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A SURVEY OF R-, U-, AND CH-CLOSED SPACES

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1. INTRODUCTION

This paper reports the progress made in the study of P-closed topological spaces, where P is Urysohn or completely Hausdorff or regular, since the appearance of the survey paper [BPS] of Berri, Porter and Stephenson. We do not address H-closed spaces, because its literature has grown enormously in the last twenty years, and the book by Porter and Woods [PW] offers a recent examination of the main lines of the subject. All topological spaces discussed in this paper have the T_1 property. Our terminology follows Engelking [En], Berri, Porter, and Stephenson [BPS], and Porter and Woods [PW]. Recall that a space is **Urysohn** (resp. **completely Hausdorff (CH)**) if every pair of distinct points are contained in disjoint closed neighborhoods (resp. can be separated by a real-valued continuous function). A P space is **P-closed** if it is closed in every P space in which it can be embedded, is **minimal P** if it has no strictly coarser P topology, and is **Katětov P** if it has a coarser minimal P topology. A regular space is **RC-regular** if it can be densely embedded in an R-closed space. The terms minimal Urysohn and minimal regular are abbreviated as MU and MR, respectively.

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Alexandroff and Urysohn [AU] first defined R-closed spaces in 1924 and asked whether such spaces are compact. In 1930, Tychonoff [T] constructed an example of a noncompact R-closed space, answering the question posed by Alexandroff and Urysohn; this example is described in 1.1. In his 1939 study of extensions, Alexandroff [A] considered free maximal regular filters. A filter base is said to be **open** if the sets belonging to it are open sets. Then, a **regular filter base** is an open filter base in which each set contains the closure of some member of the filter base. In 1941, Weinberg [We] proved that a regular space is R-closed iff every regular filter base clusters. Since a completely regular space has a compactification, it follows that a completely regular R-closed space is compact. Thus, noncompact R-closed spaces are contained in the family of regular spaces that are not completely regular. A major source of noncompact, R-closed spaces is a technique called the Jones' machine (described in 1Y in [PW]). A lack of simple examples is one of the reasons why the study of R-closed spaces is perceived to be difficult.

In 1955, Banaschewski showed that an MR space is R-closed. Berri and Sorgenfrey [BS] modified Tychonoff's example to obtain a noncompact MR space. Herrlich showed in [Her1] that Tychonoff's noncompact R-closed space is not MR.

We conclude this section with a fundamentally important example - a noncompact MR space having a subspace that is CH-closed and R-closed but not MR. Subspaces of this space are the beginning stage of the construction of many examples in P-closed spaces. First, some necessary notational definitions are presented.

Notational Definitions. For a spaces X, let $\tau(X)$ denote the set of open subsets of X. Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ be the set of positive integers, and $\mathbb{R}$ be the set of real numbers. Let I denote the unit interval with the usual topology inherited from the reals and J be the subspace $I\backslash\{1\}$. For an ordinal α, $[0,\alpha)$ (resp. $[0,\alpha]$) denotes the set of all ordinals less than (resp. less than or equal to) α, equipped with the order topology. Let ω (resp. ω_1) denote the set of all finite (resp. countable) ordinals. The non-normal, locally compact, zero-dimensional space $T = [0,\omega_1] \times [0,\omega]\backslash\{(\omega_1,\omega)\}$ is called the **Tychonoff plank.** Let $A = [0,\omega_1) \times J$ have the order topology based on the lexicographic ordering of A, i.e., A is **Alexandroff's long line.** Let $A' = A \cup \{\infty\}$ be the one point compactification of A.

1.1. Let R denote the quotient space of $T \times \mathbb{Z}$ where the points (ω_1, y, n) and $(\omega_1, y, n+1)$ are identified if n is odd, and the points $(x,\omega.n)$ and $(x,\omega,n+1)$ are identified if n is even. The image of $T \times \{n\}$ in R is denoted as T_n. Let $S = R \cup \{\pm\infty\}$. A subset $U \subseteq S$ is defined to be open if $U \cap R$ is open in R, and ∞(resp. $-\infty$) $\in U$ implies there is some $n \in \mathbb{Z}$ such that $\cup\{T_m : m \geq n\}$ (resp. $\cup\{T_m : m \leq -n\}) \subseteq U$. The space S (see [BS]) is noncompact MR, and Tychonoff's example is the subspace $\{\infty\} \cup \cup\{T_n : n \geq 0\}$ which is CH-closed, R-closed, and not MR (see [Her1]). $\square$

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2. The status of the Questions in [BPS]

In this section, we will review the progress made on questions raised in [BPS]. Our numbering system reflects the original question numbers.

Q1. *Is a regular, CH-closed space necessarily R-closed?*

Herrlich [Her2] answered this question in the negative with the following example.

2.1. Let R be the space described in 1.1. There is exactly one point p in βR such that every neighborhood of p meets each T_n. Let $X = \beta R\backslash\{p\} \cup \{p^+, p^-\}$. A subset $U \subseteq X$ is defined to be open if $U \cap (\beta R\backslash\{p\})$ is open in βR and $p^+ \in U$ (resp. $p^- \in U$) implies there is $n \in \mathbb{Z}$ such that $(U\backslash\{p^+,p^-\}) \cup cl_{\beta R}(\{\cup T_m : m \leq n\}$) (resp. $\cup\backslash\{p^+,p^-\} \cup cl_{\beta R}(\{\cup T_m : m \geq n\}))$ is a neighborhood of p in βR. The space X is regular and CH-closed but is not R-closed. $\square$

Q2. *Prove or disprove that the product of U-closed spaces is U-closed.*

Herrlich [He2] constructed U-closed spaces Y and X such that $Y \times Z$ is not U-closed. The spaces Y and Z are described in the following example.

