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DECOMPOSITIONS OF DOMAINS OF PROPER LOCAL HOMEOMORPHISMS

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ABSTRACT. Let $p : X \to Y$ be a proper surjective local homeomorphism from the Hausdorff space X to the connected first countable Hausdorff space Y. Then X is a finite disjoint union $X = X_1 \cup \ldots \cup X_n$ with each X_j connected and with $p : X_j \to X$ a surjective, proper, local homeomorphism.

1. INTRODUCTION

Recall that a continuous function $f : X \to Y$ is a proper map if and only if for every compact subset $K \subseteq Y$, it follows that $f^{-1}(K)$ is compact in X. A map $f : X \to Y$ is a local homeomorphism if and only if for each $y \in Y$ and each $x \in f^{-1}(y)$ there are open sets U and V about x and y, respectively, such that $f : U \to V$ is a homeomorphism. One of the questions investigated by the Jungck[J] is the question of when a proper, local homeomorphism is a covering projection. In the process of investigating this question he proved Theorem 1.1 below. The elementary examples following the theorem show the necessity of both the hypothesis that f is proper and the hypothesis that it is a local homeomorphism.

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Theorem 1.1. [J, 2.1] *If $f : X \to Y$ is a surjective, proper, local homeomorphism from a first countable space X to a connected Hausdorff space Y then there is a $k \in \mathbb{N}$ such that $\#(f^{-1}(y)) = k$ for all $y \in Y$.*

Example 1.2. Consider $\mathbb{R}^2$ with the standard topology. Let X be the subspace defined by $X = \{(x, y) \in \mathbb{R}^2 : y = 1 + \frac{1}{n}, n \in \mathbb{N}\} \cup \{(x, y) \in \mathbb{R}^2 : y = 1\}$. Let Y be the x-axis in the subspace topology. Let $p : X \to Y$ be the projection $p(x, y) = (x, 0)$. Then p is a proper map, and is in fact a local homeomorphism except at points on the line $y = 1$, but does not have finitely many points in $p^{-1}(x, 0)$ for any $(x, 0) \in Y$.

Example 1.3. Consider $X = \mathbb{R}_d$. That is, the real line with the discrete topology. Let $Y = \{*\}$ be a one point space. The constant map $p : X \to Y$ given by $p(x) = *$ for all $x \in X$ is a local homeomorphism that is not proper. Again note that $p^{-1}(*)$ is not finite.

Note that the above theorem is a decomposition theorem for point inverses under proper local homeomorphisms and it says that point inverses under proper maps are disjoint unions of a finite number of (connected) components. It is clear that if, in addition to the hypothesis of the theorem, X is also locally connected or X is compact then X can be written as a finite disjoint union $X = X_1 \cup \ldots \cup X_n$ of connected components where $f : X_j \to Y$ is a proper, surjective, local homeomorphism. One wonders if a decomposition theorem can be obtained without either of these strong assumptions. As the abstract indicates it is possible. Our earliest version of Theorem 1.4 required that there be a component X_0 of X such that $p(X_0) = Y$. However, at the University of North Carolina-Charlotte during the 1992 Spring Topology Conference, Jo Heath pointed out that Theorem 1.4 followed as a corollary from this apparently "weaker" result. A different, more unified proof of Theorem 1.4 is given in section 2.

Theorem 1.4. *Let Y be Hausdorff, first countable, and connected. Let X be Hausdorff. If $p : X \to Y$ is a proper, surjective, local homeomorphism then $X = X_0 \cup \ldots \cup X_n$ for some $n \geq 0$, with each X_j a component of X. Furthermore $(p|X_j) : X_j \to Y$ is a proper, surjective, local homeomorphism.*

Note that the result [J, 3.2] is an immediate corollary of Theorem 1.4. The proof given in [J] is dependent on the fact that the range of the map is "H-connected." However by substituting "connectedness" for "H-connectedness," we are able to obtain the more general result in Theorem 1.4.

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2. The proof of Theorem 1.4.

Lemma 2.1. *Let X and Y be Hausdorff and first countable. Assume that Y is connected. If $p : X \to Y$ is a proper, surjective local homeomorphism, then every component of X is open in X.*

Proof: By Theorem 1.1, since $p : X \to Y$ is a proper, surjective, local homeomorphism there is an $n \in \mathbb{N}$, such that $\#(p^{-1}(y)) = n$ for all $y \in Y$. The proof is by induction on n.

Assume that $\#(p^{-1}(y)) = 1$ for all $y \in Y$. If X is connected then it is the only component and the result follows. So assume that X is not connected. Then there is a non-empty subset $E \neq X$ that is both open and closed in X. But p is a proper, local homeomorphism. Therefore, $p(E)$ is both open and closed in Y. Since Y is connected and $E \neq \emptyset$, $p(E) = Y$. Since p is one-to-one, $E = X$. Contradiction. Thus, the assumption that X is not connected is impossible.

Now assume that the result holds whenever $1 \leq \#(p^{-1}(y)) = k < n$ for all $y \in Y$. Assume that $\#(p^{-1}(y)) = n$ for all $y \in Y$. Once again, if X is connected, the result is immediate. So, assume, that X is not connected. Then there is a pair E, F of non-empty disjoint open subsets such that $X = E \cup F$. In fact,

E and F are both open and closed. Thus, as in the first step in the induction since Y is connected, $p(E) = Y$ and $p(F) = Y$. Since, E is open and p is a local homeomorphism, the restriction $p|E$ is a local homeomorphism. Since E is closed, $p|E$ is proper. Therefore, $(p|E) : E \to Y$ is a proper surjective local homeomorphism. So by Theorem 1.1, there is a $k_E \in \mathbb{N}$ such that $\#((p|E)^{-1}(y)) = k_E$. Since, $E \neq X, k_E < n$. Therefore, by the induction hypothesis, every component of E is open E. In a similar manner, every component of F is open in F. But every component of E and every component of F is a component of X and vice versa. So the result follows. □

Proof of Theorem 1.4. Let X_0 be a component of X. Then, by Lemma 2.1, X_0 is open. Since X_0 is a component it is closed. Therefore, since Y is connected, $p(X_0) = Y$. Since $\#(p^{-1}(y)) = n$ for some $n \in \mathbb{N}$ and all $y \in N$, and the restriction of p to any component of X must be onto, it follows that there are only finitely many components of X, each of which is open and closed in X. Clearly, if X_j is one of these finitely many components, then the restriction, $p : X_j \to Y$, is a surjective, proper, local homeomorphism. □

Example 2.2 Elementary examples exist that show the necessity of the space Y being connected. Consider $\mathbb{R}^2$ with the usual topology. Let $Y = \{(0,0)\} \cup (\cup_{n=1}\{(x, \frac{1}{n}) \in \mathbb{R}^2 : x \in (\frac{1}{n+2}, \frac{1}{n})\}$ in the subspace topology. Let $X = Y$ and $p : X \to Y$ be the identity map. It is clear that p is a proper surjective local homeomorphism. However, the point $\{(0,0)]$ is a component of X but is not open in X. Note that the conclusions of neither Lemma 2.1 or Theorem 1.4 apply in this situation.

Question 2.3. In all known examples, the conclusion of Lemma 2.1 holds even if the assumption that p be proper is removed. Is it possible to prove Lemma 2.1 without this hypothesis?

References

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