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WELL-ORDERED (F) SPACES

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ABSTRACT. Elastic (hence metrisable, proto-metrisable and linearly stratifiable) spaces have well-ordered (F) and every well-ordered (F) space is monotonically normal and hereditarily paracompact. The class of well-ordered (F) spaces is shown to be closed under closed maps and the duplication and scattering processes. A product theory is developed and compact well-ordered (F) spaces are investigated. Complete details of the relationships between the 'classical' cardinal invariants for well-ordered (F) spaces are given.

1. INTRODUCTION

In this paper we begin the study of well-ordered (F) spaces. They are defined in terms of the Collins-Roscoe structuring mechanism, and as a consequence are relatively simple to handle. Recall that a T_1 topological space X has $\mathcal{W}$ satisfying (F) if $\mathcal{W} = (\mathcal{W}(x) : x \in X)$ where each $\mathcal{W}(x)$ consists of subsets of X containing x and

(F) if $x \in U$ and U is open, then there exists an open $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$ whenever $y \in V$.

Condition (F) was originally defined in [6] and is referred to as the " Collins-Roscoe structuring mechanism". Observe that (F) is only of interest if extra conditions are imposed on the $\mathcal{W}(x)$'s. As a convenient convention, if $\mathcal{P}$ is an order condition we say '$\mathcal{W}$ satisfies $\mathcal{P}$ (F)' provided each $\mathcal{W}(x)$ satisfies $\mathcal{P}$ when considered as a partial order with respect to **reverse** inclusion. Further we say 'the space X has $\mathcal{P}$ (F)' or 'X is a $\mathcal{P}$ (F) space' whenever X has $\mathcal{W}$ satisfying $\mathcal{P}$ (F). Relevant order properties include: 'chain', 'well-ordered' or 'decreasing' by which we mean that each $\mathcal{W}(x)$ is countable, say $\mathcal{W}(x) = \{W(n,x) : n \in \mathbb{N}\}$, and $W(n+1,x) \subseteq W(n,x)$ for each x and n. Evidently, a decreasing (F) space is well-ordered (F), and a well-ordered (F) space is chain (F). The following two theorems indicate both the importance of the structuring mechanism and well-ordered (F) spaces. (The definition of acyclic monotone normality is given in Section 2.)

Theorem 1. ([1] and [5]). *A space is stratifiable if and only if it has countable pseudo-character and decreasing (F).*

Theorem 2. ([15]). *A space is acyclic monotonically normal if and only if it has chain (F).*

From the second theorem it is immediate that well-ordered (F) spaces are monotonically normal, and in [5] it is shown that well-ordered (F) spaces are hereditarily paracompact. Given these restrictions the class of well-ordered (F) spaces is extremely broad. In Section 3 the authors extend Theorem 1 by demonstrating that every elastic space has well-ordered (F). Hence, if X is metrisable, proto-metrisable or linearly stratifiable then X has well-ordered (F). However, we note that the Sorgenfrey line provides an example of a monotonically normal, hereditarily paracompact space that fails to have well-ordered (F) [15]. Interestingly, it seems much easier to show that spaces are well-ordered (F) compared with showing that they are elastic. For example, the authors characterised proto-metrisability in terms of a strengthening of well-ordered open (F) (see [9]), but the proof that proto-metrisable spaces are elastic appears to be non-trivial (see [10]).

The stability of the class of well-ordered (F) spaces is investigated in Section 2. In particular, it is established that the class is closed under the scattering and duplication processes, and under closed maps. The final three sections develop a product theory for well-ordered (F) spaces, investigate compact well-ordered (F) spaces and provide a complete solution to the problem of 'classical' cardinal invariants for well-ordered (F) spaces. Applications of these results provide new properties of proto-metrisable and elastic spaces.

Conventions and notation. By a space we shall mean a T_1 topological space. The closure of a set A in a space is denoted by $\overline{A}$, and its interior by A°. If $P = (A, B)$ is a pair of sets, we shall denote A, the first element of the pair P, by P_1, and B by P_2.

2. Basic Properties and Stability

We begin by providing a characterisation of well-ordered (F) spaces in terms of monotone normality. This characterisation will be used to prove that closed images of well-ordered (F) spaces have well-ordered (F), and that elastic spaces have well-ordered (F). First recall that a monotonically normal operator $H(\cdot,\cdot)$ is said to *acyclic* [15] provided:

$\bigcap_{i=0}^{n-1} H(x_i, X \setminus \{x_{i+1}\}) = \emptyset$ whenever $n \geqslant 2, x_0, \ldots, x_{n-1}$ are distinct and $x_n = x_0$.

We shall say that a monotonically normal operator $H(\cdot,\cdot)$ on a space X is *Noetherian* provided:

$\bigcap_{i=0}^{\infty} H(x_i, X \setminus \{x_{i+1}\}) = \emptyset$ whenever $(x_n)_{n=0}^{\infty}$ is a sequence of distinct points of X.

Theorem 3. *A space X has well-ordered (F) if and only if it has a monotonically normal operator which is both acyclic and Noetherian.*

Proof: First suppose that X has $\mathcal{W}$ satisfying well-ordered (F). Define

$$H(x, U) = \{y \in X : x \in W \subseteq U \text{ for some } W \in \mathcal{W}(y)\}^\circ,$$

whenever x is in the open set U. Since $\mathcal{W}$ satisfies chain (F), $H(\cdot,\cdot)$ is an acyclic monotonically normal operator [15]. To see that $H(\cdot,\cdot)$ is Noetherian, suppose that $(x_n)_{n=0}^{\infty}$ is a sequence of distinct points, but $x \in \bigcap_{i=0}^{\infty} H(x_i, X \setminus \{x_{i+1}\})$. By the definition of $H(\cdot,\cdot)$, for each n there is $W_n \in \mathcal{W}(x)$ satisfying $x_n \in W_n \subseteq X \setminus \{x_{n+1}\}$. Since $\mathcal{W}(x)$ is ordered by inclusion we have that $W_n \subsetneq W_{n+1}$ for each n. But this contradicts the fact that $\mathcal{W}(x)$ is well-ordered by reverse inclusion.

