Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 17, 1992

Invitied Speaker

ON THE ANCEL-CANNON THEOREM

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1. Introduction

We begin by stating the basic tameness condition for codimension-one embeddings of manifolds.

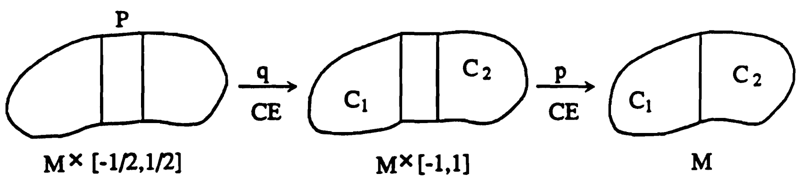
Definition 1. *If M^{n-1} is a closed manifold, an embedding $f : M^{n-1} \to N^n$ is said to be* locally flat *if for each $x \in f(M)$ there is a neighborhood U of x in N so that $(U, U \cap f(M)) \cong (\mathbb{R}^n, \mathbb{R}^{n-1})$.*

The Alexander horned sphere and its generalizations to higher dimensions show that not all codimension-one embeddings are locally flat. The purpose of this note is to provide an alternate proof of the locally flat approximation theorem of Ancel and Cannon [3] and, at the same time, to give some insight into how Quinn's Resolution Theorem works in this particular case. Here is the statement of the theorem:

Theorem 1 (Locally flat approximation theorem). *Let M^{n-1} and N^n be manifolds, M closed, $\partial N = \emptyset$, $n \geq 5$, and let $f : M \to N$ be an embedding. Let $\epsilon > 0$ be given. Then there is a locally flat embedding $\bar{f} : M \to N$ with $d(f(x), \bar{f}(x)) < \epsilon$ for each $x \in M$.*

*Partially supported by National Science Foundation Grant DMS 9003746. The author would also like to thank the University of Chicago for its hospitality while this paper was being written.

The proof of this theorem given in [3] is geometric and technically demanding. See the comments on page 61 of [3]. Ancel [1] has observed that the locally flat approximation theorem follows quickly from Quinn's Resolution Theorem ([11], [12]) and Siebenmann's CE Approximation Theorem [13]. Here is Ancel's argument:



If $M^{n-1} \subset N^n$ is a separating submanifold, let C_1 and C_2 be the two components of $N - M$. Form an ANR homology manifold $X = C_1 \cup M \times [-1,1] \cup C_2$. There is a CE map $p : X \to N$ obtained by collapsing $M \times [-1,1]$ to M. By the Resolution Theorem, there is a manifold P and a CE map $q : P \to X$. The map q can be taken to the identity on a neighborhood of $M \times 0$. Approximating the CE composition $p \circ q$ by a homeomorphism h gives the desired locally flat approximation to M.

We combine the ϵ-approximation theorem of [6] and [9] with some surgery below the middle dimension to give a proof of Theorem 1. In dimensions ≥ 5, at least, no obstructions are encountered. This removes the resolution theorem from the argument, though some surgery does enter the picture in the form of homotopy torii.

2. Local algebraic topology

The results in this section are all well-known. See [2], [7], [10], and [14]-[16]. They are included in order to keep the proof as self-contained as possible. Following [3], we restrict our attention to $M = S^{n-1}$ and $N = S^n$. The proof we give extends easily to the more general case. By abuse of notation, we will denote $f(M) \subset S^n$ by M. Let C be a component of $S^n - M$.

Proposition 1. *If $B_1 \subset B_2 \subset S^n$ are closed balls and $B_1 \cap M$ contracts to a point in $B_2 \cap M$, then $im(H_k(B_1 \cap C) \to H_k(B_2 \cap C))$ is $\mathbb{Z}$ for $k = 0$ and 0 for $k > 0$.*

Proof: We begin by considering the diagram

$$\begin{tikzcd}
H_k(B_2 - M) \arrow[r, "\cong"] & \check{H}^{n-k}(B_2, \partial B_2 \cup (M \cap B_2)) & \check{H}^{n-k-1}(\partial B_2 \cup (M \cap B_2)) \arrow[l, "\cong"'] \\
 & \check{H}^{n-k}(B_2, (B_2 - \mathring{B}_1) \cup (M \cap B_1)) \arrow[u] \arrow[d, "\cong"] & \check{H}^{n-k-1}((B_2 - \mathring{B}_1) \cup (M \cap B_1)) \arrow[l, "\cong"'] \arrow[u, "\alpha"'] \arrow[d, "\cong"] \\
H_k(B_1 - M) \arrow[r, "\cong"] & \check{H}^{n-k}(B_1, \partial B_1 \cup (M \cap B_1)) & \check{H}^{n-k-1}(\partial B_1 \cup (M \cap B_1)) \arrow[l, "\cong"']
\end{tikzcd}$$

