Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol 17, 1992

A NOTE ON COUNTABLE PRODUCTS OF LOCALLY k_ω-SPACES

CHEN HUAIPENG

1. Introduction.

Let $\mathcal{C}$ be a compact covering of a space X. Then X is said to have the weak topology with respect to $\mathcal{C}$, if $U \subset X$ is open(closed) in X whenever $U \cap C$ is open(closed) in C for each $C \in \mathcal{C}$. Following A. V. Arhangelskii [1] such a cover is called a k-system. Recall that a space X is a k-space if it has the weak topology with respect to cover consisting of all compact subset of X. Then a space having a k-system is precisely a k-space. If $\mathcal{C}$ is point(star)-countable, X is said to have a point(star)-countable k-system. If $\mathcal{C}$ is countable, then X is said to be a k_ω-space.

Lemma A. *Let X be a paracompact, locally k_ω-space. Then X is the topological sum of k_ω-spaces.*

Proof. It is a corollary of [4; Theorem 1].

Lemma B. *Let $\{C_n;\ n \in \omega\}$ be a countable k-system for a space X. Then each compact subset of X is contained in a finite union of C_n's.*

Proof. This lemma is a corollary of [4; Lemma 6], but we shall give a direct proof for the reader. Suppose that some compact subset K of X is not contained in any finite union of C_n's. Then there is a sequence $S = \{x_n;\ n \in \omega\}$ such that $x_n \in K - \cup\{C_m;\ m \leq n\}$. Then S is infinite, but each $S \cap C_n$ is at most finite, hence S is closed, discrete in X. Since $S \subset K$, S has an accumulation point in X. This is a contradiction.

We assume that all spaces are regular.

The author wishes to thank Y. Tanaka for his encouragement. The referee gives some valuable suggestions, so the author expresses heartfelt thanks.

2. Lemmas

For $x \in X$, let $(A_n) \downarrow x$ mean a decreasing sequence $\{A_n;\ n \in \omega\}$ such that $x \in \overline{A_n - \{x\}}$ for each $n \in \omega$. Recall that a space is called a strongly k'-space iff for each k-sequence $(A_n) \downarrow x$, there is a compact subset $C \subset X$ such that $x \in \overline{A_n \cap C}$ for each $n \in \omega$. Where a k-sequence $\{A_n;\ n \in \omega\}$ due to E. Michael [3; Definition 1.2] is a decreasing sequence such that $C = \cap_{n\in\omega} A_n$ is compact and each neighborhood of C contains some A_n. We denote the tightness of a topological space X by $t(X)$.

From Lemma A and [4; Proposition 7], we have

Lemma 1. *Let X be a paracompact, locally k_ω-space. Then X is locally compact if and only if X is a strongly k'-space.*

Lemma 2. *Let $\mathcal{C}_1$ be a star-countable k-system of X, and let Z have a k-system $\mathcal{C}_2$ which consists of all compact subsets of Z . If $X \times Z$ is a k-space, then $\mathcal{C}_1 \times \mathcal{C}_2 = \{C_1 \times C_2;\ C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2\}$ is a k-system of $X \times Z$.*

Proof. Take a compact subset $C \subset X \times Z$, then $C \subset \prod_1(C) \times \prod_2(C)$. where $\prod_1, \prod_2$ is the projection from $X \times Z$ onto X, Z respectively. X has a star-countable k-system, then X is the topological sum of k_ω-spaces $X_\alpha (\alpha \in \wedge)$ by [4; Theorem 1]. $\prod_1(C)$ is compact, then $\prod_1(C)$ meets only finite many X_α's. So $\prod_1(C)$ is a compact subset of a k_ω-subspace. $\prod_1(C) \subset \cup_{i\leq n} K_i$, $K_i \in \mathcal{C}_1$ by Lemma B, then $C \subset (\cup_{i\leq n} K_i) \times \prod_2(C)$ and $K_i \times \prod_2(C) \in \mathcal{C}_1 \times \mathcal{C}_2$, $i = 1, 2, \ldots, n$. Hence $\mathcal{C}_1 \times \mathcal{C}_2$ is a k-system of $X \times Z$.

