $t \sqsubset [x]$, i.e. $[x] \in U(t)$, i.e. $x \in \pi^{-1}(U(t))$. This proves Lemma 3.

Recall that M is an elementary submodel of very large $H(\theta)$ with $M^\omega \subseteq M$ and $|M| = \mathbf{c}$. Note $\pi \in M$ and $(T, \sqsubseteq) \in M$. Proposition 2(2) implies

Corollary 1. *For every $t \in T \cap M$, $\pi^{-1}(U(t) \cap M) = \pi^{-1}(U(t)) \cap M$ is extra $\tilde{U}(t) \cap M$ – c.c. in $\omega^* \cap M$.*

Let $t \in T \cap M$. Since $L(t) \subseteq M$ by Fact 4(1), we have $C(t) \cap M = \tilde{L}(t) \cup (U(t) \cap M)$. In general, let $S \subseteq T$ and $X \subseteq Y \subseteq Z (\subseteq \omega^*)$; then, if X is extra S – c.c. in Z, so is Y. Hence Corollary 1 implies

Corollary 2. *Let $t \in T \cap M$. For every S with $U(t) \cap M \subseteq S \subseteq C(t) \cap M$, $\pi^{-1}(S)$ is extra $\tilde{U}(t) \cap M$ – c.c. in $\omega^* \cap M$.*

For any $S \subseteq T$, define $E(S)$, $\hat{E}(S)$ by

$$E(S) = E(\pi^{-1}(S)) \text{ and } \hat{E}(S) = \hat{E}(\pi^{-1}(S)).$$

By Lemma 3 $\pi^{-1}(U(t))$ is $\tilde{U}(t)$–c.c. Therefore $\tilde{U}(t) \subseteq E(U(t))$. On the other hand, from the definition of $\hat{E}(\ )$ we have for any $S \subseteq T$

$$\hat{E}(S) = \{t \in T \mid \exists s \in S \ \ t \sqsubset s\} = \cup\{L(s) \mid s \in S\}.$$

Especially $\hat{E}(U(t)) = C(t)$. Thus we get

Lemma 4. *For every $t \in T$ we have*

$$\tilde{U}(t) \subseteq E(U(t)) \subseteq \hat{E}(U(t)) = C(t),$$

which means that both $E(U(t))$ and $\hat{E}(U(t))$ coincide with $C(t)$ modulo $L(t)$.

Now we are going to construct the desired partition of T as well as ω^*. Let $\triangleleft$ be a well-ordering of T. Inductively take elementary submodels $M_\alpha(\alpha < 2^{\mathbf{c}})$ of the very large $H(\theta)$ and "subtrees" $T_\alpha \subseteq T(\alpha < 2^{\mathbf{c}})$ as follows: Let t_α be the minimal element of $T \setminus \bigcup_{\beta<\alpha} T_\beta$ with respect to $\triangleleft$. Take an elementary submodel M_α such that $t_\alpha \in M_\alpha, M_\alpha^\omega \subseteq M_\alpha$ and $|M_\alpha| = \mathbf{c}$. Define

$$T_\alpha = C(t_\alpha) \cap M_\alpha = \tilde{L}(t_\alpha) \cup (U(t_\alpha) \cap M_\alpha).$$

Then T_α is a "subtree" of T in the sense that for every $t \in T_\alpha$ $L(t) \subseteq T_\alpha$, because $|L(t)| \leq \mathbf{c}$ implies $L(t) \subseteq M_\alpha$. Clearly $\{T_\alpha \mid \alpha < 2^{\mathbf{c}}\}$ covers T. Define

$$S_\alpha = T_\alpha \setminus \bigcup_{\beta<\alpha} T_\beta.$$

Then we get a partition $T = \oplus_{\alpha<2^{\mathbf{c}}} S_\alpha$. Pull this back to ω^* by the π:

$$\omega^* = \oplus_{\alpha<2^{\mathbf{c}}} C_\alpha \text{ where } C_\alpha = \pi^{-1}(S_\alpha) \subseteq \omega^* \cap M_\alpha.$$

We will show this partition is the desired one.