Conversely, suppose that $H(\cdot,\cdot)$ is an acyclic Noetherian monotonically normal operator on X. Recalling the proof of Theorem 11 of [15], fix $a \in X$ and define

$$x \sim_a y \text{ if and only if } a \in H(x, X \setminus \{y\}).$$

Acyclicity of $H(\cdot,\cdot)$ ensures that the transitive closure, $<_a$, of $\sim_a$ is an irreflexive partial order, while $H(\cdot,\cdot)$ Noetherian means that $>_a$ (i.e., $(<_a)^{-1}$) is well-founded. Thus $>_a$ has an extension, $\rhd_a$ which is a well-order. Define for each $x \in X$ $S_a(x) = \{y : y \lhd_a x\} \cup \{x\}$ and $\mathcal{W}(a) = \{S_a(x) : x \in X\}$. By the definition of the $S_a(x)$'s and the fact that $\rhd_a$ is a well-order, $\mathcal{W}(a)$ is well-ordered by reverse inclusion. It only remains to show that $\mathcal{W} = (\mathcal{W}(x) : x \in X)$ satisfies (F). This is achieved as in the proof of Theorem 11 of [15]. □

Theorem 4. *Suppose $f : X \to Y$ is a closed continuous surjection. If X has well-ordered (F), so has Y.*

Proof: Let $H_X(\cdot,\cdot)$ be an acyclic Noetherian monotonically normal operator on X. For each $A \subseteq X$ define $f^*(A) = Y \setminus f(X \setminus A)$. Observe that $f^*(A) = \{y \in Y : f^{-1}(y) \subseteq A\}$ and that, since f is closed, f^* maps open sets to open sets. Suppose $y \in Y$ and U is an open neighbourhood of y. Define

$$H_Y(y,U) = f^*(\bigcup\{H_X(x, f^{-1}(U)) : x \in f^{-1}(y)\}).$$

By the above remarks it is clear that $H_Y(y,U)$ is an open neighbourhood of y and $H_Y(y,U) \subseteq H_Y(y,U')$ whenever $U \subseteq U'$. We will show that $H_Y(\cdot,\cdot)$ is Noetherian, the proof that it is acyclic is similar. So suppose that $(y_n)_{n=0}^{\infty}$ is a sequence of distinct points of Y, but $y \in \bigcap_{i=0}^{\infty} H_Y(y_i, Y \setminus \{y_{i+1}\})$. Pick an

$x \in f^{-1}(y)$ and for each $i \in \omega$ choose $x_i \in f^{-1}(y_i)$ such that

$$x \in H_X(x_i, f^{-1}(Y \setminus \{y_{i+1}\})) \subseteq H_X(x_i, X \setminus \{x_{i+1}\}).$$

But then $x \in \bigcap_{i=0}^{\infty} H_X(x_i, X \setminus \{x_{i+1}\})$ contradicting the fact that $H_X(\cdot,\cdot)$ is Noetherian. □

We now consider two constructions which are important in this area of generalised metric spaces. The first is duplication. Recall that the Alexandroff duplicate, $\mathcal{D}(X)$, of a space X is the set $X \times \{0,1\}$ topologised so that $(x,1)$ is isolated for every $x \in X$, and so that a local base for the point $(x,0)$ $(x \in X)$ is $\{(U \times \{0,1\}) \setminus \{(x,1)\} : U \text{ open in } X \text{ and } x \in U\}$.

Theorem 5. *If a space X has well-ordered (F), then so has $\mathcal{D}(X)$.*

Proof: Suppose that $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfies well-ordered (F). For each $x \in X$ define

$$\mathcal{W}_{\mathcal{D}}(x,1) = \Big\{\{(x,1)\}\Big\} \bigcup \Big\{\{(x,1)\} \cup (W \times \{0\}) : W \in \mathcal{W}(x)\Big\},$$

and

$$\mathcal{W}_{\mathcal{D}}(x,0) = \{W \times \{0\} : W \in \mathcal{W}(x)\}.$$

Observe that for each $y \in \mathcal{D}(X)$, $\mathcal{W}_{\mathcal{D}}(y)$ is well-ordered by reverse inclusion. Now define $\mathcal{W}_{\mathcal{D}} = (\mathcal{W}_{\mathcal{D}}(y) : y \in \mathcal{D}(X))$. It is now easy to verify that $\mathcal{W}_{\mathcal{D}}$ satisfies (F) and hence $\mathcal{D}(X)$ has well-ordered (F) as required. □

If $\mathcal{C}$ is a class of topological spaces, then we define $S(\mathcal{C})$ to be the class of spaces which are obtained by the following *scattering process*: take any space in $\mathcal{C}$, isolate all the points of some subset, replace each such point by a space in $\mathcal{C}$, and repeat transfinitely, taking some subspace of the inverse limit at limit ordinals. Observe that $\mathcal{C} \subseteq S(\mathcal{C})$. We shall say that $\mathcal{C}$ is *closed under the scattering process* if $\mathcal{C} = S(\mathcal{C})$. Nyikos has shown that the class of proto-metrisable spaces is precisely the class $S(\mathcal{M})$ where $\mathcal{M}$ denote the class of metrisable spaces

[17]. The following theorem provides us with a wide variety of well-ordered (F) spaces.

