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TOTALLY BOUNDED UNIFORMITIES FOR FRAMES

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1. INTRODUCTION.

In 1975, J. R. Isbell [8] defined a uniformity for a locale, and subsequently A. Pultr defined a uniformity for a frame L in terms of covers of L [11,13]. B. Banaschewski introduced the concept of a strong inclusion for a frame [1 and 2] and proved that a frame L admits a strong inclusion if and only if L is compactifiable. In his doctoral thesis [4], J. L. Frith established a one-to-one correspondence between strong inclusions for a frame L and totally bounded covering uniformities for L. In this note we use an alternative characterization of a uniformity for a frame L in terms of order-preserving functions from L to L, which we established in [5], to obtain results analogous to those of Frith [4, p.63-67]. Our methods of proof differ from those of Frith in that his arguments use covering uniformities and for the most part ours do not. One positive aspect of our approach is that it affords an explicit construction of a base for the unique totally bounded frame uniformity associated with a given strong inclusion. The reader is referred to [9] for further information on frames and to [10] for further information on proximities.

2. PRELIMINARIES.

A *frame* is a complete lattice $(L, \leq)$ in which the infinite distributivity condition $x \wedge \bigvee\{x_\alpha : \alpha \in \Lambda\} = \bigvee\{x \wedge x_\alpha : \alpha \in \Lambda\}$ holds for each $x \in L$ and each subset $\{x_\alpha : \alpha \in \Lambda\}$ of L. Given a frame $(L, \leq)$ the greatest element of L is denoted by 1 and the least element of L is denoted by 0.

Let L and M be frames and let $g : L \rightarrow M$ be a function. Then g is a *join homomorphism* provided that g preserves the joins of arbitrary sets. A *frame homomorphism* is a join homomorphism that also preserves meets of finite sets. If $f : L \rightarrow L$ is order preserving, an element b of L is f-small provided that $b \leq f(a)$ whenever $a \wedge b \neq 0$ [5],[6]. For a given order-preserving function f, the collection of all f-small members of L is denoted by A_f. For any $b \in L$ the pseudocomplement of b, denoted by $\overline{b}$, is $\bigvee\{a \in L : a \wedge b = 0\}$. Intuitively, $\overline{b}$ is the largest $a \in L$ such that $a \wedge b = 0$.

Definition 1. A strong inclusion for a *frame* $(L, \leq)$ is a binary relation $\ll$ on L satisfying the following axioms for a, b, c, d in L.

(1) $0 \ll 0,\ 1 \ll 1$.
(2) If $a \ll b$, then $a \leq b$.
(3) If $a \leq b \ll c \leq d$, then $a \ll d$.
(4) If $a \ll c$ and $a \ll b$, then $a \ll b \wedge c$.
(5) If $a \ll c$ and $b \ll c$, then $a \vee b \ll c$.
(6) If $a \ll b$, then there exists $c \in L$ such that $a \ll c \ll b$.
(7) For each $a \in L, a = \vee\{b \in L : b \ll a\}$.
(8) If $a \ll b$ there are $a', b' \in L$ such that $a \wedge a' = 0,\ b \vee b' = 1$ and $b' \ll a'$.

A frame together with a specified strong inclusion will be called a *proximity frame.*

We call axioms (8) the Smyth symmetry axiom. It is taken from [14, Proposition 3]. A related concept has been investigated by the authors in [6]. It is an easy exercise to show that the axioms (1)-(8) above are equivalent to the axioms for a strong inclusion defined in [1].

3. The totally bounded frame uniformity for a strong inclusion.

We turn now to the alternative characterization of a totally bounded uniformity for a frame. Once we have verified that our alternative notion squares with the established concept of a totally bounded covering uniformity, we characterize the totally bounded uniformity compatible with a given strong inclusion on a frame L, and hence characterize the strong inclusion itself, in terms of a collection of well-behaved functions from L to L.