which shows that it is sufficient to compute the map α in the upper right hand corner. Noting that $\check{H}^{n-k}((M \cap B_i)/(M \cap \partial B_i)) \cong \check{H}_c^{n-k}(M \mathring{\cap} B_i) \cong H_{k-1}(M \mathring{\cap} B_i)$ for $k \neq n$, we can use the diagram

$$\begin{tikzcd}
\check{H}^{n-k}(\partial B_2) & \check{H}^{n-k}(\partial B_2 \cup (M \cap B_2)) \arrow[l] & \check{H}^{n-k}((M \cap B_2)/(M \cap \partial B_2)) \arrow[l] & H_{k-1}(M \mathring{\cap} B_2) \arrow[l, "\cong"'] \\
\check{H}^{n-k}(B_2 - \mathring{B}_1) \arrow[u, "\cong"'] & \check{H}^{n-k}((B_2 - \mathring{B}_1) \cup (M \cap B_1)) \arrow[l] \arrow[u, "\alpha"'] & \check{H}^{n-k}((M \cap B_2)/(M \cap (B_2 - \mathring{B}_1))) \arrow[l] \arrow[u] \arrow[d, "\cong"] & \\
 & & \check{H}^{n-k}((M \cap B_1)/(M \cap \partial B_1)) & \check{H}_{k-1}(M \mathring{\cap} B_1) \arrow[l, "\cong"'] \arrow[uu]
\end{tikzcd}$$

to complete the proof. $\square$

Proposition 2. *There is a sequence $\{l_i\}_{i=1}^{\infty}$ of maps $l_i : S^1 \to C$ so that*

(i) *Each l_i is an embedding.*
(ii) $\lim\limits_{i \to \infty} diam(l_i(S^1)) = 0$.

(iii) *For each $\epsilon > 0$ and $N > 0$ there is a $\delta > 0$ so that if $l : S^1 \to C$ is a loop with diam$(l(S^1)) < \delta$, then there is a disk-with-holes $D \subset D^2$ and a map $\bar{l} : D \to C$ so that diam$(\bar{l}(D)) < \epsilon$ and $\bar{l}|\partial D$ consists of l together with $l_{i_1}, \ldots, l_{i_k}$ so that $i_j \geq N$ for all j.*

(iv) *For each $\epsilon > 0$ there is an $N > 0$ such that if $i \geq N$ then $l_i(S^1) \subset N(\epsilon, M)$.*

We begin the proof with a lemma.

Lemma 1. *$\bar{C}$ is locally contractible, that is, that for every $\epsilon > 0$ there is a $\delta > 0$ so that for each $x \in \bar{C}$, $B(\delta, x) \cap \bar{C}$ contracts to a point in $B(\epsilon, x) \cap \bar{C}$.*

Proof of Lemma: Let $r : U \to M$ be a retraction from a neighborhood of M to M. Define $s : C \cup U \to C \cup M = \bar{C}$ by $s(x) = r(x)$ if $r \notin C$ and $s(x) = x$ otherwise. Clearly, s is a retraction from a neighborhood of $\bar{C}$ in S^n to $\bar{C}$.

Given $\epsilon > 0$ choose $\delta > 0$ so that if $x \in \bar{C}$, then $s(B(\delta, x)) \subset B(\epsilon, x)$. Contracting $B(\delta, x)$ in itself and composing with s gives a contraction of $B(\delta, x) \cap \bar{C}$ to a point in $B(\epsilon, x) \cap \bar{C}$. The result follows. □

Remark 1. In our special case of $M \cong S^{n-1}$ in S^n, Alexander Duality shows that $H_k(\bar{C}) \cong H_k(pt)$ for all k. By the Van Kampen Theorem, which is valid for ANR's intersecting in an ANR, $\pi_1(\bar{C})$ is trivial, so $\bar{C}$ is contractible.

Lemma 2. *If $\alpha : D^2 \to \bar{C}$ is a map with $\alpha(S^1) \subset C$ and $\mu > 0$ is given, there is a disk-with-holes $D \subset D^2$ and a map $\bar{\alpha} : D \to C$ so that $\bar{\alpha}|\partial D$ consists of $\alpha|\partial D^2$ together with a collection $l_i : S^1_i \to \bar{C}$, $i = 1, \ldots, k$, of loops so that diam$(l_i(S^1_i)) < \mu$, $l_i(S^1_i) \subset N(\mu, M)$, and $d(\alpha(x), \bar{\alpha}(x)) < \mu$ for all $x \in D$. Here, $N(\mu, M)$ denotes the set of points whose distance from M is less than μ.*

Proof: Let $\mu > 0$, and $\alpha > 0$ be given. C is dense in $\bar{C}$ and Proposition 1 shows that C is uniformly locally connected. This means that if T is any triangulation of D^2, we can approximate α arbitrarily closely by a map α' so that $\alpha''(|T^{(1)}|) \subset C$. Choosing T to be a fine triangulation and D to be a regular neighborhood of $|T^{(1)}|$ in D^2 completes the proof with $\bar{\alpha} = \alpha'|D$. □