Theorem 6. *The class of well-ordered (F) spaces is closed under the scattering process.*

Proof: Suppose that for some ordinal δ we are given:

(1) Topological spaces $(X_\alpha : \alpha \leqslant \delta)$ such that $X_0 = \{\emptyset\}$.
(2) For each $\alpha < \delta$, a subset $A_{\alpha+1}$ of X_α, and for each $a \in A_{\alpha+1}$, a space $X_{\alpha+1}(a)$ which has well-ordered (F).
(3) For each $\beta \leqslant \alpha \leqslant \delta$ a continuous surjection $j_{\alpha\to\beta} : X_\alpha \to X_\beta$.

In addition, we assume that, for each $\alpha < \delta$, the space $X_{\alpha+1}$ is obtained from X_α by replacing each point a of $A_{\alpha+1}$ by a clopen copy of $X_{\alpha+1}(a)$. Also,

$$\begin{aligned} j_{\alpha+1\to\alpha+1} &= \mathrm{id}_{X_{\alpha+1}} \\ j_{\alpha+1\to\alpha} &= \begin{cases} x & \text{if } x \in X_\alpha \setminus A_\alpha \\ a & \text{if } x \in X_{\alpha+1}(a) \end{cases} \\ j_{\alpha+1\to\beta} &= j_{\alpha\to\beta} \circ j_{\alpha+1\to\alpha} \qquad (\beta < \alpha). \end{aligned}$$

Finally, for each $\lambda \leqslant \delta$ which is a limit,

$$X_\lambda = \{(x_\alpha)_{\alpha<\lambda} : x_\alpha \in X_\alpha, \beta \leqslant \alpha < \lambda \Rightarrow j_{\alpha\to\beta}(x_\alpha) = x_\beta\}.$$

The set X_λ is endowed with the subspace topology induced by the product space $\prod_{\alpha<\lambda} X_\alpha$, and

$$\begin{aligned} j_{\lambda\to\lambda} &= \mathrm{id}_{X_\lambda}, \\ j_{\lambda\to\alpha} &= \pi_\alpha \restriction X_\lambda \qquad (\alpha < \lambda) \end{aligned}$$

where $\pi_\alpha : \prod_{\beta<\lambda} X_\beta \to X_\alpha$ is the projection map.

Since a subspace of a well-ordered (F) space has well-ordered (F), it suffices to prove that X_δ has well-ordered (F). We shall recursively define, for $\alpha \leqslant \delta$, $\mathcal{W}_\alpha = (\mathcal{W}_\alpha(x) : x \in X_\alpha)$. Assume that this has been done for each $\alpha < \gamma \leqslant \delta$ and that the following two conditions are satisfied.

Inductive hypotheses.

($I_\gamma 1$) For each $\alpha < \gamma$, $\mathcal{W}_\alpha$ satisfies well-ordered (F) for the space X_α

($I_\gamma 2$) If $\beta \leqslant \alpha < \gamma$, $x \in X_\alpha$ and $y = j_{\alpha\to\beta}(x)$, then $j^{-1}_{\alpha\to\beta}(\mathcal{W}_\beta(y)) \subseteq \mathcal{W}_\alpha(x)$. Furthermore, if $W \in \mathcal{W}_\alpha(x)$ but $W \notin j^{-1}_{\alpha\to\beta}(\mathcal{W}_\beta(y))$, then $j_{\alpha\to\beta}(W) = \{y\}$.

In the above, $j^{-1}_{\alpha\to\beta}(\mathcal{W}_\beta(y))$ denotes $\{j^{-1}_{\alpha\to\beta}(W) : W \in \mathcal{W}_\beta(y)\}$. We now define $\mathcal{W}_\gamma$ and check that ($I_{\gamma+1}1$) and ($I_{\gamma+1}2$) hold. First consider the case when γ is a successor, $\alpha + 1$ say. For each $a \in A_{\alpha+1}$, let $\mathcal{W}_{X_{\alpha+1}(a)} = (\mathcal{W}_{X_{\alpha+1}(a)}(x) : x \in X_{\alpha+1}(a))$ satisfy well-ordered (F) for the space $X_{\alpha+1}(a)$. If $x \in X_{\alpha+1}(a)$ then define

$$\mathcal{W}_{\alpha+1}(x) = \mathcal{W}_{X_{\alpha+1}(a)}(x) \cup j^{-1}_{\alpha+1\to\alpha}(\mathcal{W}_\alpha(a)).$$

If $x \notin X_{\alpha+1}(a)$ for any $a \in A_{\alpha+1}$, but $x \in X_{\alpha+1}$, then recall that $x \in X_\alpha$ and define

$$\mathcal{W}_{\alpha+1}(x) = j^{-1}_{\alpha+1\to\alpha}(\mathcal{W}_\alpha(x)).$$