Let $(L, \leq)$ be a frame and let F be a collection of order-preserving functions form L to L. We define $\leq, \wedge$ and $\vee$ pointwise on F.

Definition. Let U be a collection of order-preserving functions on a frame $(L, \leq)$. For $a, b \in L$ we write $a \ll^U b$ provided there is an $f \in U$ such that $f(a) \leq b$. The collection U is a *uniformity base* on L [5] provided that for $f \in U$ and $a, b \in L$:

(1) $a \leq f(a)$.
(2) There exists a $g \in U$ such that $g \circ g \leq f$.
(3) $a \wedge f(b) = 0$ if and only if $f(a) \wedge b = 0$.
(4) For any $f, g \in U$ there exists a join homomorphism $h \in U$ such that $h \leq f \wedge g$.
(5) The collection A_f of all f-small members of L is a cover of L.
(6) $a = \bigvee\{x \in L : x \ll^U a\}$.

The *uniformity* $\mathcal{U}$ on L *generated by* U is the collection of all order-preserving functions g such that $f \leq g$ for some $f \in U$. Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on L then $\mathcal{U}$ is *coarser* then $\mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$.

It follows from the theorem in [5] that the theory of uniformities for frames is equivalent to the theory of covering uniformities for frames given in [3], [11], [12], and [13]. The following definitions and notations are taken from [3] and [5]. A *cover* of a frame L is a subset B of L such that $\vee B = 1$. For each cover A of L and each $x \in L, Ax = \bigvee\{a \in A : \ a \wedge x \neq 0\}$.

If $\mathcal{A}$ is a collection of covers and $x, y \in L$, we write $x \ll^{\mathcal{A}} y$ provided there is an $A \in \mathcal{A}$ such that $Ax \leq y$. For covers A, B of L we write $A \prec B$ provided that for each $a \in A$ there is a $b \in B$ such that $a \leq b$. If $A \prec B$, we say that A *refines* B, and if $\{Aa : a \in A\} \prec B$ we write $A \prec^* B$. For covers A, B of $L, A \wedge B$ denotes $\{a \wedge b : a \in A, b \in B\}$. A collection $\mathcal{A}$ of covers of L is a *covering uniformity base* on L provided that

(1) If $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ such that $C \prec A \wedge B$.
(2) For any $A \in \mathcal{A}$ there is $B \in \mathcal{A}$ such that $B \prec^* A$.
(3) For any $y \in L$, $y = \bigvee\{x \in L : x \ll^{\mathcal{A}} y\}$.

The *covering uniformity* on L generated by $\mathcal{A}$ is the collection of all covers of L that are refined by some cover in $\mathcal{A}$. A covering uniformity is *totally bounded* provided that each $A \in \mathcal{A}$ is refined by a finite cover $B \in \mathcal{A}$ [3, p. 67].

Let $\mathcal{A}$ be a covering uniformity base on L. It follows from Propositions 3.1, 3.2, and 3.3 below that $\ll^{\mathcal{A}}$ is a strong inclusion on L. If $\mathcal{W}$ is a uniformity (base) on L, the corresponding covering uniformity $\mu(\mathcal{W})$ has for a base the collection of all A_f for $f \in \mathcal{W}$. If α is a covering uniformity (base), the corresponding uniformity $\mathcal{U}(\alpha)$ has for a base the collection of all functions $f_A : L \to L$ defined by $f_A(x) = Ax$ for all $A \in \alpha$.

Proposition 3.1 *Let $\mathcal{W}$ be a uniformity on a frame $(L, \leq)$ and let $a, b \in L$. Then $a \ll^{\mathcal{W}} b$ if and only if $a \ll^{\mathcal{A}} b$ where $\mathcal{A} = \mu(\mathcal{W})$.*

Proof: For any $f \in \mathcal{W}, f(a) \leq b$ if and only if $A_f a \leq b$.

