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A 2-DIMENSIONAL COMPACTUM IN THE PRODUCT OF TWO 1-DIMENSIONAL COMPACTA WHICH DOES NOT CONTAIN ANY RECTANGLE

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1. Introduction

Our terminology follows Kuratowski [Ku]. By a compactum we mean a compact metrizable space.

The aim of this note is to provide the following example which answers (for $n = 2$) a question asked by Y. Sternfeld [St1; Problem 1], [St2; 4, Problem 8].

1.1. Example *For each natural $n \geq 2$ there exist 1-dimensional compacta Z_i, $i = 1, \dots, n$, and an n-dimensional compactum $Y \subset Z_1 \times \dots \times Z_n$ such that whenever $A_1 \times \dots \times A_n \subset Y$, then all but one A_i are singletons.*

Our construction also answers another question of Sternfeld [St2; 4, Problem 7], see Sec 4. I am grateful to Y. Sternfeld for pointing out this fact to me.

2. Embedding hereditarily indecomposable continua in products of 1-dimensional continua

A continuous mapping $f : A \to B$ between compacta is monotone (zero-dimensional), if all fibers $f^{-1}(b)$ are connected (zero-dimensional).

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A continuum C is hereditarily indecomposable if for any pair of continua A, B in C with $A \cap B \neq \emptyset$, either $A \subset B$ or $B \subset A$ [Ku; 48, V].

R. H. Bing [Bi] constructed hereditarily indecomposable continua of arbitrarily large dimension.

2.1. Lemma *Let X be an n-dimensional hereditarily indecomposable continuum. There exist continuous monotone mappings $h_i : X \to Z_i$, $i = 1, \ldots, n$, onto 1-dimensional continua Z_i such that the diagonal mapping $h = (h_1, \ldots, h_n) : X \to Z_1 \times \cdots \times Z_n$ is an embedding.*

Proof. By a theorem of Hurewicz [Ku; 45, IX] there exists a zero-dimensional mapping $f : X \to I^n$ into the n-cube, and let $f = (f_1, \ldots, f_n)$, where $f_i : X \to I$.

For each $i = 1, \ldots n$, we consider the factorization $f_i = g_i \circ h_i$, where $h_i : X \to Z_i$ is a continuous monotone mappping onto Z_i, and the mapping $g_i : Z_i \to I$ is zero-dimensional, see [Ku; 47, VI, Theorem 7].

The continua Z_i are 1-dimensional, cf. [Ku; 45, Theorem 1]. We have to show that for each $z = (z_1, z_2, \ldots, z_n) \in Z_1 \times \cdots \times Z_n$, the set $h^{-1}(z) = h_1^{-1}(z_1) \cap \ldots \cap h_n^{-1}(z_n)$ contains at most one point. Let $C_i = h_i^{-1}(z_i)$ and suppose that $C_1 \cap \ldots \cap C_n \neq \emptyset$. Since C_i are subcontinua of the hereditarily indecomposable continuum X, the collection $C_1, \ldots, C_n$ is linearly ordered by the inclusion, therefore, for some j, $C_j = C_1 \cap \ldots \cap C_n$, i.e., $h^{-1}(z) = C_j$ is a continuum. On the other hand, $h^{-1}(z) \subset f^{-1}(y)$, where $y = (g_1(z_1), \ldots, g_n(z_n))$, hence $h^{-1}(z)$ is zero-dimensional. It follows that $h^{-1}(z)$ is a singleton.

2.2. Remark Sternfeld [St2; 3, p.25] gives interesting refinements of the standard factorization argument we have applied at the beginning of the proof.

3. The example.

Let us fix an $n \geq 2$, let us adopt the notation of Lemma 2.1, and let us set $Y = h(X)$.

Assume that $A_1 \times \ldots \times A_n \subset Y$ and suppose that for some $i \neq j$, $A_i = \{s_1, s_2\}$, $A_j = \{t_1, t_2\}$, $s_1 \neq s_2$, $t_1 \neq t_2$. Let us consider the continua $C_k = h_i^{-1}(s_k)$ and $D_k = h_j^{-1}(T_k)$, $k = 1, 2$. Then $C_1 \cap D_1 \neq \emptyset$, $C_1 \backslash D_1 \supset C_1 \cap D_2 \neq \emptyset$, $D_1 \backslash C_1 \supset D_1 \cap C_2 \neq \emptyset$, contradicting the fact that X is hereditarily indecomposable.

4. Remark

The 2-dimensional compactum Y in the product $Z_1 \times Z_2$ of 1-dimensional compacta described in Sec. 3 (for $n = 2$) is such that no triple $\{(s_1, t_1), (s_2, t_2), (s_1, t_2)\}$, with $s_1 \neq s_2$, $t_1 \neq t_2$, is contained in Y. By a result of Sternfeld [St2; 2, Theorem 10] this property implies that each real valued continuous function $u : Y \to R$ can be represented in the form $u(z_1, z_2) = v_1(z_1) + v_2(z_2)$, $(z_1, z_2) \in Y$, where $v_i : Z_i \to R$ are continuous functions.

This provides an answer to Problem 7, [St2; 4].

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