Let $\mathcal{W}_{\alpha+1} = (\mathcal{W}_{\alpha+1}(x) : x \in X_\alpha)$ and observe that ($I_{\gamma+1}2$) is satisfied and each $\mathcal{W}_{\alpha+1}(x)$ is well-ordered by reverse inclusion. Hence it suffices to show that, if U is an open neighbourhood of $x \in X_{\alpha+1}$, then there is an open neighbourhood V of x such that $x \in W \subseteq U$ for some $W \in \mathcal{W}_{\alpha+1}(y)$ whenever $y \in V$. Since $X_{\alpha+1}(a)$ is open in $X_{\alpha+1}$ and $\mathcal{W}_{X_{\alpha+1}(a)}$ satisfies (F) for $X_{\alpha+1}(a)$, we need only consider the case when $x \notin X_{\alpha+1}(a)$ for any $a \in A_{\alpha+1}$. There is an open set O in X_α such that $x \in O$ and $j^{-1}_{\alpha+1\to\alpha}(O) \subseteq U$. $\mathcal{W}_\alpha$ satisfies (F) for X_α and hence there is an open set V in X_α which contains x, and $x \in W \subseteq O$ for some $W \in \mathcal{W}_\alpha(z)$ whenever $z \in V$. Notice that $j^{-1}_{\alpha+1\to\alpha}(V)$ is an open neighbourhood of x in $X_{\alpha+1}$. Suppose $y \in j^{-1}_{\alpha+1\to\alpha}(V)$ and let $z = j_{\alpha+1\to\alpha}(y)$. Since $z \in V$ there is $W \in \mathcal{W}_\alpha(z)$ for which $x \in W \subseteq O$. But then

$$x \in j^{-1}_{\alpha+1\to\alpha}(W) \subseteq j^{-1}_{\alpha+1\to\alpha}(O) \subseteq U.$$

To complete the case, recall that $j^{-1}_{\alpha+1\to\alpha}(W) \in \mathcal{W}_{\alpha+1}(y)$.

Now consider the case when γ is a limit, λ say. If $(x_\alpha)_{\alpha<\lambda}$ is an element of X_λ then define $\mathcal{W}_\lambda(x) = \bigcup_{\alpha<\lambda} j_{\lambda\to\alpha}^{-1}(\mathcal{W}_\alpha(x_\alpha))$. Let $\mathcal{W}_\lambda = (\mathcal{W}_\lambda(x) : x \in X_\lambda)$. Clearly $(I_{\gamma+1}2)$ is satisfied and by a similar argument to the successor case, $\mathcal{W}_\lambda$ satisfies (F) for X_λ. Hence it only remains to show that for each $x \in X_\lambda$, $\mathcal{W}_\lambda(x)$ is well-ordered by reverse inclusion. So, suppose that A is a non-empty subset of $\mathcal{W}_\lambda(x)$. Let $\alpha < \lambda$ be minimal such that $A \cap j_{\lambda\to\alpha}^{-1}(\mathcal{W}_\alpha(x_\alpha)) \neq \emptyset$. Recall that $j_{\lambda\to\alpha}(W) = \{x_\alpha\}$ for every $W \in \mathcal{W}_\lambda(x) \setminus j_{\lambda\to\alpha}^{-1}(\mathcal{W}_\alpha(x_\alpha))$. Thus, since $j_{\lambda\to\alpha}^{-1}(\mathcal{W}_\alpha(x_\alpha))$ is well-ordered by reverse inclusion, A has a minimal element with respect to reverse inclusion, as required. □

3. Elastic spaces

Elastic spaces were introduced by Tamano and Vaughan in [20] as a natural generalisation of stratifiable spaces. Recall that a space X is *elastic* if there is a pair-base $\mathbb{P}$ on X and a relation $\sim$ on $\mathbb{P}$ such that:

(1) the relation $\sim$ is transitive and if $P, P' \in \mathbb{P}$ are such that $P_1 \cap P_1' \neq \emptyset$, then $P \sim P'$ or $P' \sim P$,

(2) if $P \in \mathbb{P}$ and $\mathbb{P}' \subseteq \{P' \in \mathbb{P} : P' \sim P\}$, then

$$\overline{\bigcup\{P_1' : P' \in \mathbb{P}'\}} \subseteq \bigcup\{P_2' : P' \in \mathbb{P}'\}.$$

We observe that the above definition differs from that given in [20]. Unfortunately the original definition of elasticity is ambiguous since it is not clear how one defines the "frame map" when there are distinct $P, P' \in \mathbb{P}$ such that $P_1 = P_1'$. Even if there is no problem defining the frame map for a space X, the same problem arises when attempting to show that subspaces of X are elastic. However, if one analyses proofs of results involving elastic spaces (particularly Theorem 2 of [20]), then it is clear that the above is the intended definition. Although elastic spaces are important, the definition is not entirely transparent, so the structure and properties of elastic spaces have not been well understood. However, the following result, together with the rest of this paper, go some way to clarifying the situation.

Theorem 7. *Every elastic space is a well-ordered (F) space.*

The proof of the above theorem makes use of the following order-theoretic lemma, the proof of which is a minor modification of the proof of Lemma 2 of [20].

Lemma 8. *Suppose that $\sim$ is a reflexive transitive relation on the set S. Then there is a reflexive, antisymmetric relation $\approx$ on S such that:*

(1) *If $x, y \in S$ and $x \sim y$, then $x \approx y$ or $y \approx x$.*
(2) *If A is a non-empty subset of S then there is an $x \in A$ such that $y \not\approx x$ whenever $y \in A \setminus \{x\}$.*
(3) *If $A \subseteq S$ and $A \subseteq \{x \in S : x \approx s\}$ for some $s \in S$, then $A \subseteq \{x \in S : x \sim s'\}$ for some $s' \in S$.*

Proof of Theorem 7: We strengthen Theorem 2.3 of [3] and Theorem 2.2 of [16] by demonstrating that elastic spaces have monotonically normal operators which are both acyclic and Noetherian. By Lemma 8, since X is an elastic space, there is a pair-base $\mathbb{P}$ on X and a reflexive, antisymmetric relation $\approx$ on S such that:

(1) If $P, P' \in \mathbb{P}$ and $P_1 \cap P'_1 \neq \emptyset$, then $P \approx P'$ or $P' \approx P$.
(2) If $\mathbb{P}' \subseteq \mathbb{P}$ is non-empty, then there is $P \in \mathbb{P}'$ such that $P' \not\approx P$ whenever $P' \in \mathbb{P}' \setminus \{P\}$.
(3) If $P \in \mathbb{P}$ and $\mathbb{P}' \subseteq \{P' \in \mathbb{P} : P' \approx P\}$, then

$$\overline{\bigcup\{P'_1 : P' \in \mathbb{P}'\}} \subseteq \bigcup\{P'_2 : P' \in \mathbb{P}'\}.$$

Recalling the proof of Theorem 2.2 of [16], suppose that U is open in X and $P \in \mathbb{P}$. Define

$$U_P = \bigcup\{V_1 : (V_1, V_2) \in \mathbb{P}, V_1 \subseteq V_2 \subseteq U \text{ and } (V_1, V_2) \approx P\},$$

and observe that $\overline{U_P} \subseteq U$. Now suppose that O is an open neighbourhood of x. Pick $P(O,x) \in \mathbb{P}$ so that $x \in P(O,x)_1 \subseteq P(O,x)_2 \subseteq O$. Define

$$O^x = P(O,x)_1 \setminus [\overline{(X \setminus \{x\})_{P(O,x)}}]$$

and notice that O^x is an open neighbourhood of x. Define, for $x \in U$ and U open,

$$H(x,U) = \bigcup\{O^x : x \in O \subseteq U \text{ and } O \text{ open}\}.$$

In [16] it is shown that $H(\cdot,\cdot)$ is an acyclic monotonically normal operator. We show that $H(\cdot,\cdot)$ is Noetherian. So, suppose that $(x_n)_{n=0}^{\infty}$ is a sequence of distinct points of X and assume, for contradiction, that $x \in \bigcap_{n=0}^{\infty} H(x_n, X \setminus \{x_{n+1}\})$. Thus for each n there is an open set O_n such that $x_n \in O_n \subseteq X \setminus \{x_{n+1}\}$ and $x \in O_n^{x_n}$; hence $x \in P(O_n,x_n)_1 \setminus \overline{[(X \setminus \{x_n\})_{P(O_n,x_n)}]}$. By (2), there is an n such that for any m,

$$P(O_m,x_m) = P(O_n,x_n) \qquad \text{or} \qquad P(O_m,x_m) \not\preceq P(O_n,x_n).$$

Now $x \in P(O_n,x_n)_1 \cap P(O_m,x_m)_1$ and thus by (1), $P(O_n,x_n) \approx P(O_m,x_m)$ for every m. In particular, $P(O_n,x_n) \approx P(O_{n+1},x_{n+1})$. Recall that

$$x \in P(O_n,x_n)_1 \subseteq P(O_n,x_n)_2 \subseteq O_n \subseteq X \setminus \{x_{n+1}\}$$

and thus $x \in (X \setminus \{x_{n+1}\})_{P(O_{n+1},x_{n+1})}$. However, this contradicts the fact that $x \in O_{n+1}^{x_{n+1}}$. Therefore $H(\cdot,\cdot)$ is Noetherian as required. □

Although the classes of elastic and well-ordered (F) spaces share many properties, they are in fact distinct. In [10] the authors provide a number of examples of well-ordered (F) spaces that are not elastic; the duplication and scattering processes playing a central role in the constructions. In particular, it is shown that the duplicate of a stratifiable space need not be elastic, that elasticity is not preserved by perfect maps, and that there is an example of a compact, first countable well-ordered (F) space that is not elastic. Indeed the results of [10] and Section 2 indicate that the class of well-ordered (F) spaces is considerably better behaved than the class of elastic spaces.

4. Products

The product of two well-ordered (F) spaces need not have well-ordered (F) (the product of the Michael line and the irrationals is not normal, so cannot have well-ordered (F)). However, while 'well-order' is not preserved by taking products, it turns out that order conditions weaker than 'well-order' are preserved.

Suppose that $\mathcal{A}$ is a collection of subsets of a set X. Lindgren and Nyikos [13] define $\mathcal{A}$ to be *Noetherian* if every ascending sequence $A_1 \subsetneq A_2 \subsetneq \ldots$ of members of $\mathcal{A}$ is finite. For each $x \in X$ define $\mathcal{A}_x = \{A \in \mathcal{A} : x \in \mathcal{A}\}$. $\mathcal{A}$ is said to be of *sub-infinite rank* if for any $x \in X$, any subcollection of $\mathcal{A}_x$ consisting of incomparable elements ($A \not\subseteq B$ and $B \not\subseteq A$) is finite. Notice that if $\cap\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is of sub-infinite rank if and only if any subcollection of $\mathcal{A}$ consisting of incomparable elements is finite.

If X is a space, we say $\mathcal{W} = (\mathcal{W}(x) : x \in X)$ satisfies *Noetherian of sub-infinite rank* (F) if $\mathcal{W}$ satisfies (F) and each $\mathcal{W}(x)$ is Noetherian and of sub-infinite rank. Note that $\mathcal{W}(x)$ is well-ordered (by reverse inclusion) if and only if it is a chain and Noetherian, and $\mathcal{W}(x)$ is a chain precisely when no finite collection of elements of $\mathcal{W}(x)$ is incomparable; hence sub-infinite rank is a weakening of chain in which 'no finite collection' has been replaced by 'no infinite collection'.