Assertion 1. *If $\beta < \alpha$, then $T_\beta \cap T_\alpha \subseteq L(t_\alpha)$.*

Proof. Let $\beta < \alpha$. Since $T_\alpha = L(t_\alpha) \cup \{t_\alpha\} \cup (U(t_\alpha) \cap M_\alpha)$ and $t_\alpha \notin T_\beta$, we need only show that $T_\beta \cap U(t_\alpha) = \emptyset$. Suppose otherwise; then there exists some $t \in T_\beta$ with $t_\alpha \sqsubset t$. Since T_β is a "subtree" as noted before, we have $t_\alpha \in T_\beta$. But this contradicts the choice of t_α.

Put $V_\alpha = \tilde{U}(t_\alpha) \cap M_\alpha$. Assertion 1 shows $V_\alpha \subseteq S_\alpha$. So, by Corollary 2, C_α is extra $V_\alpha -$ *c.c.* in $\omega^* \cap M_\alpha$. Note that since S_α's are disjoint, so are V_α's.

Assertion 2. $V_\alpha \subseteq E(S_\alpha) \subseteq \hat{E}(S_\alpha) \subseteq T_\alpha$.

Proof. Lemma 4 and Proposition 1 imply $V_\alpha \subseteq E(U(t_\alpha) \cap M_\alpha)$. Since $U(t_\alpha) \cap M_\alpha \subseteq V_\alpha \subseteq S_\alpha$, we have $V_\alpha \subseteq E(S_\alpha)$. On the otherhand, $S_\alpha \subseteq T_\alpha$ implies $\hat{E}(S_\alpha) \subseteq \hat{E}(T_\alpha) = T_\alpha$.

To show the next Assertion 3, we need to be a bit careful about M_α. (We don't have to be careful if we want to show Assertion 3 only in the case the set I is countable.) In general, let $\langle N_\alpha \mid \alpha < \mathbf{c}\rangle$ be any sequence of elementary submodels of $H(\theta)$ such that for each $\alpha < \mathbf{c}$

$$|N_\alpha| = \mathbf{c},\quad N_\alpha^\omega \subseteq N_\alpha,\ \langle N_\beta \mid \beta < \alpha\rangle \in N_\alpha,$$

and put $N = \bigcup_{\alpha<\mathbf{c}} N_\alpha$. Then it is easy to see that $\langle N_\alpha \mid \alpha < \mathbf{c}\rangle$ is a chain under inclusion (by Fact 4(1)) and that $|N| = \mathbf{c}$, $N^\omega \subseteq N$, $N_\alpha \in N$ for all $\alpha < \mathbf{c}$. This elementary submodel N of $H(\theta)$ has the "covering" property that

$$(*)\quad \forall x \subseteq N(|x| < cof(\mathbf{c}) \rightarrow \exists Y \in N\ \ x \subseteq y \text{ and } |y| = \mathbf{c}).$$

Indeed, take $\alpha < \mathbf{c}$ such that $x \subseteq N_\alpha$, then this N_α can be the y in the above $(*)$.

Now take care in choosing M_α's so that each M_α satisfies the above covering property $(*)$. Then

Assertion 3. *Let $\alpha < 2^{\mathbf{c}}$, and let I be any subset of $2^{\mathbf{c}}\backslash(\alpha+1)$ such that $|I| < cof(\mathbf{c})$. Then $E(S_\alpha)\backslash\hat{E}((\oplus_{\beta<\alpha}S_\beta) \oplus (\oplus_{\gamma\in I}S_\gamma))$ is of cardinality $\mathbf{c}$.*

Proof. Let α and I be as above. Since

$$\hat{E}((\oplus_{\beta<\alpha}S_\beta) \oplus (\oplus_{\gamma\in I}S_\gamma)) = (\bigcup_{\beta<\alpha}\hat{E}(S_\beta)) \cup (\bigcup_{\gamma\in I}\hat{E}(S_\gamma)),$$

by Assertions 1 and 2 the difference set in Assertion 3 contains

$$V_\alpha \backslash \bigcup_{\gamma\in I} T_\gamma = V_\alpha \backslash \bigcup_{\gamma\in I} L(t_\gamma).$$