Proposition 3.2. *Let α be a covering uniformity on a frame $(L, \leq)$ and let $a, b \in L$. Then $a \ll^{\alpha} b$ if and only if $a \ll^{\mathcal{W}} b$ where $\mathcal{W} = \mathcal{U}(\alpha)$.*

Proof: For any $A \in \alpha$ and $a, b \in L$ $Aa \leq b$ if and only if, $f_A(a) \leq b$.

It is evident from the previous two propositions that the relation $\ll^{U}$ used in the definition of a uniformity base on a frame L is the "uniformly below" given by B. Banaschewski and A. Pultr [3]. The following proposition, presented in terms of covering uniformities is given by Frith [4, proposition 4.19]. Since the axioms for a strong inclusion given here differ from those of Frith [4], we outline a proof of his result. This proof illustrates the way in which our approach to uniformities and strong inclusions differs from the covering approach.

Proposition 3.3. *Let $\mathcal{U}$ be a uniformity on a frame $(L, \leq)$; then $\ll^{\mathcal{U}}$ is a strong inclusion on L.*

Proof: The reader can easily verify that the first seven axioms for a strong inclusion hold. To show that the Smyth symmetry axiom holds, let $a, b \in L$ such that $a \ll^{\mathcal{U}} b$ and let $f \in \mathcal{U}$ such that $f(a) \leq b$. There exists $g \in \mathcal{U}$ such that $g \circ g \leq f$. Let $a' = \bar{a}$ and let $b' = \overline{g(a)}$. Since $b' \wedge g(a) = 0$ $g(b') \wedge a = 0$, and $g(b') \leq a'$. Hence $b' \ll^{\mathcal{U}} a'$. Evidently $a \wedge a' = 0$ and since $A_g g(a) \leq b$, $1 = \vee A_g = (\bigvee\{x \in A_g : x \wedge g(a) \neq 0\}) \vee (\bigvee\{x \in A_g : x \wedge g(a) = 0\}) \leq b \vee \overline{g(a)} = b \vee b'$.

Definition. A uniformity $\mathcal{U}$ on a frame $(L, \leq)$ is *totally bounded* provided that for each $g \in \mathcal{U}$ there is a finite cover B of L such that $f_B \in \mathcal{U}$ and $f_B \leq g$. (Corollary 3.1 states that the assumption that $f_B \in \mathcal{U}$ may be omitted from the preceding definition.)

The definition of total boundedness of a uniformity is justified by the following proposition.

Proposition 3.4. *Let $(L, \leq)$ be a frame. If α is a totally bounded covering uniformity on L, then $\mathcal{U}(\alpha)$ is a totally bounded uniformity; if $\mathcal{W}$ is a totally bounded uniformity on L, then $\mu(\mathcal{W})$ is a totally bounded covering uniformity on L.*

Proof: Let α be a totally bounded covering uniformity on L and let $g \in \mathcal{U}(\alpha)$. There exists $A \in \alpha$ such that $f_A \leq g$ and there is a finite $B \in \alpha$ such that $B \prec A$. Then $f_B \in \mathcal{U}(\alpha)$ and $f_B \leq g$.

Let $\mathcal{W}$ be a totally bounded uniformity and let $A \in \mu(\mathcal{W})$. There ia $g \in \mathcal{W}$ such that $A_g \prec^* A$. Let $f \in \mathcal{W}$ such that $f \circ f \circ f \leq g$. There is a finite cover B of L such that $f_B \in \mathcal{W}$

and $f_B \leq f$. For each $b \in B$, $A_{f_B} b \leq f_B(b) = Bb$ and so $A_{f_B} \prec \{Bb : b \in B\}$. It follows that $\{Bb : b \in B\}$ is a finite cover of L belonging to $\mu(\mathcal{W})$. Moreover each $b \in B$ is f_B-small and hence is both f-small and g-small. To see that $\{Bb : b \in B\}$ refines A_g let $b \in B$ and let $z \in A_f$ such that $z \wedge f(b) \neq 0$. Since $Bb = f_B(b) \leq f(b)$ it suffices to show that $f(b) \leq A_g b$. It may be seen that $f(b)$ is a g-small set, for if $a \in L$ and $a \wedge f(b) \neq 0$, then $f(a) \wedge b \neq 0$ and since b is f-small, $b \leq f \circ f(a)$ and so $f(b) \leq f \circ f \circ f(a) \leq g(a)$. Consequently $f(b) \leq A_g b$.