In [13] Lindgren and Nyikos consider spaces with bases which are Noetherian and of sub-infinite rank. It is shown that such spaces are hereditarily metacompact, and that the finite product of spaces, each of which has a Noetherian base of sub-infinite rank, likewise has a Noetherian base of sub-infinite rank. We now sketch an analogous (but more general) theory for spaces with $\mathcal{W}$ satisfying Noetherian of sub-infinite rank (F). The following lemma is fundamental to the development (see Lemma 4 of [18] and Exercise 7, p.181 of [2]).

Lemma 9. *If $\mathcal{A}$ is a collection of subsets of a set X and $\cap\mathcal{A} \neq \emptyset$, then the following are equivalent:*

(1) *$\mathcal{A}$ is Noetherian and of sub-infinite rank.*
(2) *If $\mathcal{A}'$ is a subset of $\mathcal{A}$, then there is a finite subset C of $\mathcal{A}'$ such that every member of $\mathcal{A}'$ is contained in some member of C.*
(3) *Every infinite subset of $\mathcal{A}$ contains an infinite descending sequence.*

Suppose that $X_1, X_2, \ldots, X_n$ are spaces and $\mathcal{W}_i$ satisfies Noetherian of sub-infinite rank (F) for X_i. For each $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ define $\mathcal{W}(x_1, \ldots, x_n) = \{W_1 \times W_2 \times \cdots \times W_n : W_i \in \mathcal{W}_i(x), 1 \leqslant i \leqslant n\}$. By (3) of Lemma 9, it is clear that $\mathcal{W}(x_1, \ldots, x_n)$ is Noetherian and of sub-infinite rank. Furthermore, if $\mathcal{W} = (\mathcal{W}(x) : x \in \prod_{i=1}^n X_i)$ then it is easy to check that $\mathcal{W}$ satisfies (F) for X. Thus we have the following lemma.

Lemma 10. *The finite product of spaces, each of which has $\mathcal{W}$ satisfying Noetherian of sub-infinite rank (F), has $\mathcal{W}$ satisfying Noetherian of sub-infinite rank (F).*

The following lemma extends Theorem 4 of [5].

Lemma 11. *If the space X has $\mathcal{W}$ satisfying Noetherian of sub-infinite rank (F), then X is hereditarily metacompact.*

Proof: As the hypotheses are hereditary, it suffices to show that X is metacompact. So suppose that α is an ordinal and $\mathcal{U} = \{U_\beta : \beta < \alpha\}$ is an open cover of X. For each $\beta < \alpha$ define

$$V_\beta = \bigcup\{V(x, U_\beta) : x \in U_\beta \setminus \cup\{U_\gamma : \gamma < \beta\}\}$$

($V(\cdot,\cdot)$ is the operator associated with condition (F)). Now $\mathcal{V} = \{V_\beta : \beta < \alpha\}$ is an open refinement of $\mathcal{U}$. If $\mathcal{V}$ were not point finite then there would exist an x and an increasing sequence of ordinals $(\alpha_n)_{n=1}^\infty$ such that $x \in V_{\alpha_n}$ for each n. So, there are points y_n in $U_{\alpha_n} \setminus \bigcup\{U_\beta : \beta < \alpha_n\}$ such that $x \in V(y_n, U_{\alpha_n})$. Condition (F) tells us that there exists $(W_n)_{n=1}^\infty$ in $\mathcal{W}(x)$ such that $y_n \in W_n \subseteq U_{\alpha_n}$. Notice that if $m < n$ then $y_n \notin U_{\alpha_m}$ and hence $W_n \not\subseteq W_m$. Hence $\{W_n : n \in \mathbb{N}\}$ is an infinite subcollection of $\mathcal{W}(x)$ which does not contain an infinite descending sequence, contradicting the fact that $\mathcal{W}(x)$ is Noetherian and of sub-infinite rank. □

An immediate consequence of our results is the following.

Theorem 12. *The finite product of well-ordered (F) or elastic or proto-metrisable spaces is hereditarily metacompact.*

If $\mathcal{B}$ is a Noetherian base of sub-infinite rank for a space X, then by defining $\mathcal{W}(x) = \{B \in \mathcal{B} : x \in B\}$, we see that X has $\mathcal{W}$ satisfying Noetherian of sub-infinite rank (F). However, as indicated above, the converse fails. The following fact will be important: Suppose $\mathcal{A}$ is a Noetherian collection of sub-infinite rank and $\cap\mathcal{A} \neq \emptyset$, then if $\mathcal{A}'$ is an uncountable subset of $\mathcal{A}$ then there exist $A_\alpha \in \mathcal{A}'$ $(\alpha \in \omega_1)$ such that $A_\alpha \subsetneq A_\beta$ whenever $\beta < \alpha$.

Example 13. *There is a stratifiable (hence well-ordered (F)) space which does not have a Noetherian base of sub-infinite rank.*

Let X be the space obtained from the space $\omega \times (\omega + 1)$ by identifying the end points (n, ω). The quotient map is closed, so X is Lasnev and in particular stratifiable. By the above fact, no local base about the identified point can be Noetherian and of sub-infinite rank. Hence X does not have a Noetherian base of sub-infinite rank. □

A slight modification of the above proof yields that X also fails to have a point-additively Noetherian base (see [18]). We note that it is possible to develop a product theory of spaces with $\mathcal{W}$ satisfying *additively Noetherian* (F) which is analogous to the theory Nyikos developed for spaces with additively Noetherian bases [18]

5. Compactness

We begin by remarking that there are non-trivial compact well-ordered (F) spaces; recall there is a compact, first countable, well-ordered (F) space that fails to be elastic [10]. However, compact well-ordered (F) spaces are Eberlein compact. As a consequence, compact elastic spaces are Eberlein compact.