2.2. Let D_1, D_2, and D_3 be three pairwise disjoint dense subsets of J whose union is J and $0 \in D_1$. Let Y be A' (A' is defined before 1.1) with the finer topology generated by $\tau(A') \cup \{[0,\omega_1) \times D_2,\ [0,\omega_1) \times D_3 \cup \{\infty\}\}$. To the subspace $T_1 \cup T_2$ of the space R defined in 1.1, we add a point t. Let $Z = T_1 \cup T_2 \cup \{t\}$. A subset $U \subseteq Z$ is defined to be open if $U \cap (T_1 \cup T_2)$ is open in $T_1 \cup T_2$ and $t \in U$ implies for some $(\alpha, n) \in [0,\omega_1) \times [0,\omega), (\alpha,\omega_1) \times (n,\omega] \times \{1\} \subseteq U$. The spaces Y and Z are U-closed, but $Y \times Z$ is not U-closed. $\square$

Q3. *Prove or disprove that the product of MU spaces is MU.*

Stephenson [St1] constructed MU spaces whose product is not U-closed.

2.3. Let $X = T_1 \cup T_2 \cup T_3 \cup \{a, b\}$ where T_n is defined in 1.1. A subset $U \subseteq X$ is defined to be open if $U \cap (T_1 \cup T_2 \cup T_3)$ is open in $T_1 \cup T_2 \cup T_3$ and a (resp. b) $\in U$ implies for some $(\alpha, n) \in [0,\omega_1) \times [0,\omega), (\alpha,\omega_1) \times (n,\omega] \times \{1\}$ (resp. $(\alpha,\omega_1] \times (n,\omega) \times \{3\}$) $\subseteq U$. Let R_1, R_2, R_3, R_4, and R_5 be five pairwise disjoint dense subsets of a compact space K whose union is K. Let $Y(K) = (R_1 \times \{1,5,9\}) \cup (R_2 \times \{2,8\}) \cup (R_3 \times \{3,7,11\}) \cup (R_4 \times \{4,10\}) \cup (R_5 \times \{6,12\})$ with the topology generated by $\{(U \times \{i\}) \cap Y(K) : i \in \{1,3,5,7,9,11\},\ U \in \tau(K)\} \cup \{(U \times \{i-1,i,i+1\}) \cap Y(K) : i \in \{2,4,6,8,10\},\ U \in \tau(K)\} \cup \{(U \times \{1,11,12\}) \cap Y(K) : U \in \tau(K)\}$.

Let R_1, R_2, R_3, R_4, and R_5 be five pairwise disjoint dense subsets of I whose union is I and $1 \in R_3$ and form $Y(I)$. Form $Y(A')$(A' is defined before 1.1) using $S_i = [0,\omega_1) \times R_i$

for $i = 1,2,4,5$ and $S_3 = ([0,\omega_1) \times R_3\backslash\{1\}) \cup \{\infty\}$. The spaces $X, Y(I)$, and $Y(A')$ are MU but $X \times Y(I) \times Y(A')$ is not U-closed. □

Q4. *Prove or disprove that the product of CH-closed spaces is CH-closed*

Stephenson [St3] constructed a CH-closed space X such that $X \times X$ is not CH-closed; the space X is described in the following example.

2.4. By 9.15 in [GJ] there is a space G such that $\mathbb{N} \subseteq G \subseteq \beta\mathbb{N}$, every infinite subset of $\beta\mathbb{N}$ has a limit point in G, and $G \times G$ contains an infinite closed discrete subset D such that $D \subseteq \mathbb{N} \subseteq \mathbb{N}$. Let B be the set $\beta\mathbb{N}$ with the topology generated by $\tau(\beta\mathbb{N}) \cup \{\beta\mathbb{N}\backslash(G\backslash\mathbb{N})\}$. Let X be the quotient space of the subspace $G \times \{1\} \cup B \times \{2\}$ of $B \times \{1,2\}$ where $(y,1)$ and $(y,2)$ are identified for $y \in G\backslash\mathbb{N}$. The space X is GH-closed but $X \times X$ is not. □

Q5. *Prove or disprove that the product of R-closed spaces is R-closed.*

Q6. *Prove or disprove that the product of MR spaces is MR.*

Pettey [Pe2] solved Q5 and Q6 by constructing an MR space M whose product with itself is not even R-closed. The space M is described in 4.3.

Q7. *Find a necessary and sufficient condition for a Urysohn space to be embedded (or densely embedded) in an MU space.*

To the best of our knowledge, both parts of this question remain unanswered.

Q8. *Find a necessary and sufficient condition that a regular space can be embedded in an R-closed space.*

Q9. *Find a necessary and sufficient condition that a regular space can be embedded (or densely embedded) in an MR space.*

Dow and Porter [DP1] completely resolved Q8 and the non-dense portion of Q9 by showing that every regular space can be embedded as a closed subspace in an MR space. The topological dense embedding question is still open. Harris [Ha1] has characterized RC-regular spaces as those regular spaces whose topology is generated by a generalized proximity, called an RC-proximity. Since the topology of an RC-regular space can be determined by several RC-proximities, Harris' characterization is, in some sense, not topological; this observation is the motivation for Q18. Porter and Votaw [PV] showed that every regular space can be densely embedded in a regular space which is nearly R-closed. (A regular space is **nearly R-closed** if every regular end converges; a regular filter $\mathcal{F}$ on a space X is a **regular end** if whenever U and V are open sets in X such that $cl_X U \cap cl_X V = \emptyset$ and $\mathcal{F}$ meets U, it follows that $X \backslash cl_X V \in \mathcal{F}$.)

Q10/Q11. *Is a minimal P space of second category when P is Urysohn or first countable Urysohn?*

Stephenson [St1] solved both Q10 and Q11 by constructing the non-second category space $Y(I)$ described in 2.3.

Q12. *Is a regular, CH-closed space necessarily of second category?*

This question seems to be open.