Lemma 3. *Let X be a paracompact, locally k_ω-space. If X is not locally compact, then there are an $x_o \in X$ and a net sequence $\{S_n;\ n \in \omega\}$ satisfying the following condition (C).*

(C): 1) S_n converges to x_o, $x_o \notin S_n$, $n < \omega$.

2) Each compact subset C of X meets only finitely many S_n's.

Proof. By Lemma A, X is the topological sum of k_ω-spaces $X_\alpha(\alpha \in \Lambda)$. We can assume that each X_α has a countable increasing k-system $\mathcal{C}_\alpha$. X is not a locally compact space, then X is not a strongly k'-space by Lemma 1. Then there is a k-sequence (A_n) in X_α and an $x_o \in X$ such that $(A_n) \downarrow x$ but for each compact K of X_α, there is an $n < \omega$ with $x_o \notin \overline{A_n \cap K}$. Let $\mathcal{C} = \cup\{\mathcal{C}_\alpha; \mathcal{C}_\alpha$ is a countable k-system of $X_\alpha\}$. Then for each $C_n \in \mathcal{C}_\alpha$ there is an A_m with $x_o \notin \overline{A_m \cap C_n}$. Since $x_o \in \overline{A_1 - \{x_o\}}$, there is a net $S_o \subset A_1 - \{x_o\}$ such that S_o converges to x_o. Since S_o is not closed in X_α, there is a $C_{n_1} \in \mathcal{C}_\alpha$ such that $S_o \cap C_{n_1}$ is not closed in C_{n_1}. Then there is an A_{m_1} such that $x_o \notin \overline{C_{n_1} \cap A_{m_1}}$. Since $x_o \notin \overline{A_{m_1} - \{x_o\}}$, there is a net $S_1 \subset A_{m_1} - \{x_o\}$ such that S_1 converges to x_o, $S_1 \cap C_{n_1} = \emptyset$. Since S_1 is not closed in X, there is a C_{n_2} such that $S_1 \cap C_{n_2}$ is not close in C_{n_2}. On the other hand, $S_1 \cap C_{n_1} = \emptyset$ therefore $n_1 < n_2$. For C_{n_2}, there is an $m_2 < \omega$ with $x_o \notin \overline{A_{m_2} \cap C_{n_2}}$. By induction, there is a sequence $\{S_n;\ n < \omega\}$ such that :

1) S_n converges to x_o, $x_o \notin S_n$, $n < \omega$.
2) Each compact subset C of X meets only finitely many S_n's.

In fact. If $C \in \mathcal{C} - \mathcal{C}_\alpha$, then $C \cap S_n = \emptyset$, for $n < \omega$. If $C \in \mathcal{C}_\alpha$, $C = C_{n_j}$, then $C \cap S_i = \emptyset$, for $i \geq j$. Namely, each $C \in \mathcal{C}$ meets only finitely many S_n's. By Lemma B and Lemma 2, for each compact subset K of X, there are $C_1, C_2, \ldots, C_n$ in $\mathcal{C}$ such that $K \subset \cup_{i \leq n} C_i$. But the $C_i (i \leq n)$ meet only finitely many S_n's, then so does K.

For a space X we denote $\sup\{|D|;\ D$ is a discrete closed set of $X\}$ by $D(X)$.

Lemma 4. *Suppose X is a paracompact, locally k_ω-space and*

X is not compact. Then X contains a discrete closed subset D such that $|D| = D(X)$.

Proof. By Lemma A, X is the topological sum of k_ω-spaces $X_\alpha(\alpha \in \Lambda)$. But X is not compact, so we can assume that any X_α is not compact. Then, for each $\alpha \in \Lambda$, there exists $D_\alpha \subset X_\alpha$ with $|D_\alpha| = D(X_\alpha) = \aleph_0$. Let $D = \cup_{\alpha\in\Lambda} D_\alpha$, then $|D| = D(X)$.

Remark. If X is a compact space, Lemma 4 will fail. We assume that any paracompact, locally k_ω-space is not compact in all lemmas below.