Let $\ll$ be a strong inclusion on a frame $(L, \leq)$. A cover $A = \{a_i : i \in \Lambda\}$ is a *proximity cover* provided there is a cover $B = \{b_i : i \in \Lambda\}$ such that $b_i \ll a_i$ for each $i \in \Lambda$. It is easy to see that if $\{A_i : 1 \leq i \leq n\}$ is a finite collection of proximity covers then $\wedge\{A_i : 1 \leq i \leq n\}$ is also a proximity cover.

If D is a cover of $L, D^* = \{\vee A : A \subseteq D$ and for all $x, y \in A, x \wedge y \neq 0\}$. Although $B^* \prec A$ and $B \prec^* A$ say different things if $C^* \prec B$ and $B^* \prec A$ then $C \prec^* A$.

The proof of the following lemma is based upon [3, Lemma 4].

Lemma 3.1. *Let $\ll$ be a strong inclusion on a frame $(L, \leq)$ and let E be a finite proximity cover. Then there is a finite proximity cover D such that $D^* \prec E$.*

Proof. We first establish the lemma in the special case that E is a two-element cover of L. Let $E = \{b_1, b_2\}$ be a proximity cover. There are covers $\{w_1, w_2\}, \{x_1, x_2\}, \{z_1, z_2\}$ such that for $i = 1, 2$, $w_i \ll x_i \ll z_i \ll b_i$. Since $x_1 \ll z_1$ by Smyth symmetry there are $x_1', z_1' \in L$ such that $x_1 \wedge x_1' = 0$, $z_1 \vee z_1' = 1$ and $z_1' \ll x_1'$. Similarly there are $x_2', z_2' \in L$ such that $x_2 \wedge x_2' = 0, z_2 \vee z_2' = 1$ and $z_2' \ll x_2'$. Set $D = \{x_1 \wedge b_2,\ x_2 \wedge b_1,\ x_1 \wedge x_2',\ x_1' \wedge x_2\}$. Since $1 = x_1 \vee x_2 = [x_1 \wedge (z_2 \vee z_2')] \vee [x_2 \wedge (z_1 \vee z_1')] \leq [x_1 \wedge (b_2 \vee x_2')] \vee [x_2 \wedge (b_1 \vee x_1')] = \vee D$, D is a cover. Because $x_i \leq z_i \leq b_i$ for $i = 1, 2$, is easy to verify that $D^* \prec \{b_1, b_2\}$. To see that D is a proximity cover, let $C = \{w_1 \wedge z_2,\ w_2 \wedge z_1,\ w_1 \wedge z_2',\ z_1' \wedge w_2\}$. Then $w_1 \wedge z_2 \ll$

$x_1 \wedge b_2$, $w_2 \wedge z_1 \ll x_2 \wedge b_1$, $w_1 \wedge z_2' \ll x_1 \wedge x_2'$ and $z_1' \wedge w_2 \ll x_1' \wedge x_2$. Moreover, $1 = w_1 \vee w_2 = [w_1 \wedge (z_2 \vee z_2')] \vee [w_2 \wedge (z_1 \vee z_1')] = \vee C$.