Q13. *Is there a noncompact minimal perfectly normal space?*

Recall (see 5.2 in [BPS]) that a space is minimal perfectly normal iff it is perfectly normal and countably compact. Using Martin's axiom with the negation of the continuum hypothesis, Weiss [Ws] has shown that such spaces are compact. On the other hand, assuming the set axiom diamond, Ostaszewski [O] has shown the existence of a noncompact, perfectly normal, countably compact space. Thus, Q13 is independent of the usual ZFC axioms of set theory.

Q14. [Ba] *Is a space in which each closed set is R-closed necessarily compact?*

This notoriously difficult problem is still open and has attracted the attention of several researchers. A resolution will probably require the development of new machinery or insights. It is well known that a Hausdorff space is compact if and only if every closed set is H-closed. Several results and problems listed in this survey were motivated by Q14. These include 3.11, 3.12, 4.6, 4.7, Q26, Q32, Q33, Q34, and Q42.

Q15. *Is a space in which each closed set is U-closed necessarily compact?*

This question is still open.

Q16. *Is a U-closed space necessarily Katětov Urysohn? Is there only one MU topology coarser than a U-closed, Katětov Urysohn topology?*

Porter [Po1] constructed a first countable semiregular U-closed space for which there exists neither a coarser MU topology nor a coarser minimal first countable Urysohn topology; this example is described below. The second part of the question remains open.

2.6. Let R_1, R_2, R_3, and R_4 be pairwise disjoint dense subsets of $I = [0,1]$ such that $I = \cup\{R_i : 1 \leq i \leq 4\}$ and $X = \cup\{R_i \times \{i\} : i \in \{1,2,3,4\}\} \cup R_1 \times \{5\}$. The topology on X is generated by $\{(U \times \{i\}) \cap X : i \in \{1,3,5\},\ U \in \tau(I)\} \cup \{(U \times \{i-1, i, i+1\}) \cap X :\ i \in \{2,4\},\ U \in \tau(I)\}$. The space X is first countable, semiregular, and U-closed, but X is not Katětov P where P is Urysohn or first countable Urysohn. □

Q17. *Is an R-closed space necessarily Katětov regular? Is there only one MR topology coarser than an R-closed, Katětov regular topology?*

The second part of the question remains open. The first part was solved by Dow and Porter [DP1], where they constructed an R-closed space which has no coarser minimal regular topology. Here is a description of their space.

2.7. Let S be the space defined in 1.1. The subspace $Z = \{(x,\gamma) \in S \times [0,\omega] :$ if $\gamma \in [0,\omega)$, then $x = +\infty$ or $x = (\alpha,\beta,n)$ where $n \geq -\gamma\}$ is R-closed but not Katětov regular.

3. Recent Developments - Embeddings and Subspaces

In this section, we present results and concepts concerning embeddings and subspaces that have arisen since the appearance of [BPS]. Open questions/problems are concurrently presented using a continuation of the numbering system of the second section. We start this section by stating an embedding result by Dow and Porter; this result was noted after Q9 as a solution to Q8 and a partial solution to Q9.

3.1. [DP1] *Every regular space can be embedded as a closed subspace of an MR space.*

A regular space is **strongly minimal regular,** or SMR, if the complements of R-closed subspaces form an open base. The noncompact, MR space S described in 1.1 is also SMR.

3.2. [St2] *An SMR space is MR.*

Pettey, in response to a question by Stephenson [St2], constructed an example of an MR space which is not SMR.

3.3. [Pe1] Let F denote the closed subspace $T_0 \cap T_{-1}$ of the space S described in 1.1. Let M denote the space obtained from the product $S \times [0, \omega + 1]$ by first identifying the point (∞, ω) with the point $(-\infty, \omega)$ and then for each point of t of F identifying (t, ω) with $(t, \omega + 1)$. The quotient space M is MR but not SMR.

In response to the question [St4, Wi] of whether every R-closed space is the continuous image of an MR space, Friedler and Pettey established this much stronger result.

3.4. [FP1] *Every R-closed space is the image of an SMR space by a perfect, open retraction.*

This theorem when combined with 3.1 produces an improvement of 3.1

3.5 *Every regular space can be embedded as a closed subspace of an SMR space.*

The topological dense embedding problem for R-closed spaces is still open. Herrlich demonstrated in [Her1] that there are regular spaces that are not RC-regular.

Q18. [DP1] *Find a topological characterization of RC-regular spaces.*

The dense embedding problem seems rather difficult. Perhaps it will be easier to first solve a very restricted case.

Q19. *Characterize those regular spaces that have one-point R-closed extensions.*

Note that a regular space has a one-point R-closed extension iff some free regular filter meets every other free regular filter. This is equivalent to some free regular filter being in every free maximal regular filter. Nevertheless, other (more useful) characterizations would be desirable.

A regular space is **locally R-closed** (resp. **rim R-closed**) if it has an open base with R-closed closures (resp boundaries). We use this definition of locally R-closed space instead of the one presented in [DP1]: a regular space in which each point has a quasi-base of R-closed neighborhoods. Dow and Porter established this next result.

3.6. [DP1] *A rim R-closed, R-closed space is locally R-closed and SMR.*

In light of well-known theorems about locally compact spaces and locally H-closed space (see [En, 3.5.11] and [PW, 7.3(b)]), an obvious question is whether a regular space is locally R-closed iff it has a one-point R-closed extension. Note, however, that if S is the space described in 1.1, then $S\backslash(\{\omega_1\} \times [0,\omega) \times \{0\})$ is a locally R-closed, RC-regular space without a one-point R-closed extension. Furthermore, a simple modification of an example given by Dow and Porter [DP1, p.51] yields a regular space that is not locally R-closed but does have a one-point SMR extension.

Q20. [DP1] *Prove or disprove that a regular space is RC-regular if it is rim R-closed or locally R-closed or both.*

A natural question is whether the regular continuous image of an RC-regular space is also RC-regular. Not only is this false, but the regular continuous perfect or open image is not necessarily RC-regular. To show this we start with the description of an example by Herrlich of a regular space which is not RC-regular.