So it suffices to show that $U(t_\alpha) \cap M_\alpha \backslash \bigcup_{\gamma\in I} L(t_\gamma)$ is of cardinality $\mathbf{c}$. Note first that the set I need not belong to M_α. (In case $I \in M_\alpha$, the proof is easy.) Put $J = \{\gamma \in I \mid U(t_\alpha) \cap M_\alpha \cap L(t_\gamma) \neq \emptyset\}$; then $U(t_\alpha) \cap M_\alpha \backslash \bigcup_{\gamma\in I} L(t_\gamma)$ is identical with $U(t_\alpha) \cap M_\alpha \backslash \bigcup_{\gamma\in J} L(t_\gamma)$. For each $\gamma \in J$ choose $s_\gamma \in M_\alpha$ such that $t_\alpha \sqsubset s_\gamma \sqsubset t_\gamma$. Put $\sigma = \{s_\gamma \mid \gamma \in J\}$. Since $|\sigma| < cof(\mathbf{c})$, by the covering property $(*)$ of M_α we can find a subset $\sum \in M_\alpha$ such that $\sigma \subseteq \sum \subseteq U(t_\alpha)$ and $|\sum| = \mathbf{c}$. Since t_α produces $2^{\mathbf{c}}$-many, incomparable types (as remarked at the beginning of this section), we can find a type u such that $t_\alpha \sqsubset u$ and u is incomparable with any $s \in \sum$. Since t_α, $\sum \in M_\alpha$, we can assume $u \in M_\alpha$ by the elementarity of M_α. Then we get $U(t_\alpha) \cap M_\alpha \setminus \bigcup_{\gamma \in J} L(t_\gamma) \supseteq U(u) \cap M_\alpha$, which proves Assertion 3.

Summarizing the above construction, we get the following theorem, in which the property (2) is deduced from Assertion 3 and Fact 2.

Theorem 2. *Cover T by a sequence $\langle M_\alpha \mid \alpha < 2^{\mathbf{c}} \rangle$ of elementary submodels of the sufficiently large $H(\theta)$ such that for each $\alpha < 2^{\mathbf{c}}$*

(i) $M_\alpha^\omega \subseteq M_\alpha$, $|M_\alpha| = \mathbf{c}$, $\langle M_\beta \mid \beta < \alpha \rangle \in M_\alpha$;

(ii) M_α has the "covering" property that

$$\forall x \subseteq M_\alpha \ (|x| < cof(\mathbf{c}) \ \rightarrow \exists y \in M_\alpha \ \ x \subseteq y \text{ and } |y| = \mathbf{c}).$$

Using this sequence, we can define a partition of T

$$T = \oplus_{\alpha < 2^{\mathbf{c}}} S_\alpha \text{ with } V_\alpha \subseteq S_\alpha \subseteq M_\alpha, \quad |V_\alpha| = |S_\alpha| = \mathbf{c}$$

which induces a partition of ω^*

$$\omega^* = \oplus_{\alpha < 2^{\mathbf{c}}} C_\alpha \text{ where } C_\alpha = \pi^{-1}(S_\alpha), \quad \pi : \omega^* \rightarrow T = \omega^* / \sim .$$

with the following properties: for each $\alpha < 2^{\mathbf{c}}$

(1) C_α is a countably compact, T-saturated (hence, dense) subset of ω^* with cardinality $\mathbf{c}$. (Precisely, C_α is extra $V_\alpha - c.c.$ in $\omega^* \cap M_\alpha$, especially, is $V_\alpha - c.c.$)

(2) C_α is *not* embeddable into $(\oplus_{\beta<\alpha} C_\beta) \oplus (\oplus_{\gamma \in I} C_\gamma)$, where I is any subset of $2^{\mathbf{c}} \setminus (\alpha + 1)$ of cardinality $< cof(\mathbf{c})$.

Remark 3. Consider any kind of partition of ω^* such that $\omega^* = \oplus_{\alpha < 2^{\mathbf{c}}} C_\alpha$, $|C_\alpha| = \mathbf{c}$ for each $\alpha < 2^{\mathbf{c}}$. Then, each C_α is embeddable into any "tail" $\oplus_{\xi < \gamma < 2^{\mathbf{c}}} C_\gamma$ where $\xi < 2^{\mathbf{c}}$, because this tail contains a copy of ω^* by Fact 1. Since $|C_\alpha| = \mathbf{c}$, one can find a subset $K \subseteq 2^{\mathbf{c}} \setminus (\xi + 1)$ of cardinality $\mathbf{c}$ such that C_α *is* embeddable into $\oplus_{\gamma \in K} C_\gamma$. It may well happen that this set K can be chosen to be of cardinality $cof(\mathbf{c})$ rather than $\mathbf{c}$. So, we can conclude that the partition of ω^* in Theorem 2 is the best possible non-homogeneous one.

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