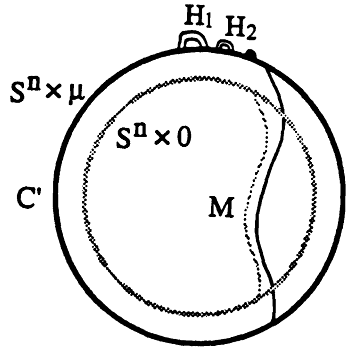
Proof of Proposition 2. Let $\{a_i\}_{i=1}^{\infty}$, $a_i : D^2 \to \bar{C}$ be dense in the space of maps $\alpha : D^2 \to \bar{C}$ with $\alpha(S^1) \subset C$. By Lemma 2.5 there is a countable collection $\{D_{ij}\}$ of disks-with-holes, $D_{ij} \subset D^2$ and maps $\bar{\alpha}_{ij} : D_{ij} \to C$ so that $\bar{\alpha}_{ij}|\partial D_{ij}$ consists of $\alpha_i|\partial D^2$ together with a collection $l'_{ijk} : S^1 \to \bar{C}, k = 1, \ldots, n_{ij}$ of loops so that $diam(l'_{ijk}(S^1)) < \frac{1}{i+j}$, $l'_{ijk}(S^1) \subset N(\frac{1}{i+j}, M)$, and $d(\alpha_i(x), \bar{\alpha}_{ij}(x)) < \frac{1}{i+j}$ for all $x \in D_{ij}$. Letting $\{l_i\}$ be $\cup_{ijk} l'_{ijk}$ completes the proof. □

We will also need the following local version of the Hurewicz theorem, which appears as Proposition 3.1 of [9].

Proposition 3 (Eventual Hurewicz Theorem). *For each $k \geq 0$ there is an integer $n_k > 0$ such that if $A_1 \subset A_2 \subset \cdots \subset A_{n_k}$ is a sequence of connected polyhedra with $i_\# : \pi_1(A_j) \to \pi_1(A_{j+1})$ and $i_* : H_l(A_j) \to H_l(A_{j+1})$ equal to zero for all $j < n_k$ and for all l between 0 and k, then each map of a k-dimensional complex into A_1 is homotopic to a constant map in A_{n_k}.*

3. Taming embeddings by small surgeries

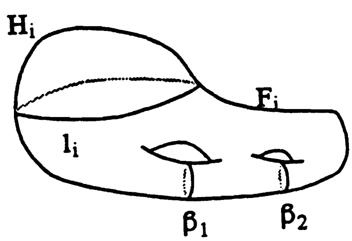
In this section, we consider S^n to be standardly embedded in $\mathbb{R}^{n+k}$, k large. For $\mu > 0$ some small number, thicken S^n radially to obtain a copy of $S^n \times [0, \mu] \subset \mathbb{R}^{n+k}$. Let $\{l_i\}$ be a collection of loops as in the statement of Proposition 2. Since the loops l_i have small diameter, they are canonically contractible in S^n and there is a canonical trivialization of the normal bundle to each l_i. Attach 2-handles $H_i \cong D^2 \times D^{n-1}$ along each $l_i(S^1) \times D^{n-1} \times \mu$ extending the given framing to obtain a "cobordism" W' from S^n to a new space N'. Note that neither W' nor N' is a manifold. We may assume that the H_i's are embedded in $\mathbb{R}^{n+k}$ in such a way that $diam(H_i) < 2\, diam(l_i(S^1))$. Here is a schematic picture:

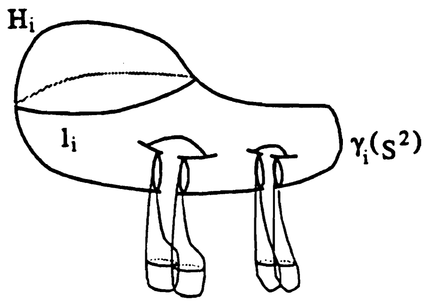


Since each $l_i(S^1)$ represents the trivial element in $H_1(C)$, $H_2(N)$ has a generator corresponding to each l_i. We need to show that these homology elements are represented by $\gamma_i : S^2 \to C'$ with $diam(\gamma_i(S^2))$ small for each i. Here, $C' \subset N'$ is the result of doing surgeries on C.

Proposition 4. *For every $\epsilon > 0$ there is a $\delta > 0$ so that if $diam(l_i(S^1)) < \delta$, then γ_i can be chosen with $diam(\gamma_i) < \epsilon$.*

Proof: Given $\epsilon > 0$, Choose δ_3 as in Proposition 2 for $\frac{\epsilon}{5}$. Choose δ_2 as in the same proposition for δ_3, and choose δ as in Corollary 2.1 for δ_2. Thus, if $diam(l_i(S^1)) < \delta, l_i$ extends to a map $\bar{l}_i : F_i \to C$ where F_i is an orientable surface with one boundary component and $diam(\bar{l}_i(F_i)) < \delta_2$. Let $\{\beta_j\}$ be a collection of curves in F_i such that $F_i - \cup\{\beta_j\}$ is a disk-with-holes. $\square$