3.7. Let T_i be the subspace defined in 1.1, $R_n = T_0 \cup \ldots \cup T_n$, and $Y = \cup\{R_n \times \{n\} : n \in \omega\} \cup \{\infty\}$. Now $U \subseteq Y$ is defined to be open if $U \cap (R_n \times \{n\})$ is open in $R_n \times \{n\}$ for $n \in \omega$ and $\infty \in U$ implies $\cup\{T_k \times \{m\} : m \geq k \geq n\} \subseteq U$ for some $n \in \omega$. Herrlich [Her1] showed that Y is a regular space but not RC-regular.

From the subspace $(T \times [0,\omega)) \cup ([0,\omega_1] \times [0,\omega] \times \{\omega\})$ of the compact space $[0,\omega_1] \times [0,\omega] \times [0,\omega]$ identify the points (ω_1, y, n) and $(\omega_1, y, n+1)$ if n is odd, and the points (x,ω,n) and $(x,\omega,n+1)$ if n is even to obtain a space denoted as Z^*; let Z denote the subspace $T \times [0,\omega]$ of Z^*. Now, R_n is the subspace $T \times [0,n]$ of Z. The space Z^* is R-closed and so the product $Z^* \times [0,\omega]$ is also R-closed. From $Z^* \times [0,\omega]$, shrink the compact set $[0,\omega_1] \times [0,\omega] \times \{\omega\} \times \{\omega\}$ to a point ∞ to obtain an R-closed space denoted as X^*.

The dense subspace $(Z\times[0,\omega))\cup\{\infty\}$ of X^* is denoted as X_1; so X_1 is RC-regular. The subspace $\bigcup\{R_n\times\{n\}:n\in\omega\}\cup\{\infty\}$ is the space Y, i.e., Herrlich's example of a regular space which is not RC-regular. There is a perfect retraction form X_1 onto Y.

For $n\in\omega$, let H_n denote the subspace $R_n\cup([0,\omega_1)\times[0,\omega)\times[n+1,\omega])$ of Z. Let X_2 denote the dense subspace $\bigcup\{H_n\times\{n\}:n\in\omega\}\cup\{\infty\}$ of X_1; so X_2 is RC-regular. There is an open retraction with compact point inverses from X_2 onto Y. □

3.8. *The space Y is not the image of an RC-regular space under an open perfect mapping.*

Proof: Assume that X is an RC-regular space and f is a continuous open perfect surjection onto Y. Let X^* denote an R-closed extension of X, and let A denote the set $X^*\backslash X$.

For each n in $\mathbb{N}$, every free regular filter base that traces on $R_n\times\{n\}$ necessarily traces on each of the sets $T\times\{i\}\times\{n\}$ for $1\le i\le n$. Therefore, since f is open ad closed (and thus takes regular filter bases on X to regular filter bases on Y), if x is a point of $A\cap cl_{X^*}f^{\leftarrow}[R_n\times\{n\}]$, then $x\in cl_{X^*}f^{\leftarrow}[T\times\{i\}\times\{n\}]$ for $1\le i\le n$.

Each of the sets $f^{\leftarrow}[R_n\times\{n\}]$ is open and closed with respect to X but is not R-closed and thus cannot be closed with respect to X^*. Thus, for each n in $\mathbb{N}$ we can choose a point x_n of $A\cap cl_{X^*}f^{\leftarrow}[R_n\times\{n\}]$.

Since $T\times\{1\}\times\mathbb{N}$ is a closed subset of Y and does not contain ∞, there is an open (with respect to X^*) neighborhood V of $f^{\leftarrow}(\infty)$ that misses $f^{\leftarrow}[T\times\{1\}\times\mathbb{N}]$. Since $\{x_1,x_2,x_3,\dots\}\subseteq cl_{X^*}f^{\leftarrow}[T\times\{1\}\times\mathbb{N}]$, it follows that V also misses $\{x_1,x_2,x_3,\dots\}$. Because $f^{\leftarrow}(\infty)$ is compact, there is an open neighborhood W of $f^{\leftarrow}(\infty)$ such that $cl_{X^*}W\subseteq V$. Let $U=Y\backslash f[X\backslash W]$. Then U is an open neighborhood of ∞ and thus contains $T\times\{n\}\times\{n\}$ for all but finitely many n. But $f^{\leftarrow}[U]\subseteq W$, so it follows that $x_n\in cl_{X^*}f^{\leftarrow}[T\times\{n\}\times\{n\}]\subseteq cl_{X^*}f^{\leftarrow}[U]\subseteq cl_{X^*}W\subseteq V$ for all but finitely many n. We therefore have a contradiction. □

The above result motivates the next question.

Q21. *If a regular space is the open perfect image of an RC-regular space, it is also RC-regular?*

Harris [Ha1] asked if comparable RC proximities give rise to comparable R-closed embeddings and if there is a largest R-closed extension of an RC-regular space, in the spirit of the Stone-Čech compactification or Katětov extension. Sharma and Naimpally answered both questions in the negative with the same example.

3.9. [SN] let S be Berri and Sorgenfrey's example of a non-compact, MR space described in 1.1, and let W be the dense subspace of S consisting of points none of whose coordinates are infinite limit ordinals. Then βW and S are non-comparable R-closed extensions of W although their corresponding RC-proximities are comparable. The space W has no largest R-closed extension. □

The solutions to the standard extension of continuous functions problem in the setting of R-closed or U-closed spaces are unknown.

Q22. *Let Y be an R-closed extension of a space X and $f : X \to Z$ be a continuous function where Z is R-closed. Find a necessary and sufficient condition for f to have a continuous extension to Y.*

Q23. *Let Y be a U-closed extension of a space X and $f : X \to Z$ be a continuous function where Z is U-closed. Find a necessary and sufficient condition for f to have a continuous extension to Y.*

Although most examples of regular, non-completely regular spaces are quite complicated, there is a simple example due to Mysior.

