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PRESERVATION PROPERTIES OF TRI-QUOTIENT MAPS WITH SIEVE-COMPLETE FIBRES

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Abstract: We show that regular images of regular monotonic p-spaces under tri-quotient maps with sieve-complete fibres are of pointwise countable type, and are first countable if the domain has also a base of countable order. It is also shown that such maps are always countable-compact-covering. None of the above remains true if one replaces "tri-quotient" by "bi-quotient."

0. Introduction

The concept of a tri-quotient map was introduced in [M 77]. It is a generalization of both the concepts of an open map and a perfect map. One would therefore expect that topological properties preserved by both open maps and perfect maps are also preserved by tri-quotient maps. In this paper, we establish some results of this kind.

Our focus is on tri-quotient maps with sieve-complete fibres (not necessarily for a fixed sieve on the domain) which have regular monotonic p-spaces or spaces with a base of countable order as their domains. These spaces behave well with respect to perfect maps and open maps in the presence of uniform completeness of the fibres (see [WW 67] and [WW 73]. A summary of results can be found in [ČN]).

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The class of monotonic p-spaces is rather extensive; in particular it includes the classes of sieve-complete spaces and spaces having a base of countable order as well as the class of p-spaces introduced by Arhangel'skiĭ in [A 65], which in turn includes all locally compact spaces and all Čech-complete spaces.

Our Theorems 3.1 and 3.2 state that under certain conditions a tri-quotient map is countable-compact-covering (3.2) or even inductively perfect (3.1). Countable-compact-covering maps are of interest in the present context, because they are often tri-quotient (see [M 81, 1.2(c)] for a sufficient condition). Theorem 3.1 generalizes Theorem 1.4 of [M 81], which concerns only functions whose domain is a metric space.

The paper is organized as follows: In section 1, we establish terminology and some basic facts used throughout our arguments. Section 2 is devoted to a technical lemma, and section 3 to our major results. In section 4 we discuss some examples that show limitations of possible generalizations of our results.

1. Terminology and basic facts

Our terminology is mostly standard. All spaces considered in this paper are regular. It is understood that regularity implies T_0. All maps are continuous. We shall write $f^{-1}y$ instead of $f^{-1}\{y\}$. A space X is of *pointwise countable type*, if for every point $x \in X$ there exists a compact $C \subseteq X$ such that $x \in C$ and $\chi(C, X) = \aleph_0$.

A family $\mathcal{U}$ of open subsets of a space X is an *outer base* for a subspace Y of X if for every open $V \subseteq X$ such that $Y \subseteq V$ there exists a $U \in \mathcal{U}$ such that $Y \subseteq U \subseteq V$. A sequence $(U_n)_{n\in\omega}$ of subsets of X is *closurewise decreasing* if $cl(U_{n+1}) \subseteq U_n$ for every $n \in \omega$.

We shall frequently make use of König's Lemma which states that every tree of height ω with finite levels has an infinite branch.

Suppose $f : X \to Y$ is a fixed map from a space X onto a space Y, and $E \subseteq Y$. A subspace C of X is called a *cover* *of* E if the image $f[C] = E$. If every (countable) compact subset E of Y has a compact cover, then we say that the map f is *(countable-)compact-covering*. A surjection $f : X \to Y$ is *inductively perfect* if there is a cover $X' \subseteq X$ of Y such that the restriction $f|X'$ is perfect.

1.0 Definition: (a) A surjective map $f : X \to Y$ is *tri-quotient* if one can assign to each open U in X an open U^* in Y such that:

(i) $U^* \subseteq f[U]$,
(ii) $X^* = Y$,
(iii) $U \subseteq V$ implies $U^* \subseteq V^*$,
(iv) If $y \in U^*$ and $\mathcal{W}$ is a cover of $f^{-1}y \cap U$ by open subsets of X, then there is a finite $\mathcal{F} \subseteq \mathcal{W}$ such that $y \in (\bigcup \mathcal{F})^*$.

We call $U \mapsto U^*$ a *t-assignment* for f.

(b) A surjective map $f : X \to Y$ is *bi-quotient*, if for every $y \in Y$ the following condition holds:

$(!)_y$ If $\mathcal{W}$ is an open cover of $f^{-1}y$, then there is a finite $\mathcal{F} \subseteq \mathcal{W}$ such that $y \in int(f[\bigcup \mathcal{F}])$.

1.1 Fact: (a) All open maps and all perfect maps are tri-quotient.
(b) Every tri-quotient map is bi-quotient.

Proof: For (a), see [M 77, Theorem 6.5]. (b) is obvious.