Since $diam(\beta_j) < \delta_2, \beta_j$ bounds a disk-with-holes of diameter $< \delta_3$ where the holes are bounded by l_k's, k large. Extending using the corresponding H_k's, we see that each loop β_j bounds a disk of diameter $< \delta_3$. Cutting along these disks give a disk homologous to F_i which has diameter $< \delta_2 + 2\delta_3$ in C' and which is bounded by l_i. The corresponding $\gamma_i : S^2 \to C'$ has diameter $< 2\delta_1 + \delta_2 + 2\delta_3 < \epsilon$. □

Since $n \geq 5$, the $\gamma_i(S^2)$'s have trivial normal bundles. Continuing the construction, we attach thin 3-handles $G_i \cong D^3 \times D^{n-2}$ to the $\gamma_i(S^2)$'s to obtain a "cobordism" W'' from S^n to a space $N'' = C'' \cup M \cup (S^n - \bar{C})$.

Proposition 5. *For every $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in C''$, then $B(\delta, x) \cap C''$ contracts to a point in $B(\epsilon, x) \cap C''$.*

Proof: Suppose not. Then there is a fixed $\epsilon > 0$ so that no $\delta > 0$ suffices. By compactness, there is an $x \in C'' \cup M$ so that for every $\delta > 0, B(\delta, x) \cap C''$ does not contract in $B(\epsilon, x) \cap C''$. By the Eventual Hurewicz Theorem, this can only occur if there is a problem with the local fundamental group or the local homology at x.

Local π_1 is clearly no problem. Any loop in C'' near x can be moved into $C - \cup H_i$ by general position. Therefore, by Proposition 2, it bounds a disk-with-holes in C. Capping off using H_k's, k large, gives a small disk.

Similarly, there can be no problems with local homology. If $H_*(B(\delta, x) \cap C) \to H_*(B(\epsilon, x) \cap C)$ is trivial, the image of $H_*(B(\delta, x) \cap C'')$ in $H_*(B(\epsilon, x) \cap C'')$ is generated by the $\gamma_i(S^2)$'s and their duals. Choosing the gap between ϵ and δ large enough that $G_i \subset B(\epsilon, x)$ whenever H_i meets $B(\delta, x)$ guarantees that the relevant $\gamma_i(S^2)$'s and their duals die in $H_*(B(\epsilon, x) \cap C'')$. □

We are now in a position to apply the Černavskii-Seebeck Theorem of [9], p. 579. (Details of the papers of Černavskii and Seebeck referred to in [9] have never appeared.) Here is its statement:

Theorem 2. *Suppose that C is a noncompact n-manifold without boundary, $n \geq 5$, and that M is an $(n-1)$-dimensional manifold without boundary. Suppose that $X = C \cup M$ is a locally compact metric space such that $C \cap M = \emptyset, C$ is dense in X, and M is $(n-1)$-LCC in X. Then X is an n-manifold with boundary M.*

The $(n-1)$-LCC condition is trivially implied by Proposition 5, so the theorem applies to show that $C'' \cup M$ is a manifold with boundary. Of course, we could just as easily have done surgery to *both* complementary domains of M in S^n. In that case, we obtain a cobordism W'' from S^n to a manifold N''. Note that if $\mu > 0$ is given in advance, we can perform the construction above using 2- and 3-handles of diameter $< \mu$.

4. THE PROOF OF THE MAIN THEOREM

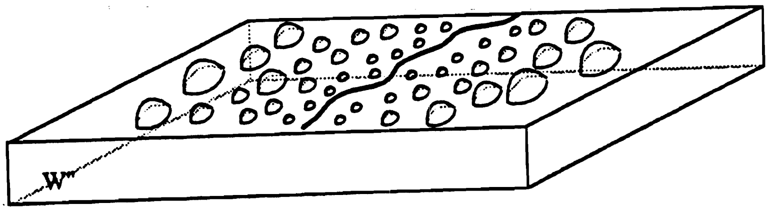
We wish to exploit the following "ϵ-approximation theorem," which combines Theorems 2-4 of [9].

Theorem 3. *If M^n is a closed n-manifold, $n \geq 5$, and $\epsilon > 0$ is given, then there is a $\delta > 0$ so that if $f : M \to N$ is a map from M to a connected n-manifold such that $diam(f^{-1}(x)) < \delta$ for every $x \in N$, then there is a homeomorphism $h : N \to M$ such that $d(h \circ f, id) < \epsilon$.*

We proceed to construct a map $g : S^n \to N''$ which has small point-inverses. Using the Eventual Hurewicz Theorem as in the proof of Proposition 5 gives us:

Proposition 6. *If k is an integer greater than zero and real numbers $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < 1$ are given, $\mu > 0$ can be chosen so that adding handles of diameter $< \mu$ has above yields a manifold N'' such that $B(\lambda_i, x) \cap N''$ contracts to a point in $B(\lambda_{i+1}, x) \cap N''$ for each $x \in N''$ and $i < k$.*

Proof: Given $\delta > 0$, let $\lambda_i = 3^i\delta, 0 \leq i \leq 2n-2$. Choose $\mu > 0$, $3\mu < \delta$, and construct W'' using handles of diameter $< \mu$ so that $B(\lambda_i, x) \cap N''$ contracts to a point in $B(\lambda_{i+1}, x) \cap N''$ for each $x \in N''$ and $i \leq 2n+1$.