3.10. [M] Let X denote the upper half plane together with a distinct point a. All points (x, y) with $y > 0$ are isolated.

For $x \in \mathbb{R}$, a neighborhood of $(x, 0)$ contains$(\{(x,y) : 0 \leq y < 2\} \cup \{(x+y, y) : 0 < y < 2\}) \backslash F$ for some finite subset F of X. A neighborhood of a contains $\{a\} \cup \{(x,y) : x > r\}$ for some $r \in \mathbb{R}$. The space X is regular but not completely regular. □

Q24. *For the space X described in 3.10, is X or $X \times X$ RC-regular?*

There are many nondense subspace problems which remain unsolved.

Q25. [Hal] *Characterize the R-closed subspaces of an R-closed space.*

Q26. *Is an R-closed space in which the closure of every open set is R-closed necessarily compact?*

Q27. *Is a U-closed space in which the closure of every open set is U-closed necessarily compact?*

Another subspace problem is whether a countable decreasing chain of nonempty, R-closed subspaces is nonempty. This has been answered, in the negative, by Pettey [Pe5].

Q28. *Does there exist a (countable) nested chain of nonempty U-closed (MU) spaces with an empty intersection?*

The next result shows that any space satisfying the conditions of Q14 or Q15 must be at least countably compact.

3.11. *If every countable, closed subset of a space X is R-closed or U-closed, then X is countably compact.*

Proof: It is well known that a space is countably compact iff it has no infinite closed discrete subspaces, and since infinite discrete spaces are neither R-closed nor U-closed, the conclusion is immediate. □

This last result overlaps with the following result by Scarborough and Stone. First recall that a space is **feebly compact** if every countable open cover has a finite subfamily whose union is dense. Clearly, a countably compact space is feebly compact.

3.12. [SS] *An R-closed space is feebly compact.*

4. Recent Developments - Products and Categorizations

Due to the number of new concepts that have arisen in the last twenty years, there are a number of unresolved questions that focus on creating a classification system for topological spaces. We start this section with a useful product result by Pettey.

4.1. [Pe2] *Let $\{X_\alpha : \alpha \in A\}$ be a nonempty family of nonempty R-closed (resp. MR) spaces. Then $\Pi\{X_\alpha : \alpha \in A\}$ is R-closed (resp. MR) iff $\Pi\{X_\alpha : \alpha \in A\}$ is feebly compact.*

In addition to solving Q6, Pettey has shown that the product of SMR spaces need not be R-closed; this example is described below.

In [Pe2], Pettey defined a space X to be ***-feebly compact** if for every countable family $\{U_n : n \in \omega\}$ of nonempty open sets of X, there is an infinite subset $A \subseteq \omega$ and a compact subspace K of X such that every neighborhood of K meets U_n for all but finitely many $n \in A$. The importance of *-feebly compact in the class of R-closed spaces is indicated by the following result by Pettey.

4.2. [Pe2] *(a) A feebly compact k space is *-feebly compact, and a *-feebly compact space is feebly compact.*

*(b) If $\{X_\alpha : \alpha \in A\}$ is a family of *-feebly compact space, then $\Pi\{X_\alpha : \alpha \in A\}$ is feebly compact.*

*(c) If $\{X_\alpha : \alpha \in A\}$ is a family of *-feebly compact, R-closed (resp. MR, SMR) spaces, then $\Pi\{X_\alpha : \alpha \in A\}$ is also R-closed (resp. MR, SMR).*

*(d) If $\{X_\alpha : \alpha \in A\}$ is a family of *-feebly compact, R-closed (resp MR) spaces and Y is R-closed (resp. MR), then $\Pi\{X_\alpha : \alpha \in A\} \times Y$ is R-closed (resp. MR).*

Stephenson's result [St2] that a product of first countable R-closed (resp. MR) spaces is R-closed (resp. MR) is an immediate consequence of 4.2(a,c). It is unknown if 4.2(d) is true for SMR spaces even when there are only two factors.

Q29. [Pe2] *Is the product of an SMR space and a *-feebly compact SMR space necessarily SMR?*

4.3. [Pe2] This is an example of an SMR space M such that $M \times M$ is not R-closed. Let D and G be the spaces described in 2.4. Let U denote the set of first and second coordinates of points of D. Now U is a subset of $\mathbb{N}$ and hence is open in G. Now, by 3.5, $G \setminus U$ is a closed subset of some SMR space, but Pettey proves in 4.4 of [Pe2] that any Tychonoff space is a closed subset of a *-feebly compact, SMR space. Let Z denote a *-feebly compact, SMR space which contains $G \setminus U$ as a closed set; let h denote the embedding of $G \setminus U$ in Z. Let M denote the space obtained by attaching G to Z through h. The space M is SMR and D is an infinite clopen discrete subset of $M \times M$; so, $M \times M$ is not feebly compact and, by 4.1, $M \times M$ is not R-closed.

4.4. *Let M be the space described in* 4.3. *Then $M \times M$ is RC-regular.*

Proof: Let M, Z, G, U, and D be the space described in 4.3. Since $M \times Z$ and $Z \times M$ are R-closed by 4.2(d), their union, which is $(M \times M) \setminus (U \times U)$, is also R-closed. Let $H = (G \times G) \setminus (U \times U)$. Since $G \times G$ is closed subspace of $M \times M$, it is sufficient to show that $G \times G$ can be densely embedded in a regular Hausdorff space Y such that H is closed in Y, and, for every regular filter base $\mathcal{F}$ on Y, if there exists $F \in \mathcal{F}$ such that $F \cap H = \emptyset$, then the restriction of $\mathcal{F}$ to $Y \setminus H$, $\{F \setminus H : F \in \mathcal{F}\}$, has a cluster point in Y. (For if K is the space obtained by attaching Y to $(M \times M) \setminus (U \times U)$ along H, then the space K is R-closed and $M \times M$ is a dense subspace of K.) Let $Y = \beta(G \times G) \setminus (cl_{\beta(G \times G)} H \setminus H)$. Then clearly $G \times G$ is dense in Y and H is closed in Y. Suppose $\mathcal{F}$ is a regular filter base on