Theorem 14. *Compact well-ordered (F) spaces are Eberlein compact (that is to say homeomorphic to a weakly compact subset of a Banach space).*

Proof: By Theorem 12, if X is a compact well-ordered (F) space, then X^2 is hereditarily metacompact. Thus, if Δ denotes the diagonal of X^2, then $X^2 \setminus \Delta$ is metacompact, and hence by Theorem 2.2 of [11][1], X is Eberlein compact. □

In [21] Williams and Zhou introduced the class of Extremely Normal (EN) spaces. Amongst other results they proved that for scattered locally compact spaces, EN is equivalent to hereditarily paracompact. Although EN spaces are monotonically normal and hereditarily paracompact the authors have been unable to prove that EN spaces have well-ordered (F), or even chain (F). However we can add to Williams and Zhou's result for scattered locally compact spaces. The theorem is in contrast to the situation for compact well-ordered (F) spaces.

Theorem 15. *Let X be a scattered locally compact space. Then the following are equivalent:*

(1) *X is hereditarily paracompact.*
(2) *X is EN.*
(3) *X has well-ordered (F).*
(4) *X is elastic.*

Recall that (4) $\Rightarrow$ (3) $\Rightarrow$ (1) and hence it suffices to prove that hereditary paracompact, scattered, locally compact spaces are elastic. The proof of Williams and Zhou's result depends

[1]The authors wish to thank H. Junnila for bringing this result to their attention.

on the fact that the one-point compactification of a locally compact EN space is EN. An easy modification of Williams and Zhou's proof yields that the proof of Theorem 15 will be complete once we have established the following result.

Theorem 16. *The one-point compactification of a locally compact elastic space is elastic.*

Proof: Suppose that X is a locally compact elastic space and let $X^+ = X \cup \{\infty\}$ denote the one-point compactification of X. By the proof of Theorem 5.1.27 of [7] we see that $X = \bigcup_{\lambda \in I} X_\lambda$ where

(1) each X_λ is closed and open in X,
(2) $X_\lambda \cap X_\mu = \emptyset$ whenever $\lambda, \mu \in I$ and $\lambda \neq \mu$, and
(3) for each λ there are open sets $\{U_n^\lambda\}_{n=0}^\infty$ such that $X_\lambda = \bigcup_{n=0}^\infty U_n^\lambda$ and $\overline{U_n^\lambda}$ is compact for every n.

We can assume that $\overline{U_n^\lambda} \subseteq U_{n+1}^\lambda$ for every n. Let $(\mathbb{P}, \sim)$ be an elastic pair-base for X. Notice that we can assume $\mathbb{P} = \bigcup_{\lambda \in I} \mathbb{P}^\lambda$ where $P_2 \subseteq X_\lambda$ whenever $P \in \mathbb{P}^\lambda$. For each compact subset K of X pick $n(K) \in \mathbb{N}$ and $\mathcal{F}_K$, a finite subset of I, so that $K \subseteq \bigcup_{\lambda \in \mathcal{F}_K} U_{n(K)}^\lambda$. Define

$$P^K = \left(X^+ \setminus \bigcup_{\lambda \in \mathcal{F}_K} \overline{U_{n(K)}^\lambda},\ X^+ \setminus \bigcup_{\lambda \in \mathcal{F}_K} U_{n(K)}^\lambda \right)$$

and set $\mathbb{P}^+ = \mathbb{P} \cup \{P^K : K \subseteq X \text{ and } K \text{ compact}\}$. Observe that $\mathbb{P}^+$ is a pair-base for X^+. Now define $\sim^+$ on $\mathbb{P}^+$ so that $\sim^+ \restriction_{\mathbb{P} \times \mathbb{P}}$ is $\sim$ and

$P^K \sim^+ P^{K'}$	whenever K, K' are compact subsets of X,
$P^K \sim^+ P$	whenever $P \in \mathbb{P}$ and $K \subseteq X$ is compact.

Notice that $(\mathbb{P}^+, \sim^+)$ is an elastic pair-base for X^+ provided that $\mathbb{P}^\infty = \{P^K : K \subseteq X \text{ and } K \text{ compact}\}$ is cushioned (i.e., if $\mathbb{A} \subseteq \mathbb{P}^\infty$ then $\overline{\bigcup\{A_1 : A \in \mathbb{A}\}} \subseteq \bigcup\{A_2 : A \in \mathbb{A}\}$). This is clear from the following claim.

Claim. *For each $x \in X$ there is an open neighbourhood U_x of x such that for any compact $K \subseteq X$, if $U_x \cap P_1^K \neq \emptyset$ then $x \in P_2^K$.*

To see this fix $x \in X$ and let $\mu \in I$ be such that $x \in X_\mu$. Set $m = \min\{n : x \in U_n^\mu\}$ and let $U_x = U_m^\mu$. Suppose that $K \subseteq X$ is compact and that $U_x \cap P_1^K \neq \emptyset$. If $\mu \notin \mathcal{F}_K$ then $X_\mu \subseteq P_1^K$ and hence $x \in P_2^K$. So, consider the case when $\mu \in \mathcal{F}_K$. Now, $U_x \cap P_1^K \neq \emptyset$ implies that $U_m^\mu \cap (X_\mu \setminus \overline{U_{n(K)}^\mu}) \neq \emptyset$ and hence $m > n(K)$. Thus, by the definition of m, $x \notin U_{n(K)}^\mu$ and therefore $x \in X_\mu \setminus U_{n(K)}^\mu \subseteq P_2^K$ as required. □

We note the following related result which may be of interest.