Triangulate S^n so that the diameter of each simplex is less than μ. Define a map $g : S^n \to N''$ inductively, as follows. For each vertex $v \in S^n$, let $g(v)$ be a point in N'' with $d(v, g(v)) < \lambda_0$. If vertices v_0 and v_1 bound a 1-simplex in $S^n, g(v_0)$ and $g(v_1)$ are within λ_1 of each other and can be connected by a path of length $< \lambda_2$. This allows us to extend g over $\langle v_0, v_1 \rangle$. Similarly, we extend over the remaining skeleta, obtaining a map g so that $d(x, g(x)) < 3^{2n+2}\delta$ for all $x \in S^n$. This guarantees that the point-inverses of g have diameter $< 2 \cdot 3^{2n+2}\delta$. Applying the ϵ-approximation theorem completes the proof of the Ancel-Cannon Theorem. □

5. Remarks on the proof of the ϵ-approximation theorem

The referee has suggested that we include a few words about the proof of the ϵ-approximation theorem. The main step in the proof is the following, which is stated on p. 583 of [6].

Theorem 4. *Let M^n, $n \geq 5$, be a closed topological manifold with a fixed topological metric d. Then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $f : N \to M$ is a δ-equivalence over M, then f is ϵ-homotopic to a homeomorphism.*

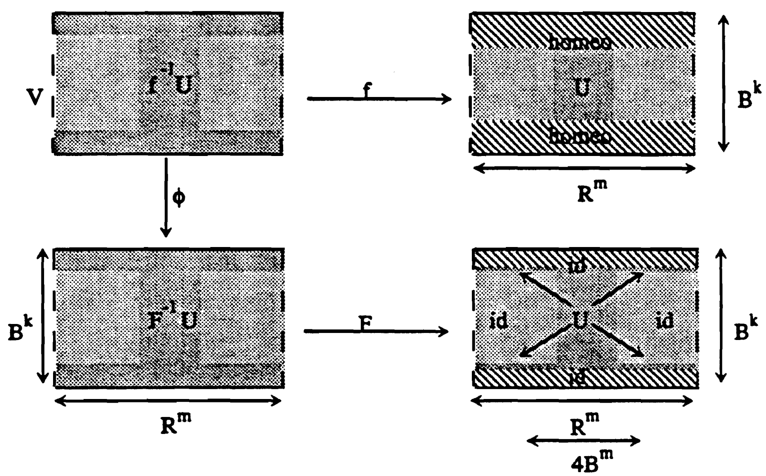
By a *δ-equivalence* $f : X \to Y$, we mean a homotopy equivalence f with homotopy inverse g and homotopies $h_t : g \circ f \simeq id_X$, $k_t : f \circ g \simeq id_Y$ so that the tracks $\{k_t(x)\}$ and $\{f \circ h_t(x)\}$ have diameter $< \delta$.

We sketch the proof in dimensions ≥ 6. The proof is a torus argument modeled on the proof of Siebenmann's CE Approximation Theorem. The first step in the proof is a (rather!) technical lemma.

Lemma 3 (Handle Lemma). *Let V^n be a topological manifold, $n \geq 5$, and let $f : V \to B^k \times \mathbb{R}^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times \mathbb{R}^m)$ and f is a homeomorphism over $(B^k - \frac{1}{2}\overset{\circ}{B}{}^k) \times \mathbb{R}^m$. For every $\epsilon > 0$ there is a $\delta > 0$ so that if f is a δ-equivalence over $B^k \times 3B^m$ and $m \geq 1$, then:*

(i) *There exists an ϵ-equivalence $F : B^k \times \mathbb{R}^m \to B^k \times \mathbb{R}^m$ such that $F = id$ over $(B^k - \frac{5}{6}\overset{\circ}{B}{}^k) \times \mathbb{R}^m \cup B^k \times (\mathbb{R}^m - 4\overset{\circ}{B}{}^m)$, and*

(ii) *There exists a homeomorphism $\phi : f^{-1}(U) \to F^{-1}(U)$ such that $F \circ \phi = f|f^{-1}(U)$, where $U = (B^k - \frac{5}{6}\overset{\circ}{B}{}^k) \times \mathbb{R}^m \cup B^k \times 2B^m$.*



Remark 2. In general, the torus trick takes a piece of a given map and extends it in such a way that it has nice properties

near infinity. Thus, one might expext that the Handle Lemma should take a δ-equivalence and turn it into an ϵ-equivalence F which is a homeomorphism near infinity so that F agrees with f on a core and near the coundary and so that F is a global ϵ-equivalence which is a homeomorphism near infinity. This is inappropriate, however, since we cannot assume that the original map f is a δ-equivalence over more than a core, so we have no reason to believe that V is even homotopy equivalent to $B^k \times R^m$. The best we can hope for, then, is that the parts of V over a core and near the boundary embed in a minifold homeomorphic to $B^k \times R^m$ where the desired extension exists. This is the content of our Handle Lemma.