Y such that there exists $F \in \mathcal{F}$ with $F \cap H = \emptyset$, and let $\mathcal{B}$ be the restriction of $\mathcal{F}$ to $Y \backslash H$. Let $\mathcal{B}' = \{B \cap (G \times G) : B \in \mathcal{B}\}$. Then $\mathcal{B}'$ is a regular filter base in $G \times G$ and each member of $\mathcal{B}'$ lies in $U \times U$. If $\mathcal{B}'$ has a cluster point in $G \times G$, then this same point is obviously a cluster point of $\mathcal{B}$ in Y. If $\mathcal{B}'$ has no cluster point in $G \times G$, then we may obtain a new regular Hausdorff space X by adding a new point q to the space $G \times G$ and letting $\{V \cup \{g\} : V \in \mathcal{B}'\}$ be a neighborhood base at q. Since $U \times U$ is a discrete open subspace of $G \times G$ and since no member of $\mathcal{B}'$ meets H, X is completely regular and $q \notin cl_X H$. Then βX is a Hausdorff compactification of $G \times G$, and therefore, there is a continuous function f from $\beta(G \times G)$ onto βX such that the restriction of f to $G \times G$ is the identity. Since q is a cluster point of $\mathcal{B}'$, some point p of $f^{\leftarrow}(q)$ must be a cluster point of $\mathcal{B}'$ and thus of $\mathcal{B}$. Since q is not in $cl_X H$, p is not in $cl_{\beta(G \times G)} H$. Hence, $p \in Y$. So $\mathcal{B}$ has a cluster point in Y. □

Q30. [Hal] *Is the product of two R-closed spaces RC-regular?*

In response to a question by Harris [Hal] of whether the product of RC-proximity spaces is an RC-proximity, Friedler established the next result.

4.5 . [Fr1] *For every R-closed, noncompact Hausdorff space X, there is a compact Hausdorff space Y such that the product proximity on $X \times Y$ is not an RC-proximity.*

Q31. *If the product of regular spaces is RC-regular, is each factor RC-regular?*

A regular space is said to be R-**functionally compact (R-FC)** if every continuous function from it to a regular space is a closed map. Pettey has constructed a noncompact MR space whose every regular continuous image is MR.

The next proposition is new; its proof follows from 3.1.17 in [En].

4.6. *Let Y be a regular space which has every closed subset R-closed. Then the following are equivalent:*

(a) Y is compact,

(b) every closed subset of $X \times Y$ is R-closed for every compact Hausdorff space X, and

(c) $X \times Y$ is RFC for every compact Hausdorff space X.

Q32. [Pe4] *Does every RFC space have the property that every closed subspace is R-closed?*

Q33. [Pe4] *Does every RFC space have the property that every regular continuous image is SMR?*

In response to a question in [FP1] of whether a regular space is RFC if every regular continuous image of a space is MR, Pettey found an example of an SMR, non-RFC space whose every regular continuous image is SMR; a description of this space follows

4.7. [Pe4] Let B denote the deleted big square $[0,\omega_1] \times [0,\omega_1] \setminus \{(\omega_1,\omega_1)\}$. Let Q be the quotient space obtained from $B \times [0,\omega)$ by identifying the point (α,ω_1,n) with the point $(\omega_1,\alpha,n+1)$ for each n in $[0,\omega)$ and each α in $[0,\omega_1)$. The image of each $B \times \{n\}$ is denoted as B_n, and the image of each point $(\omega_1,\alpha,0)$ of the subset $\{\omega_1\} \times [0,\omega_1) \times \{0\}$ of $B \times \{0\}$ is denoted as b_α. Let $Z = Q \cup \{\infty\}$. A subset U of Z is defined to be open if $U \cap Q$ is open in Q and $\infty \in U$ implies $B_n \subseteq U$ for all but finitely may n in $[0,\omega)$. Finally, let Y denote the quotient space obtained from $Z \times [0,\omega_1] \setminus \{(\infty,\omega_1)\}$ as follows: for each α in $[0,\omega_1)$ and each β in $[0,\omega_1]$, identify the point (b_α,β) with the point (∞,α). Then every regular T_1 continuous image of Y is SMR but Y is not RFC.

In the above example, there is an uncountable chain of nonempty SMR subspaces of Y (each of which is a topological copy of Y) such that the intersection of these subspaces is empty.

Q34. [BB, FP1] *Is every RFC space necessarily compact?*

Q35. *Does every MU space have a base of open sets with U-closed complements?*

Q36. *A Urysohn space is U-functionally compact (UFC) if every continuous function onto a Urysohn space is closed. Is there a noncompact UFC space?*

Q37. [St4] *Is it true that every U-closed space is a retract of an MU space (under an open, perfect map)?*

If Q37 has an affirmative answer, then every Urysohn space can be embedded in a MU space, and this would answer part of Q7.

In 1983, Pettey established the next result.

4.8. [Pe3] *Every locally R-closed, R-closed space is the retract, under an open and perfect map, of an R-closed, rim R-closed space.*

This result answers a question posed by Dow and Porter [D-P1] of whether every R-closed, rim R-closed space is compact, since the space S of 1.1 is noncompact, locally R-closed, and R-closed.

It is known [Pe3] that the locally R-closed property is preserved by open continuous mappings.

Q38. *If a space Y is a retract of a locally R-closed space, then is Y also locally R-closed?*

The analogous question about locally H-closed spaces has been answered in the affirmative by Girou [Gi]. Furthermore, as is shown below, the answer to Q38 is yes when the less restrictive definition of locally R-closed (see the paragraph after Q19) is used.