Now let $A = \{a_1, a_2, \ldots, a_n\}$ be an arbitrary finite proximity cover and let $X = \{x_1, \ldots, x_n\}$ be a cover such that $x_i \ll a_i$ for $i = 1, 2, \ldots, n$. Choose $i, 1 \leq i \leq n$. There are b_i, c_i in L such that $x_i \ll b_i \ll c_i \ll a_i$. Set $A_i = \{\bar{b}_i, a_i\}$. Then A_i is a two element proximity cover and so we may choose a proximity cover B_i such that $B_i^* \prec A_i$. Note that $x_i \wedge \bar{b}_i = 0$. For $i = 1, 2, \ldots, n$ define A_i as above. Since X is a cover, $\bigwedge_{i=1}^{n} \bar{b}_i = 0$ and so $\bigwedge_{i=1}^{n} A_i \prec A$. Let $D = \bigwedge_{i=1}^{n} B_i$. As we have already noted, D, being the finite meet of proximity covers is itself a proximity cover and D is the proximity cover we require since $D^* \prec \bigwedge_{i=1}^{n} A_i \prec A$.

Lemma 3.2 *Let $(L, \leq)$ be a frame and let $\ll$ be a strong inclusion on L. Let $B = \{f_C : C$ is a finite proximity cover of $L\}$. For $a, b \in L$, $a \ll b$ is and only if there exists $f \in B$ such that $f(a) \leq b$.*

Proof: Suppose there is a finite proximity cover $C = \{c_i\}_{i=1}^{n}$ such that $f_C(a) \leq b$. Let $A = \{a_i\}_{i=1}^{n}$ be a cover such that $a_i \ll c_i$ for $i = 1, \ldots, n$. Let $j \in \{1, 2, \ldots, n\}$ such that $a_j \wedge a \neq 0$. Then $a_j \ll c_j \leq Ca$. Hence $Aa \ll Ca \leq b$ and so $a \ll b$.

Now suppose $a \ll b$. There exists $c \in L$ such that $a \ll c \ll b$. By Smyth symmetry there are $a', c', c'', b'' \in L$ such that $a \wedge a' = 0$, $c \vee c' = 1$, $c' \ll a'$, $c \wedge c'' = 0$, $b'' \vee b = 1, b'' \ll c''$. Let $C = \{b, \bar{a}\}$. Since $1 = c \vee c' \leq c \vee a' \leq c \vee \bar{a} \leq b \vee \bar{a}$, C is a finite cover. To see that C is a proximity cover note that $A = \{c', c\}$ is a cover such that $c \ll b$ and $c' \ll a' \leq \bar{a}$. Thus $f_C \in B$ and $f_C(a) = b$.

If $\ll$ is any strong inclusion on a frame $(L, \leq)$ we say that a uniformity $\mathcal{U}$ is compatible with $\ll$ provided that for all $a, b \in L$, $a \ll b$ is and only if $a \ll^{\mathcal{U}} b$, and we say that $\mathcal{U}$ belongs to the proximity class $\pi(\ll)$.

Theorem 3.1. *Let $(L, \leq)$ be a frame and let $\ll$ be a strong inclusion on L. Let $B = \{f_C : C$ is a finite proximity cover of $L\}$. Then B is a base for a uniformity $\mathcal{P}$ on L compatible with $\ll$. Moreover $\mathcal{P}$ is the coarsest member of $\pi(\ll)$ and is the only totally bounded member of $\pi(\ll)$.*

Proof: It is evident that B satisfies conditions (1), (3), and (5) of the definition of a uniformity base and it follows from Lemma 3.2 that B satisfies condition (6) as well. Because the finite meet of proximity covers is a proximity cover, if f_C and f_D belong to B, then $f_{C \wedge D} \in B$. Hence condition (4) obtains. To see that condition (2) obtains let $f \in B$, and let E be a finite proximity cover such that $f = f_E$. By Lemma 3.1, there is a finite proximity cover D such that $D^* \prec E$. By definition $f_D \in B$. We show that $f_D \circ f_D \leq f$. Let $x \in L$. Then $f_D \circ f_D(x) = \vee\{d \in D :$ there exists $d' \in D$ such that $d \wedge d' \neq 0$ and $d' \wedge x \neq 0\} \leq D^*x \leq Ex = f(x)$.