Proof: We construct the following diagram:

$$
\begin{tikzcd}
B^k \times \mathbb{R}^m \arrow[rr, "F"] & & B^k \times \mathbb{R}^m \\
B^k \times \mathbb{R}^m \arrow[u, hook, "i"] \arrow[rr, "F'"] \arrow[d, "id\times e"'] & & B^k \times \mathbb{R}^m \arrow[u, hook, "i"'] \arrow[d, "id\times e"] \\
B^k \times T^m & W_3 \arrow[l, "h"'] \arrow[r, "f_3"] & B^k \times T^m \\
& W_2 \arrow[u, hook] \arrow[r, "f_2"] \arrow[d, hook] & (B^k \times T^m) - \mathring{D}^n \arrow[u, hook] \arrow[d, hook] \\
& W_1 \arrow[r, "f_1"] & (B^k \times T^m) - (\frac{2}{3}B^k \times \{x_0\}) \\
& W_0 \arrow[u, hook] \arrow[r, "f_0"] \arrow[d, "\beta'"'] & B^k \times T_0^m \arrow[u, hook] \arrow[d, "id\times\beta"] \\
& V \arrow[r, "f"', "\text{homeo over } \partial"] & B^k \times \mathbb{R}^m
\end{tikzcd}
$$

(i) W_0 is constructed by taking the pullback. W_0 is a manifold and f_0 is an δ_1-equivalence away from the hole in the torus. One way to see this is to pull back the mapping cylinder projection from the mapping cylinder of f to V. This is the mapping cylinder of f_0 and the 5δ-SDR from $M(f)$ to V lifts to a δ_1-SDR from $M(f_0)$ to W_0 away from the hole.

(ii) Since f is a homeomorphism over the boundary, we can put the plug in over $B^k - \frac{2}{3}B^k) \times T^m$. This gives us W_1 and f_1.

(iii) Parameterize the end of $(B^k \times T^m) - (\frac{2}{3}B^k \times \{x_0\})$ as $S^{n-1} \times [0,1)$ and choose D^n to be a disk in $B^k \times T^m$ containing $(\frac{2}{3}B^k \times \{x_0\})$. For δ sufficiently small, we can use the Splitting Theorem below to find $W_2 \subset W_1$ and a homotopy equivalence of pairs $f_2 : (W_2, \partial W_2) \to ((B^k \times T^m) - \overset{\circ}{D}{}^n, \partial D^n)$. Moreover, we can take $f_2 = f_1$ outside of a small neighborhood of ∂D^n. Note that at this stage we have lost some control, since f_2 is an uncontrolled homotopy equivalence over ∂D^n and D^n is not small.

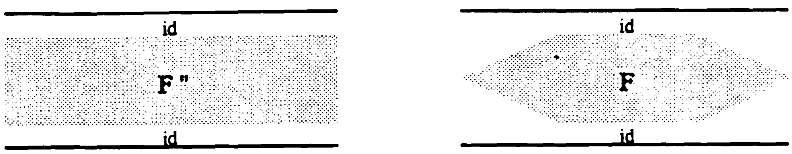
(iv) We cone off ∂W_2 and extend to $f_3' : W_3 \to B^k \times T^m$. We regain the lost control by Stretching out a collar on a disk $2D^n \supset D^n$ and squeezing D^n to be small. The result is a δ_2-equivalence $f_3 : W_3 \to B^k \times T^m$.

(v) Choose $h : W_3 \to B^k \times T^m$ to be a homeomorphism agreeing with f_3 over $(B^k - \frac{5}{6}\overset{\circ}{B}{}^k) \times T^m$ and homotopic to f_3. The existence of h is a consequence of topological surgery theory.

(vi) Pass to the universal cover and get F', which is bounded and equal to the identity over $(B^k - \frac{5}{6}\overset{\circ}{B}{}^k) \times \mathbb{R}^m$.

(vii) Let $\rho : \mathbb{R}^m \to 4\overset{\circ}{B}{}^m$ be a radial homeomorphism which is the identity on $2B^m$. Conjugating F' by $id \times \rho$ squeezes

F' to a homeomorphism $F'' = \rho \circ F' \circ \rho^{-1} : B^k \times 4\overset{\circ}{B}{}^m \to B^k \times 4\overset{\circ}{B}{}^m$ which comes closer and closer to commuting with projection near the boundary. Squeezing in the B^k-direction – this is essentially an Alexander isotopy – gives an $F : B^k \times 4\overset{\circ}{B}{}^{\to}\ B^k \times 4\overset{\circ}{B}{}^m$ which extends by the identity to $B^k \times \mathbb{R}^m$.