4.9. *Suppose X is a regular space having a quasi-base of R-closed neighborhoods (i.e., for every point x and every neighborhood U of x, there is an R-closed neighborhood N of x such that $N \subseteq U$). Then every retract of X has a quasi-base of R-closed neighborhoods.*

Proof: Let r be a retraction of X onto a subspace Y of X. Let y be a point of Y and U an open neighborhood of y in Y. Choose N to be an R-closed neighborhood of y in X such that $N \subseteq r^{\leftarrow}[U]$. Then $y \in (Y \cap int_X N) \subseteq int_Y r[N] \subseteq r[N] \subseteq U$. Since $r[N]$ is necessarily R-closed, this completes the proof. □

The following two problems are old and well-known in the folklore literature; they are questions that most researchers in this area have tried at one time or another. The two problems are motivated by the facts that an H-closed space is minimal Hausdorff iff it is semiregular and a CH-closed space is minimal CH iff it is completely regular (iff it is compact).

Q39. *Find a property P which does not imply R-closed for which a space is R-closed and has property P iff it is MR.*

Q40. *Find a property Q which does not imply U-closed for which a space is U-closed and has property Q iff it is MU.*

We now focus on the classification of R-closed and U-closed spaces by various cardinality functions. The examples of noncompact, R-closed spaces so far presented have not been separable nor first countable. Using the continuum hypothesis, Stephenson [St2] gave an example of a noncompact, separable, first countable R-closed space: Hechler [Hec] modified this example and eliminated the continuum hypothesis assumption. This example is now presented.

4.10. Let $X = \mathbb{N} \cup \mathcal{M}$ where $\mathcal{M}$ is an infinite maximal almost disjoint family on $\mathbb{N}$. A set $U \subseteq X$ is defined to be open if $M \in \mathcal{M} \cap U$ implies $M \backslash U$ is finite. The space X is the well-known Ψ space (see [GJ, PW]) which is first countable, zero-dimensional, locally compact, feebly compact, and not normal. Also, $\mathcal{M}$ can be selected so that $|\mathcal{M}| = \mathfrak{c}$ and there is some $\mathcal{A} \subseteq \mathcal{M}$ such that $|\mathcal{A}| = \mathfrak{c}$ and for each open set U of X, $|U \cap \mathcal{A}| > \omega$ implies $|clU \cap \mathcal{M} \backslash \mathcal{A}| > \omega$. Let $\mathcal{B}$ and $\mathcal{C}$ be disjoint subsets of $\mathcal{A}$ such that $|\mathcal{B}| = |\mathcal{C}| = \mathfrak{c}$. Let $f : \mathcal{B} \rightarrow \mathcal{M} \backslash \mathcal{A}$ and g :

$\mathcal{C} \to \mathcal{M}\backslash\mathcal{A}$ be bijections. Let Y denote the quotient space of $X \times \mathbb{Z}$ where these pairs of points are identified: $(g(x), i)$ and $(x, i+1)$ for $x \in \mathcal{C}$, (x, i) and $(g(x), i-1)$ for $x \in \mathcal{C}$, $(f(x), i)$ and $(x, i-1)$ for $x \in \mathcal{B}$, (x, i) and $(f(x), i+1)$ for $x \in \mathcal{B}$, (x, i) and $(g^{\leftarrow}(f(x)), i+2)$ for $x \in \mathcal{B}$, and (x, i) and $(f^{\leftarrow}(g(x)), i-2)$ for $x \in \mathcal{C}$.Let $Z = Y \cup \{\pm\infty\}$ and denote the image of $X \times \{n\}$ in Y as X_n. A subset $U \subseteq Z$ is defined to be open if $U \cap Y$ is open in Y, and $\infty \in U$ implies there is $n \in \mathbb{Z}$ such that $(\mathbb{N} \cup \mathcal{B}) \times \{n\} \cup (X\backslash\mathcal{C}) \times \{n+1\} \cup \bigcup\{X_m : m \geq n+2\} \subseteq U$, and $-\infty \in U$ implies there is $n \in \mathbb{Z}$ such that $(\mathbb{N} \cup \mathcal{C}) \times \{n\} \cup (X\backslash\mathcal{B}) \times \{n-1\} \cup \bigcup\{X_m : m \leq n-2\} \subseteq U$. The space Z is first countable, separable, feebly compact, and SMR but is not countably compact.

The character of a space X, denoted as $\psi(X)$, is the least cardinal λ such that every point of X has a neighborhood base of cardinality less than or equal to λ. A well known result of Ahangel'skii for compact Hausdorff space ($|X| \leq 2^{\psi(X)}$ whenever X is compact Hausdorff) has been extended to H-closed spaces in [DP3] and shown to be false for R-closed spaces in [DP2]. The following question about U-closed spaces remains open.

Q41. *Prove or disprove that if X is U-closed, $|X| \leq 2^{\psi(X)}$.*

We do not know the answer to Q41 even in the class of first countable spaces nor do we know the answer to Q14 in the more restrictive setting of first countable spaces.

Q42 *Is a first countable space compact when every closed subset is R-closed?*

By 3.11, a space in which every closed subset is R-closed is countably compact. If the first countable hypothesis of Q42 is replaced by perfect (i.e., every closed set is a G_δ), then Q42 has an affirmative answer under the assumption of Martin's axiom and the negation of the continuum hypothesis. This follows by a result of Weiss [Ws], who has established that a perfect, regular, countably compact space is compact under the assumption of Martin's axiom and the negation of the continuum hypothesis.

Another question about first countable spaces is the following:

Q43. [Pol] *Is every first countable CH-closed space Katětov (first countable)-CH?*

Recall that the weigh wZ of a space Z is the least cardinal of a base of Z. If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection, then $wY \leq wX$; this result is also true when X is H-closed and Y is minimal Hausdorff [Fr2]. We do not know the answer in the setting of R-closed spaces.

Q44. *If X and Y are R-closed spaces and $f : X \to Y$ is a continuous surjection, is it true that $wY \leq wX$?*

We conclude this section with diagrams indicating the known relationships between some of the regular classes discussed so far.