We now show that $\mathcal{P}$ is the coarsest uniformity in $\pi(\ll)$. It follows from Lemma 3.2 that $\mathcal{P} \in \pi(\ll)$. Let $\mathcal{V}$ be any uniformity belonging to $\pi(\ll)$ and let $h \in \mathcal{P}$. Then there is a finite proximity cover $C = \{c_i : 1 \leq i \leq n\}$ such that $f_C \leq h$. Let $A = \{a_i : 1 \leq i \leq n\}$ be a cover of L such that $a_i \ll c_i$ for $i = 1, 2, \ldots, n$. There are $g_i \in \mathcal{V}, 1 \leq i \leq n$, such that $g_i(a_i) \leq c_i$. Let $g = \bigwedge_{n=1}^{n} g_i$. Since $g \in \mathcal{V}$, it suffices to show that $g \leq h$. Let $x \in L$ and suppose that j is a positive integer, $j \leq n$ such that $c_j \wedge x = 0$. Then $g_j(a_j) \wedge x \leq c_j \wedge x = 0$ and so $a_j \wedge g(x) \leq a_j \wedge g_j(x) = 0$. Hence $g(x) \leq \bigvee\{a_i \in A : a_i \wedge x \neq 0\} \leq \bigvee\{c_i \in C : c_i \wedge x \neq 0\} = f_C(x) \leq h(x)$.

It is obvious that $\mathcal{P}$ is totally bounded and so it only remains to prove that $\mathcal{P}$ is the only totally bounded member of $\pi(\ll)$. Let $\mathcal{W}$ be a totally bounded uniformity belonging to $\pi(\ll)$. It suffices to show that $\mathcal{W} \subseteq \mathcal{P}$. Let $g \in \mathcal{W}$. Then $A_g \in \mu(\mathcal{W})$ and by Proposition 3.4 there is a finite cover $C \in \mu(\mathcal{W})$ such that $C \prec A_g$. In the comments preceding [3, Proposition 4] the authors prove that there is a finite cover $D \in \mu(\mathcal{W})$ such that

$D \prec^* C$. For each $d \in D$ choose $c_d \in C$ such that $Dd \leq c_d$ and set $R = \{c_d : d \in D\}$. Since $D \prec R$, $R \in \mu(\mathcal{W})$. Clearly R is a finite refinement of C and $f_R \leq f_{A_g} \leq g$. We show that R is a proximity cover. Since $D \in \mu(\mathcal{W})$, it follows from the remarks preceding Proposition 3.1 that $f_D \in \mathcal{U}(\mu(\mathcal{W}))$ and by [5, proposition 2.3], $\mathcal{U}(\mu(\mathcal{W})) = \mathcal{W}$. Moreover for each $d \in D, f_D(d) = Dd \leq c_d$ and so $d \ll c_d$. Therefore R is a proximity cover of L.

Corollary 3.1. *Let $(L, \leq)$ be a frame and let $\mathcal{W}$ be a uniformity on L. The following statements are equivalent:*

(i) $\mathcal{W}$ is totally bounded,

(ii) For each $g \in \mathcal{W}$, A_g has a finite subcover, and

(iii) For each $g \in \mathcal{W}$, there exists a finite cover B of L such that $f_B \leq g$.

Proof: (i) $\Rightarrow$ (ii). Let $g \in \mathcal{W}$. There is a finite cover B of L such that $f_B \in \mathcal{U}$ and $f_B \leq g$. Let $b \in B$ and suppose that $a \wedge b \neq 0$. Then $b \leq f_B(a) \leq g(a)$ and so b is g-small.

(ii) $\Rightarrow$ (iii). Let $g \in \mathcal{W}$ and let B be a finite subcover of A_g. Let $x \in L$; then $f_B(x) = \bigvee\{b \in B : b \wedge x \neq 0\} \leq g(x)$.