(viii) The construction of ϕ proceeds as usual. We simply note that F contains a copy of f over B^n and extend near the boundary using f to identify a neighborhood of the boundary in V with a neighborhood of the boundary in the range.

This completes the proof of the Handle Lemma. □

The next step in the proof is to use Siebenmann's inversion trick to prove the following Handle Theorem. the idea here is to reverse the roles of 0 and ∞ and use the Handle Lemma a second time. The first time gives us a map agreeing with f on a core which is a homeomorphism near infinity. Inverting and applying the Handle Lemma again gives us a map agreeing with f on a band and which is a homeomorphism over a core.

Theorem 5 (Handle Theorem). *Let V^n be a topological manifold, $n \geq 5$, and let $f : V \to B^k \times \mathbb{R}^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times \mathbb{R}^m)$ and f is a homeomorphism over $(B^k - \frac{1}{2}\overset{\circ}{B}{}^k) \times \mathbb{R}^m$. For every $\epsilon > 0$ there is a $\delta > 0$ so that if f is a δ-equivalence over $B^k \times 3B^m$, then there exists a proper map $\bar{f} : V \to B^k \times \mathbb{R}^m$ such that*

(i) *$\bar{f}$ is an ϵ-equivalence over $B^k \times 2.5B^m$*

(ii) *$\bar{f} = f$ over $[(B^k - \frac{2}{3}circB^k) \times \mathbb{R}^m] \cup [B^k \times (\mathbb{R}^m - 2\overset{\circ}{B}{}^m)]$,*

(iii) *$\bar{f}$ is a homeomorphism over $B^k \times B^m$.*

Proof: The case $m = 0$ follows from the generalized Poincaré Conjecture and coning, so we assume $m \geq 1$.

We apply the Handle Lemma to obtain F as above and compactify F by the identity to obtain a homeomorphism $B^k \times S^m \to B^k \times S^m$. We take out $B^k \times$ south pole and its inverse image and apply the handle lemma again, parameterizing $B^k \times \mathbb{R}^m$ so that there is an overlap where we still have the original f. We compactify again.

The result is a new space $\bar{V}$ with a global ϵ-equivalence $\bar{F} : \bar{V} \to B^k \times \mathbb{R}^m$ so that $\bar{F}$ is a homeomorphism over $B^k \times B^m$ and $\bar{F} = f$ over $B^k \times (3B^m - 2\overset{\circ}{B}{}^m) \cup (B^k - \frac{7}{8}\overset{\circ}{B}{}^k) \times 3B^m$. Using the Splitting Theorem and the Generalized Poincaré conjecture, we can find an $S^{n-1} \subset f^{-1}(B^k \times (3B^m - 2\overset{\circ}{B}{}^m) \cup (B^k - \frac{7}{8}\overset{\circ}{B}{}^k) \times 3B^m)$ which bounds a ball in V containing $f^{-1}(\frac{1}{2}B^k \times B^m)$. By coning, we can identify $\bar{F}^{-1}(B^k \times 3B^m)$ with a subset of V, completing the proof. □

Proof of α-approximation: The proof of the α-approximation theorem is now an easy handle induction. We begin by taking a small handle decomposition of M. A 0-handle is a closed n-ball. Taking an open collar on the boundary, we have a $B^0 \times \mathbb{R}^n$. The Handle Theorem produces a new δ_1-equivalence which is a homeomorphism over a neighborhood of the original handle. After doing this for all 0-handles, each 1-handle is a $B^1 \times B^{n-1}$ meeting the 0-handles in $\partial B^1 \times B^{n-1}$. Adding a collar, we have a δ_1-equivalence over $B^1 \times 3B^{n-1}$ and a homeomorphism over a neighborhood of $\partial B^1 \times \mathbb{R}^m$. Applying the Handle Theorem gives a δ_2-equivalence which is a homeomorphism over a neighborhood of the original 1-handle. The induction continues until the Poincaré Conjecture and the Alexander trick allow us to cone off at the last stage. The degree of approximation is governed by the sizes of the original handles. □

Remark 3. Of course, this requires that we know that topological manifolds in dimensions ≥ 6 have small handle decompositions. This is one of the results of the Kirby-Siebenmann program [11, p.104]. A way of avoiding this is to use a handle decomposition of $\mathbb{R}^n$ to prove the result over coordinate patches and then use the following strong version of local contractibility of the homeomorphism group.

Theorem 6([8]). *Let M^n be a topological manifold. If C is a compact subset of M and U is an open neighborhood of C in M, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $h : U \to M$ is an open embedding with $d(h(x), x) < \delta$, then there is a homeomorphism $\bar{h} : M \to M$ so that $\bar{h}|C = h|C$, $\bar{h}|(M - U) = id$, and $d(\bar{h}(x), x) < \epsilon$.*

Here is how the piecing together process works. Cover M by finitely many balls $\{B_i\}_{i=0}^{p}$ and for each i, let $B_{i,0} \supset \mathring{B}_{i,0} \supset B_{i,1} \cdots \supset \mathring{B}_{i,n-1} \supset B_{i,n} = B_i$ be a sequence of nested neighborhoods of B_i. We prove inductively that for every $\epsilon > 0$ there is a $\delta > 0$ so that every δ-equivalence $f : N \to M$ is ϵ-close to an ϵ-equivalence f_i which is a homeomorphism over $B_{1,i} \cup \cdots \cup B_{i,i}$.