Lemma 5. *Suppose X is a paracompact, locally k_ω-space. Then X^ω contains a closed metric subspace T such that $|T| = D(X)$, and T has only one accumulation point.*

Proof. Since X is a paracompact, locally k_ω-space, by Lemma 4 there is a discrete closed subset D in X such that $|D| = D(X)$. Take an $x' \in D^\omega$, let $x' = (x_1, x_2 \ldots .)$, $B_n = \{y = (x_1, \ldots, y_n, x_{n+1}, \ldots); \; y_n \in D - \{x_n\}\}$, and let $T = \{x'\} \cup (\cup_{n<\omega} B_n)$, then T is a closed subset of D^ω and x' is the only accumulating point of T and $|T| = D(X)$.

Let X be a paracompact, locally k_ω-space, and let $\{S_n; \; n < \omega\}$ be a net sequence satisfying (C) in Lemma 3. Then we say that the net sequence $\{S_n; \; n < \omega\}$ satisfies (C′) if $|S_n| \leq D(X)$ for each $n < \omega$.

Lemma 6. *Let X be a paracompact, locally k_ω-space. If X^ω is a K-space, then X contains no net sequences satisfying (C′).*

Proof. By Lemma 5, X^ω contains a metric closed subset $T = (\cup_{n<\omega} B_n) \cup \{x'\}$, $|B_n| = D(X)$ such that T has only one accumulating point x'. Since $X \times T$ is a closed subset of $X \times X^\omega$ and $X \times X^\omega$ is homeomorphic to k-space X^ω, $X \times T$ is a k-space. Suppose X contains a net sequences $\{S_n; \; n < \omega\}$ satisfying condition (C′). For each $n < \omega$, $|S_n| \leq D(X) = |B_n|$. Let $f_n : S_n \to B_n$ be an injective, and let $A = \{(x, f_n(x)); \; n < \omega$ and $x \in S_n\}$. 1) A is not closed in $X \times T$. In fact $(x_o, x') \notin A$.

On the other hand, for each $U \times V$, where U is a neighborhood of x_o, in X and V is a neighborhood of x' in T, $U \times V \supset U \times [\{x'\} \cup (\cup_{i \geq n} B_i)]$. Take $m > n$, then $S_m \cap U \neq \emptyset$. Take an $x \in S_m \cap U$, then $(x, f_m(x)) \in A \cap U \times [\{x'\} \cup (\cup_{i \geq n} B_i)]$, that is $(x_o, x') \in \overline{A}$. 2) A is k-closed in $X \times T$; X has a star-countable k-system by [4; Theorem 1]. In view of Lemma 2 it suffices to show that for each $C \in \mathcal{C}$ and each convergent sequence S together with the limit point x', $(C \times S) \cap A$ is closed in $C \times S$. In fact, by condition (C), we can prove that for some $m < \omega$, $A \cap (C \times S) = \{(x, f_i(x));\ i \leq m,\ x \in S_i\} \cap (C \times S)$. Since B_n is a discrete closed subset of T for each $n < \omega$, $S \cap B_n$ is finite for each B_n. Thus $A \cap (C \times S)$ is finite.

Lemma 7. *Let X be a paracompact, locally k_ω-space, and let X^ω be a k-space. For each $x_o \in X$, if a net S converges to x_o and $|S| \leq D(X)$, then there is a compact set $C \subset X$ with $x_o \in C$ and $x_o \notin \overline{S - C}$.*

Proof. Let net S converge to x_o and $|S| \leq D(X)$. Take a clopen X_α in X and a countable increasing k-system $\mathcal{C}_\alpha$ of X_α. We may assume $S \subset X_\alpha$ and $S \not\ni x_o$. Suppose $x_o \in \overline{S - C_n}$ for each $C_n \in \mathcal{C}_\alpha$. Since S is not closed, there is a $C_{n_1} \in \mathcal{C}_\alpha$ such that $C_{n_1} \cap S$ is not closed in C_{n_1}. Let $S_1 = S - C_{n_1}$, then $x_o \in \overline{S - C_{n_1}} = \overline{S_1}$ and $x_o \notin S_1$. Hence there is a $C_{n_2} \in \mathcal{C}_\alpha (n_2 > n_1)$ such that $C_{n_2} \cap S_1$ is not closed in C_{n_2}. Let $S_2 = S - C_{n_2} = S_1 - C_{n_2}$, $x_o \in \overline{S_2}$ and $x_o \notin S_2$. By induction, there is a net sequence $\{S_n;\ n < \omega\}$ satisfying condition (C′). But X is a paracompact, locally k_ω-space and X^ω is a k-space, this is a contradiction to Lemma 6.