(iii) $\Rightarrow$ (i). Let $g \in \mathcal{W}$. There exists an $h \in \mathcal{W}$ such that $h \circ h \circ h \leq g$ and h is a join homomorphism from L to L. Let B be a finite cover of L such that $f_B \leq h$. Put $C = \{h(b) : b \in B\}$. Then C is a finite proximity cover of L. It follows from Theorem 3.1 that $f_C \in \mathcal{W}$; hence it suffices to show that $f_C \leq g$. Let $a \in L$; then

$$\begin{aligned} f_C(a) &= \vee\{c \in C : c \wedge a \neq 0\} \\ &= \vee\{h(b) : b \in B \text{ and } h(b) \wedge a \neq 0\} \\ &= \vee\{h(b) : b \in B \text{ and } h(a) \wedge b \neq 0\} \\ &= h(\vee\{b \in B : b \wedge h(a) \neq 0)\} \\ &= h(Bh(a)) = h(f_B(h(a))) \leq h \circ h \circ h(a) \leq g(a). \end{aligned}$$

Corollary 3.2. *Let $(L, \leq)$ be a frame and let α be a covering uniformity on L. Then α is totally bounded if and only if for each $A \in \alpha$ there is a finite cover B of L such that $B \prec A$.*

Proof: In view of Proposition 3.4 and the fact that $\alpha = \mu(\mathcal{U}(\alpha))$ [5, Proposition 2.3] it suffices to show that $\mathcal{U}(\alpha)$ is totally bounded. Let $g \in \mathcal{U}(\alpha)$. Then there is an $A \in \alpha$ such that $f_A \leq g$. Let B be a finite cover of L such that $B \prec A$. Then $f_B \leq f_A$ and it follows from Corollary 3.1 that $\mathcal{U}(\alpha)$ is totally bounded.

REFERENCES

1. B. Banaschewski, *compactifications of Frames*, Math Nachr. **149** (1990), 105-115.
2. B. Banaschewski, *Frames and compactifications,* "Contributions to Extension Theory of Topological Structures," Academic Press, New York, 1969, 29-33.
3. B. Banaschewski and A. Pultr, *Samuel compactification and completion of uniform frames*, Math. Proc. Camb. Phil. Soc. **108** (1) (1990), 63-78.
4. J. L. Frith, *Structured Frames*, Ph.D. Thesis, The University of Cape Town, Cape Town (1986).
5. P. Fletcher and W. Hunsaker, *Entourage uniformities for frames,* Monatsh. Math. **112** (1991), 271-279.
6. P. Fletcher and W. Hunsaker, *Symmetry conditions in terms of open sets,* Topology Appl. **45** (1992), 39-47.
7. P. Fletcher and W. F. Lindgren, "Quasi-uniform spaces," Marcel Dekker, New York and Basel, 1982.
8. J. R. Isbell, *Atomless parts of space,* Math. Scand. **31** (1972), 5-32.
9. P. Johnstone, "Stone Spaces," Cambridge University Press, Cambridge, 1982.
10. S. Naimpally and B. Warrack, "Proximity Spaces," Cambridge Tracts in Mathematics and Mathematical Physics **59**. Cambridge, 1970.
11. A. Pultr, *Pointless Uniformities I. Complete regularity*, Comm. Math. Univ. Carolinae **25** (1984), 91-104.
12. A. Pultr, *Pointless Uniformities II (Dia)metrization,* Comm. Math. Univ. Carolinae **25** (1984), 105-120.
13. A. Pultr, *Some recent topological results in locale theory*, in "General Topology and its Relations to Modern Analysis and Algebra VI (1986)," Heldermann Verlag, Berlin, 1988, 451-467.
14. M. B. Smyth, *Stable compactification I.*, J. London Math. Soc., (2) **45** (1992) no. 2, 321-340.

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