Theorem 17. *The one-point compactification of a locally compact well-ordered (F) space has well-ordered (F).* □

6. Cardinal Invariants

Finally we explore the relationships between the most common cardinal invariants in elastic and well-ordered (F) spaces. For the definitions of the following cardinal invariants see [12]: let L denote Lindelöf degree, c cellularity, d density, nw network weight, w weight; and if f is any cardinal invariant then let hf be the hereditary version of f.

A combination of results in [8], [19] and [21] yields the following theorem.

Theorem 18. *If X is a monotonically normal space then*

(1) $L(X) \leqslant (c(X) = hc(X) = hL(X)) \leqslant (d(X) = hd(X)) \leqslant nw(X) \leqslant w(X)$,
(2) $d(X) \leqslant c(X)^+$, *and*
(3) *the Souslin Hypothesis is equivalent to: every CCC monotonically normal space is separable.* □

As well-ordered (F) spaces are monotonically normal, their cardinal invariants satisfy the above. Here we show that if X is a well-ordered (F) space then in addition $d(X) = nw(X)$.

The basic tactic is to manipulate the $\mathcal{W}(x)$'s so that we obtain a suitable bound on $|\mathcal{W}(x)|$ and then apply the following lemma. (This lemma is, of course, the property of condition (F) which led to its investigation by Collins and Roscoe [6, Lemma 3].)

Lemma 19. *If X has $\mathcal{W}$ satisfying (F) then* $nw(X) \leqslant d(X).\sup_{x \in X} |\mathcal{W}(x)|$.

Proof: Take D to be a dense subset of X of cardinality $d(X)$. Define $\mathcal{B} = \{W \in \mathcal{W}(x) : x \in D\}$. Clearly $|\mathcal{B}| \leqslant d(X).\sup_{x \in X} |\mathcal{W}(x)|$ and condition (F) tells us that $\mathcal{B}$ is a network for X. □

Write $\overline{\mathcal{W}}(x)$ for $\{\overline{W} : W \in \mathcal{W}(x)\}$ and $\overline{\mathcal{W}}$ for $(\overline{\mathcal{W}}(x) : x \in X)$.

Lemma 20. *If X has $\mathcal{W}$ satisfying well-ordered (F), then $\overline{\mathcal{W}}$ also satisfies well-ordered (F).*

Proof: Certainly each $\overline{\mathcal{W}}(x)$ is well-ordered by reverse inclusion. Recall that well-ordered (F) spaces are monotonically normal and hence regular. Finally, it is clear that if X is a regular space with $\mathcal{W}$ satisfying (F), then $\overline{\mathcal{W}}$ also satisfies (F). □

We note that the following theorem applies to elastic spaces.

Theorem 21. *If X is a well-ordered (F) space then*

(1) $L(X) \leqslant (c(X) = hc(X) = hL(X)) \leqslant (d(X) = hd(X) = nw(X)) \leqslant w(X)$,

(2) $d(X) \leqslant c(X)^+$.

Proof: By Theorem 18, it suffices to show that $nw(X) \leqslant d(X)$. Let $\mathcal{W} = (\mathcal{W}(x) : x \in X)$ satisfy well-ordered (F) and fix $x \in X$. As $\overline{\mathcal{W}}(x)$ is well-ordered by reverse inclusion, $\{X \setminus \overline{W} : \overline{W} \in \overline{\mathcal{W}}(x)\}$ is a collection of open sets well-ordered by containment. But this forces:

$$|\overline{\mathcal{W}}(x)| = |\{X \setminus \overline{W} : \overline{W} \in \overline{\mathcal{W}}(x)\}| \leqslant hL(X) \leqslant d(X).$$

Thus, by Lemma 19, $nw(X) \leqslant d(X)$ as required. □

The following corollary was proved for elastic spaces by Borges [4].

Corollary 22. *Separable well-ordered (F) spaces are stratifiable.*

Proof: A separable well-ordered (F) space has, by the preceeding theorem, a countable network and is therefore a σ-space. Recall that monotonically normal σ-spaces are stratifiable. □

On the other hand countable cellularity leads us into set theory.

Corollary 23. *The Souslin Hypothesis is equivalent to: every CCC well-ordered (F) space is stratifiable, and to: every CCC elastic space is stratifiable.*

Proof: Recall that the Souslin Hypothesis is equivalent to: every CCC monotonically normal space is separable. So assuming the Souslin Hypothesis every CCC well-ordered (F) space is separable; hence by the preceeding Corollary, stratifiable. For the converse, assume there is a Souslin tree. The branch space of this tree is a CCC non-separable non-archimedean space (and in particular, not stratifiable). However non-archimedean spaces are proto-metrisable and therefore elastic [10]. □

Of course the above result demonstrates that we cannot prove in ZFC that $c(X) = d(X)$ for well-ordered (F) spaces. We finish with two examples showing that no further relationships exist in Theorem 21.

Example 24. *There is an elastic (hence well-ordered (F)) space X such that $L(X) < c(X)$.*

Let X be the set $\omega_1 + 1$ whose topology is the order topology on $\omega_1 + 1$ refined so that all points except ω_1 are isolated. X is non-archimedean and hence elastic. Finally observe that $L(X) = \omega$ while $c(X) = \omega_1$. □

Example 25. *M^c Auley's Bow-tie space B [14] is an elastic (hence well-ordered (F)) space such that* $d(B) = \omega < 2^\omega = w(B)$. □

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