Lemma 8. *Let X be a paracompact, locally k_ω-space, and let X^ω be a k-space. Then for each $x_o \in X$, there is a compact subset $C \subset X$ with $x_o \in C$ such that $X - C$ contains no net S which converges to x_o and $|S| \leq D(X)$.*

Proof. Suppose there is an $x_o \in X$ such that for each compact $C \subset X$ there is a net $S \subset X - C$ with S converging to x_o and

$|S| \leq D(X)$. Take a clopen subset X_α of X and a countable increasing k-system $\mathcal{C}_\alpha$ of X_α such that $S \subset X_\alpha$. Take a compact subset $C \subset X$, then there is a net $S_1 \subset X - C$ which converges to x_o and $|S_1| \leq D(X)$. By Lemma 7 and Lemma B there is a $C_{n_1} \in \mathcal{C}_\alpha$ with $x_o \notin \overline{S_1 - C_{n_1}}$. There is a net $S_2 \subset X_\alpha - C_{n_1}$ which converges to x_o and $|S_2| \leq D(X)$. Again Lemma 7 there is a $C_{n_2} \in \mathcal{C}_\alpha (n_2 > n_1)$ such that $x_o \notin \overline{S_2 - C_{n_2}}$. By induction, we can obtain a net sequence $\{S_n;\ n < \omega\}$ satisfying condition (C′). This is a contradiction to Lemma 6.

3. Results.

Theorem 1. *Let X be a paracompact, locally k_ω-space, and let X^ω be a k-space. Then for each $x_o \in X$, x_o has a compact neighborhood otherwise there is a compact subset $C \subset X$ such that $x_o \in \overline{X - C}$, and for any net S in $X - C$ converging to x_o, $|S| > D(X)$.*

Proof. Take an $x \in X$. If there is a compact subset $C \subset X$ with $x \in C$ and $x \notin \overline{X - C}$, then C is a compact neighborhood of x. Let $x \in \overline{X - C}$ for each compact subset $C \subset X$ and $x \in C$. But by Lemma 8 there is a compact subset $C_o \subset X$ with $x \in C_o$ such that $X - C_o$ contains no net S which converges to x and $|S| \leq D(X)$. If net $S \subset X - C_o$ converges to x, then $|S| > D(X)$.

Corollary 1. *Let X be a paracompact, locally k_ω-space, and let $t(X) \leq D(X)$. Then X^ω is a k-space if and only if X is locally compact.*

Proof. "If". Since X is locally compact, X^ω is a k-space by [1; Theorem 3.7 and 3.9].

"Only if". Since X is a paracompact, locally k_ω-space and X^ω is a k-space, if $t(X) \leq D(X)$, X is locally compact by Theorem 1.

Remark The author [2] proved that under (CH), there is a k_ω-space X which is not locally compact (even a k'-space), but

X^ω has a k-system. In fact, let Z be the topological sum of countably many copies of $[0, \omega_2]$ with the order topology. Let X be the quotient space obtained from Z by identifying all the ω_2's. Then X^ω is a k-space. Let $Y = \sum_{\alpha \in \wedge} X_\alpha$. Here X_α is a copy X for each $\alpha \in \wedge$ and $| \wedge | = \aleph_2$. We notice that for each $y \in Y = \sum_{\alpha \in \wedge} X_\alpha, y$ has a clopen neighborhood X_α which is a copy of X. Hence both X and Y have the same local structure. But Y is not locally compact and $t(Y) = D(Y) = \aleph_2$. Then Y^ω is not a k-space by Corollary 1. Thus we see that Y^ω is not a k-space even if X and Y have the same local structure and X^ω is a k-space.