Let CPT denote the class of compact Hausdorff spaces, CRC denote regular spaces for which all closed sets are R-closed, and RC denote R-closed spaces. We have these two diagrams:

$$CPT \subseteq CRC \subseteq RFC \subsetneq MR \subsetneq RC \text{ and}$$

$$CRC \subsetneq SMR \subsetneq MR$$

We do not know if $CPT = CRC$ (Q14) or if $CRC = RFC$ (Q32).

5. Recent Developments - Related Areas

The first area we examine is the class of hyperspaces. For a space X, let 2^X denote the set of all nonempty closed subsets of X with the topology generated by the base $\{\langle U_1, \dots, U_n\rangle : n \in \mathbb{N},\ U_i \in \tau(X) \text{ for } i = 1, \dots, n\}$ where if $A_1, \dots, A_n \subseteq X, \langle A_1, \dots, A_n\rangle = \{F \in 2^X : F \subseteq \cup\{A_i : 1 \leq i \leq n\},\ F \cap A_i \neq \emptyset \text{ for } 1 \leq i \leq n\}$. The investigation of hyperspaces is motivated by Q14 and the well-known fact that 2^X is compact iff X is compact. Friedler, Dickman, and Krystock established the next result.

5.5. [FDK] *For a Hausdorff space X, X is compact iff 2^X is R-closed.*

A question in [FDK] of whether 2^X has the property that every regular filter clusters (i.e., R-closed without the regularity property) whenever X is R-closed is answered in the negative by using the next result and the R-closed space M of 4.3. (Since $M \times M$ is not R-closed, by 5.2, there is a regular filter on 2^M which does not have a cluster point.)

5.2. [FP2] *If X is regular and every regular filter on 2^X has a cluster point, then 2^X is feebly compact and X^n is R-closed for each $n \in \mathbb{N}$.*

Q45. [FP2] *If X is R-closed and 2^X is feebly compact, then does every regular filter on 2^X have a cluster point?*

Q46. [FP2] *What are necessary and sufficient conditions on a space X for every regular filter on 2^X to have a cluster point?*

The second area of investigation is the class of $S(\alpha)$ spaces. Let $\alpha > 0$ be an ordinal. Two filters $\mathcal{F}$ and $\mathcal{G}$ on a space X are R(α)-separated (resp., U(α)-separated) if there are open families $\{U_\beta : \beta < \alpha\} \subseteq \mathcal{F}$ and $\{V_\beta : \beta < \alpha\} \subseteq \mathcal{G}$ such that $U_0 \cap V_0 = \emptyset$ (*resp.*, $clU_0 \cap clV_0 = \emptyset$) and for $\gamma + 1 < \alpha$, $clU_{\gamma+1} \subseteq U_\gamma$ and $clV_{\gamma+1} \subseteq V_\gamma$. A space X is R(α) (resp., U(α)) if for distinct points $x, y \in X$, the neighborhood filters $\mathcal{N}_x$ and $\mathcal{N}_y$ are R(α)-separated (resp., U(α)-separated). For $\alpha \geq \omega$, R(α) and U(α) are equivalent concepts. So, for $\alpha \geq \omega$, let S(α) = R(α) = U(α) and for $0 < n < \omega$, let S(2n-1) = R(n) and S(2n) = U(n). Now the property of Hausdorff is the same as S(1), Urysohn the same as S(2), a S(α +1) space is S(α), a regular space is S(ω) (but not necessarily S(ω +1)), and a CH space is S(α) for $\alpha < \omega_1$ (but not necessarily S(ω_1)).

5.3. [PV] *let $\alpha > 0$ be an ordinal.*

(1) *A minimal $S(\alpha)$ space is $S(\alpha)$-closed and semiregular.*

(2) *A compact space is minimal* $S(\alpha)$ *for* $\alpha < \omega_1$.
(3) *An* $S(\alpha)$ *space can be densely embedded in an* $S(\alpha)$*-closed space.*
(4) *A minimal* $S(\alpha)$ *space is regular when* α *is a limit ordinal.*
(5) *An* $S(\alpha+1)$*, minimal* $S(\alpha)$ *space is regular.*
(6) *A CH, minimal* $S(\alpha)$ *space is regular whenever* $\alpha \leq \omega_1$.
(7) *A space is R-closed iff it is* $S(\omega)$*-closed and regular.*
(8) *A space is* MR *iff it is minimal* $S(\omega)$.

Many of the questions in this survey are analogous to open questions for S(α) spaces.

The third area of investigation is in the setting of subcompact spaces. J. de Groot defined a regular space X to be a **subcompact** if there is a base $\mathcal{B}$ of open sets of X such that if $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base, then $\cap\mathcal{F} \neq \emptyset$. Every locally compact Hausdorff space is subcompact relative to a base of open sets with compact closures and in metrizable spaces, subcompactness is equivalent to completeness, so this concept yields a general setting for the Baire category theorem. Subcompact spaces are preserved by products [dG] and by open maps if either the domain or range is metrizable or if the range is a Moore space [AL]. Every R-closed space is subcompact and an infinite discrete space is subcompact but not R-closed. The following questions, due to Aarts and Lutzer, are apparently still open.

Q47. *If* X *is subcompact and* Y *is a dense* G_δ *subset of* X*, is* Y *subcompact?*

Q48. *Do open mappings between regular spaces preserve subcompactness?*

Q49. *Do perfect, irreducible mappings preserve subcompactness?*

For a Hausdorff space X, let $S(X)$ be the Stone space of the Boolean algebra of all regular open sets of X. The absolute of X is the subspace $\{\mathcal{U} \in S(X) : \mathcal{U}$ has a cluster point$\}$;

a more detailed discussion of absolutes is provided in [PW]. The absolute of an R-closed space is both feebly compact and subcompact.

Q50. *Characterize the absolute of an R-closed space.*

Q51. *Characterize the absolute of a U-closed space.*

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