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IF L IS THE RATIONAL LONG LINE, THEN $(L \oplus 1)^\omega$ IS HOMOGENEOUS

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ABSTRACT. We establish that the direct sum of the rational long line and an isolated point is a zero-dimensional first countable space whose ω-power *is* homogeneous. The method is more general and shows that adding an isolated point to a "very" homogeneous space will not affect the homogeneity of the ω-power.

In 1988, Gary Gruenhage raised the question of whether the ω-power of any first countable zero-dimensional space must be homogeneous. Gruenhage and Zhou have written a paper entitled "Homogeneity of X^ω" in which they prove, among other things, that the ω-power of any first countable zero-dimensional space with a dense set of isolated points *is* homogeneous.

Definition 1. *Let P be a countable dense-in-itself linearly ordered space with a first element. Let L be $\omega_1 \times P$ with the lexicographic ordering and the induced topology. We call L the* rational long line.

The rational long line L was used in the 1990 thesis of David McIntyre. Robin Knight conjectured in November 1990 that $L \oplus \{1\}$ is an example of a zero-dimensional first countable space whose ω-power is not homogeneous.

Theorem 1. *$(L \oplus 1)^\omega$ is homogeneous.*

The method of establishing homogeneity of $(L \oplus 1)^\omega$ requires little of L other than a kind of strong homogeneity. We present the specific result, believing that this best reveals the simple but non-trivial ideas used in this argument. The reader can verify that L can be any homogeneous space which admits Q, h, $\{Q_i : i \in \omega\}$ and $\{h_i : i \in \omega\}$ satisfying conditions (1) to (5) below.

Let $X = (L \oplus 1)^\omega$.

Lemma 1. *There is an autohomeomorphism f of X such that $f(\vec{0}) \neq \vec{0}$.*

Proof. Let $1 \in Q \subset L$ and $\{Q_n : n > 0\}$ be such that

(1) Q and each Q_n is clopen
(2) $\{Q_n : n > 0\} \cup \{1\}$ is a partition of $\mathbb{Q}$
(3) $(\forall i > 0) h_i : Q \to Q_i$ is a homeomorphism
(4) $h : (L - Q) \oplus 1 \to L \oplus 1$ is a homeomorphism
(5) $\{Q_n : n > 0\} \to 1$

We also make some definitions:

- $X_i = \{x \in X : x(0) = 0, x(i) \in Q, (\forall j < i) x(j) \notin Q\}$(for $i > 0$)
- $Y_i = \{x \in X : x(0) \in Q_i\}$(for $i > 0$)
- $X^* = \{x \in X : x(0) = 0, (\forall i \in \omega) x(i) \notin Q\}$
- $Y^* = \{x \in X : x(0) = 1\}$
- $Z = \{x \in X : x(0) \notin Q \cup 1\}$

The family $\{X_i : i > 0\} \cup \{Y_i : i > 0\} \cup \{X^*, Y^*, Z\}$ partitions X.

$$\text{Define } \pi_i : X_i \to Y_i \text{ by } \pi_i(x)(j) = \begin{cases} h_i(x(i)) & \text{if } j = 0 \\ h(x(j)) & \text{if } 0 < j < i \\ x(j+1) & \text{if } j \geq i \end{cases}$$

$$\text{Define } \rho_i : Y_i \to X_i \text{ by } \rho_i(x)(j) = \begin{cases} 0 & \text{if } j = 0 \\ h_i^{-1}(x(0)) & \text{if } j = i \\ h^{-1}(x(j)) & \text{if } 0 < j < i \\ x(j-1) & \text{if } j > i \end{cases}$$

$$\text{Define } \pi : X^* \to Y^* \text{ by } \pi(x)(j) = \begin{cases} 1 & \text{if } j = 0 \\ h(x(j)) & \text{if } j > 0 \end{cases}$$

$$\text{Define } \rho : Y^* \to X^* \text{ by } \rho(x)(j) = \begin{cases} 0 & \text{if } j = 0 \\ h^{-1}(x(j)) & \text{if } j > 0 \end{cases}$$

Let $f = \bigcup\{\pi_i \ : \ i \in \omega\} \cup \bigcup\{\rho_i \ : \ i \in \omega\} \cup \pi \cup \rho \cup id \restriction Z$.

Checking the details: We can calculate, by cases on j, that $\rho_i(\pi_i(x))(j) = x(j)$, $\pi_i(\rho_i(x))(j) = x(j)$, $\rho(\pi(x))(j) = x(j)$ and $\pi(\rho(x))(j)$. Thus since ρ_i, π_i, ρ and π are well-defined, they are bijections. Since each of these functions is continuous (this can be checked coordinate-wise), they are all homeomorphisms.