Corollary 2. *Let X be the topological sum of k_ω-spaces X_α, $\alpha \in \wedge$. Suppose that $| \wedge | \geq sup\{t(X_\alpha), \alpha \in \wedge\}$. Then X^ω is a k-space if and only if X is locally compact.*

Theorem 2. *Let X be a locally k_ω-space. Then X is locally compact if and only if $X \times Z$ is a k-space for every metric space Z.*

Proof. The "Only if" part follows from the following well known result due to D. E. Cohen: Every product of a locally compact space and a k-space is a k-space

"If" Suppose X is not a locally compact space. We look for a metric space Z such that $X \times Z$ is not a k-space. Take a net sequence $\{S_n;\ n < \omega\}$ satisfying condition (C) in Lemma 3. It is possible, since X is a locally k_ω-space and X is not a locally compact space. Take an $x_o \in X$ such that x_o has not any compact neighborhood. let F be the closed k_ω-neighborhood of x_o, and let $\mathcal{C}$ be the countable increasing k-system. We may assume $S_n \subset F$ and S_n converges to x_o for each S_n. Let $Z = [\cup_{i<\omega}(S_i \times \{i\})] \cup \{x_o, 0)\}$ and topologize Z as follows: let $\{(x, i)\}$ be open for each $(x, i) \in \cup_{i<\omega}(S_i \times \{i\})$, and let $\{V_n;\ n < \omega\}$ be countable local base at $(x_o, 0)$ here $V_n = \cup_{i \geq n}(S_i \times \{i\}) \cup \{x_o, 0)\}$. Then Z is a metric space which is not locally compact. We show that $X \times Z$ is not a k-space. Let $A = \{(x,\ (x, n));\ x \in S_n,\ n < \omega\}$, then $(x_o,\ (x_o, 0)) \notin A$.

Take a neighborhood U of x_o in X, then $(U \times V_n) \cap A \neq \emptyset$. Thus $(x_o, (x_o, 0)) \in \overline{A}$.

Let $\mathcal{C}'$ be the collection of all compact subsets of Z.

Suppose $X \times Z$ is a k-space, then $F \times Z$ is a closed k-subspace of $X \times Z$ and $\mathcal{C} \times \mathcal{C}'$ is a k-system of $F \times Z$ by Lemma 2. On the other hand, for each $C_n \times K \in \mathcal{C} \times \mathcal{C}'$. Pick $y \in (C_n \times K) - [(C_n \times K) \cap A]$. 1) $y = (x, (x', n)), x \neq x'$, let U_1 and U_2 be disjoint neighborhood of x and x', then $U_1 \times \{(x', n)\}$ is a neighborhood of $(x, (x', n))$ in $F \times Z$. Then $(C_n \times K) \cap A \cap [U_1 \times \{(x, n)\}] = \emptyset$. 2) $y = (x', (x_o, 0))$. Take n_o such that $n_o > n$ and $S_m \cap C_n = \emptyset$ for each $m > n_o$. Let U_1 be a neighborhood of x'. If there is a $t \in (U_1 \times V_{n_o}) \cap (C_n \times K) \cap A$, then $t = (x, (x, i))$, $x \in S_i$, $t \in U_1 \times V_{n_o}$. Then $i \geq n_o$, $S_i \cap C_n = \emptyset$, then $x \notin C_n$. Thus $(x, (x, i)) \notin C_n \times K$, it is a contradiction. A is k-closed in $F \times Z$ by 1), 2). This is a contradiction.

REFERENCES

1. A.V. Arhangelskii, *Factor mappings of metric spaces*, Soviet. Math. Dokl., **5** (1965) 368-371.
2. H. Chen, *An answer to a conjecture on the countable products of k-space*, to appear in Proc. Amer. Math. Soc.
3. E. Michael, *A quintuple quotient quest*, General Topology and Appl., **2** (1972) 91-138.
4. Y. Tanaka, *Point-countable k-systems and products of k-spaces*, Pacific Jour. Math. **101** (1982) 199-208.

Quinhai Normal University
Xining China