The case $i = 1$ is easy. We take a handle decomposition of $\mathring{B}_{1,0}$ and use the handle induction above to get a homeomorphism over $B_{1,1}$.

The case $i = 2$ is representative of the general case. We have homeomorphisms h and k over $B_{1,1}$ and $B_{2,1}$ which are close to our original f. We therefore have an homeomorphism $h \circ k^{-1} : \mathring{B}_{1,1} \cap \mathring{B}_{2,1} \to M$ which is close to the identity. By local contractibility of the homeomorphism group, we can find $\bar{h} : M \to M$ agreeing with $h \circ k^{-1}$ on $\mathring{B}_{1,2} \cap \mathring{B}_{2,2}$ which is close to the identity and which is equal to the identity outside of

$\mathring{B}_{1,1} \cap \mathring{B}_{2,1}$. Defining a new homeomorphism over $\mathring{B}_{1,2} \cup \mathring{B}_{2,2}$ to be h over $\mathring{B}_{1,2}$ and $\bar{h} \circ k$ over $\mathring{B}_{2,2}$ completes the inductive step. $\square$

Of course, we're still left using the classification of topological homotopy torii in high dimensions, but this trick of blending homeomorphisms using local contractibility is often useful.

Theorem 7 (Splitting Theorem). *Let W^n be a manifold, $n \geq 5$ and $\partial W = \emptyset$, and let $f : W \to S^{n-1} \times \mathbb{R}$ be a proper map which is an ϵ-equivalence over $[-2,2]$ via the projection map $p : S^{n-1} \times \mathbb{R} \to \mathbb{R}$. If $\epsilon > 0$ is sufficiently small, then there is an $(n-1)$-sphere S subset $f^{-1}(S^{n-1} \times [-1,1])$ such that $f|S : S \to S^{n-1} \times \mathbb{R}$ is a homotopy equivalence, S is bicollared, and S separates the component of W containing $f^{-1}(S^{n-1} \times [-1,1])$ into two components, one containing $f^{-1}(S^{n-1} \times \{-1\})$ and one containing $f^{-1}(S^{n-1} \times \{1\})$.*

Proof: In dimensions $n \geq 6$, this is similar to the proof of Siebenmann's thesis. Split by transversality over $S^{n-1} \times \{0\}$ and do surgery to make the map

$$H_k(f^{-1}(S^{n-1} \times [0, 1 - \tfrac{1}{n-k+3}]), f^{-1}(0)) \to H_k(f^{-1}(S^{n-1} \times [0,1]), f^{-1}(0))$$

the zero map.

The only real novelty here is that at each stage we must extend the map so that the surgered boundary manifold is the new inverse image of zero. This is done by applying a homotopy h_t which drags the image of the handle into $S^{n-1} \times \{0\}$ and then poking the interior of the handle across $S^{n-1} \times \{0\}$ inside a collar neighborhood of $S^{n-1} \times \{0\}$. $\square$

Remark 4. Given the basic tools of topological surgery – handle decompositions, transversality, and especially periodicity – the proof of the torus geometry we need isn't too hard.

It suffices for our argument to show that a homotopy $B^k \times T^m$ rel ∂ becomes standard after passage to a finite cover. After passing to a finite cover, we can use a relative version of Siebenmann's thesis to split open over T^{m-1} and reduce to the same problem for $B^{k+1} \times T^{m-1}$ rel ∂ and for $B^k \times T^{m-1}$. The first factor is no problem. We just induct on down to the case of B^n rel ∂, which we solve by an Alexander trick.

The second is more of a problem because of low-dimensional difficulties. Here is where $\mathcal{S}(B^{k+3} \times T^{m-1}, \partial)$, which pushes up the dimension, avoiding low-dimensional difficulties. See [17] for a nice explanation.

Here, $\mathcal{S}(B^k \times T^m, \partial)$ is the *structure set*, whose elements are homotopy equivalences $f : (N, \partial) \to (B^k \times T^m, \partial)$ with $f|\partial$ a homeomorphism, where f and f' are said to be equivalent if there is a homeomorphism $\phi : N \to N'$ so that f is homotopic to $f' \circ \phi$. Note that $\mathcal{S}(B^k \times T^m, \partial)$ has a group structure for $k \geq 1$. This same argument would work in **PL**, except that the periodicity isomorphism fails for $k = 3$.

The referee has pointed out that there is an another proof of the Ancel-Cannon Theorem in [4] which also uses the Černavskii-Seebeck Theorem.

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