Each X_i, Y_i as well as Z are clopen. Each ρ_i, π_i as well as ρ, π and $id \restriction Z$ are homeomorphisms. $X^* \cup \bigcup\{X_i : i \in \omega\}$ and $Y^* \cup \bigcup\{Y_i : i \in \omega\}$ are clopen. X^* and Y^* are closed.

Suppose $x_i \in X_i$ and $x \in X^*$ and $x_i \to x$. Now $(\forall j) x_i(j) \to x(j)$ and $(\forall j > 0)(\forall i > j)\pi_i(x_i)(j) = h(x_i(j))$ and also, if $j > 0$, $h(x(j)) = \pi(x)(j)$. Thus $(\forall j > 0)\pi_i(x_i)(j) \to \pi(x)(j)$ and so $\pi_i(x_i) \to \pi(x)$.

Suppose $y_i \in Y_i$ and $y \in Y^*$ and $y_i \to y$. Now $(\forall j) y_i(j) \to y(j)$ and $(\forall j > 0)(\forall i > j)\rho_i(y_i)(j) = h^{-1}(y_i(j))$ and $(\forall j > 0)\rho(y)(j) = h^{-1}(y(j))$. Now $(\forall j)h^{-1}(y_i(j)) \to h^{-1}(y(j))$ so that $(\forall j > 0)\rho_i(y_i)(j) \to \rho(y)(j)$ and $\rho_i(y_i) \to \rho(y)$.

Lemma 2. *There is an autohomeomorphism j of X such that $j(\vec{1})(0) = 0$.*

Proof. For each $x \in X - \{\vec{0}\}$, let $i(x)$ be the least i such that $x(i) \neq 0$.

$$\text{Define } j : X \to X \text{ by } j(x)(i) = \begin{cases} 0 & \text{if } i = 0, i(x) \text{ even} \\ x(i-1) & \text{if } i > 0, i(x) \text{ even} \\ x(i+1) & \text{if } i \in \omega, i(x) \text{ odd} \\ 0 & \text{if } i(x) \text{ undefined} \end{cases}$$

Checking the details: We can calculate, by cases on j, that $j^2(x)(i) = x(i)$ and so j is a bijection. Let $X_i = \{x \in X : i(x) = i\}$. Each X_i is clopen. Since j is continuous on each X_i (we can check this coordinate-wise), it suffices to assume that $x_i \in X_i$ so that $x_i \to \vec{0}$ and check that $j(x_i) \to \vec{0}$. This can be done coordinate- wise, finding, for each j, some $k(j)$ such that $(\forall i > k(j))x_i(j) = 0$. Now, letting $k^*(j) = \max\{k(j-1), k(j+1)\}$, we get, for each j, $(\forall i > k^*(j))j(x_i)(j) = 0$

Lemma 3. *For any $x \in X$ such that $rng(x) \subset \{0,1\}$, there is an autohomeomorphism j of X such that $(j(x)^{-1})(0)$ is infinite and coinfinite.*

Proof. Suppose that $x^{-1}(0)$ is finite. Partition ω into infinitely many infinite sets $\{A_i : i \in \omega\}$ such that $(\forall i > 0)A_i \subset x^{-1}(1)$. Note that $|A_0 \cap x^{-1}(1)| = \omega$. Find an autohomeomorphism j_i of $(L \oplus 1)^{A_i}$ which is a copy of j from lemma 2.
Suppose that $x^{-1}(1)$ is finite. Partition ω into infinitely many infinite sets $\{A_i : i \in \omega\}$ where $x^{-1}(1) \subset A_0$. Note that $|A_0 \cap x^{-1}(0)| = \omega$. Find an autohomeomorphism j_i of $(L \oplus 1)^{A_i}$ which is a copy of j from lemma 1.

$$j = id \restriction (L \oplus 1)^{A_0} \times \prod\{j_i : i > 0\}$$

Proof of Theorem. By coordinate-wise homeomorphisms using homogeneity of L, we can map any x to another x' whose range is contained in $\{0,1\}$. By Lemma 3 and applying this fact again, we can map any point to a $x \in X$ such that $rng(x) \subset \{0,1\}$ and $x^{-1}(0)$ infinite and coinfinite. Let $x_0 \in X$ be defined by letting $x_0(j) = 0$ if j is even and $x_0(j) = 1$ if j is odd. By permuting the coordinates, we can get any point mapped to x_0.

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