1.2 Definition: Let X be a space and $X' \subseteq X$. A sequence $\mathcal{S} = (\langle \mathcal{A}_n, D_n, \pi_n \rangle)_{n \in \omega}$ will be called a *sieve for X' in X* if for every n, $\mathcal{A}_n = \{A_d\}_{d \in D_n}$ is a cover of X' by open subsets of X, and $\pi_n : D_{n+1} \to D_n$ is such that:

1) $X' \cap A_d = \bigcup \{X' \cap A_{d'} : \pi_n(d') = d\}$ for every $d \in D_n$, and

2) $A_d \subseteq A_{\pi_n(d)}$ for every $d \in D_{n+1}$.

A sequence $(A_{d_n})_{n \in \omega}$, where $d_n \in D_n$ and $\pi_n(d_{n+1}) = d_n$ will be called a *thread of* $\mathcal{S}$. The sieve $\mathcal{S}$ is *complete for X'* if for every thread $(A_{d_n})_{n\in\omega}$, every filter base $\mathcal{F}$ on X' which meshes with $\{X' \cap A_{d_n} : n \in \omega\}$ clusters in X'. (Recall that a filter base $\mathcal{F}$ on X *clusters at x in X* if $x \in cl_X(F)$ for all $F \in \mathcal{F}$. Two collections of sets $\mathcal{F}$ and $\mathcal{A}$ *mesh* if every $F \in \mathcal{F}$ intersects every $A \in \mathcal{A}$.) X is *sieve-complete* if there is a complete sieve for X in X.

1.3 Remark: In our treatment of sieves, we allow that $A_d = A_{d'}$ for $d \neq d'$. However, in order to simplify notation we always assume that the sets D_n are pairwise disjoint.

1.4 Fact: (a) A metrizable space X is sieve-complete iff it is completely metrizable.

(b) If a closed subspace X' of X is sieve-complete, then there exists a complete sieve for X' in X.

Proof: (a) is an easy consequence of e.g., [M 77, Theorem 3.2].

(b) Let $\mathcal{S}' = (\langle \mathcal{A}'_n, D_n, \pi_n\rangle)_{n\in\omega}$ be a complete sieve for X' in X'. Let $U = X \backslash X'$, and define for $d \in D_n$ and $A'_d \in \mathcal{A}'_n$: $A_d = A'_d \cup U$. Let $\mathcal{A}_n = \{A_d : d \in D_n\}$. Then $\mathcal{S} = (\langle \mathcal{A}_n, D_n, \pi_n\rangle)_{n\in\omega}$ is a complete sieve for X' in X.

1.5 Definition: Let $(B_n)_{n\in\omega}$ be a sequence of subsets of a topological space X. We define the following properties:

(md) If $x \in \bigcap_{n\in\omega} B_n$, then $\{B_n : n \in \omega\}$ is a base at x.

(mp) If $\bigcap_{n\in\omega} B_n \neq \emptyset$, and $\mathcal{F}$ is a filter base on X which meshes with $\{B_n : n \in \omega\}$, then $\mathcal{F}$ clusters in X.

(mc) If $\mathcal{F}$ is a filter base which meshes with $\{B_n : n \in \omega\}$, then $\mathcal{F}$ clusters in X.

Let (m) denote any of the properties (md), (mp), (mc). A sequence $(\mathcal{B}_n)_{n\in\omega}$ of bases for a regular space X is called an *(m)-sequence* if every closurewise decreasing sequence $(B_n)_{n\in\omega}$, where $B_n \in \mathcal{B}_n$ for every $n \in \omega$, has (m).

A space X is said to be a *monotonic p-space* if it has an (mp)-sequence of bases, and is said to have a *base of countable order* if it has an (md)-sequence of bases.

1.6 Fact: (a) A paracompact T_2-space is metrizable iff it has a base of countable order.

(b) Every (md)-sequence of bases is an (mp)-sequence of bases. Hence every space with base of countable order is a monotonic p-space.

(c) A monotonic p-space with G_δ-diagonal has a base of countable order.

(d) The following are equivalent for a closurewise decreasing $(B_n)_{n\in\omega}$ of subsets of a space X :

  (i) $(B_n)_{n\in\omega}$ has (mp).

  (ii) For every centered family $\mathcal{A}$ such that for all $n \in \omega$ there exists an $A \in \mathcal{A}$ such that $A \subseteq B_n$, if $\bigcap_{n\in\omega} B_n \neq \emptyset$, then $\bigcap\{cl_X(A) : A \in \mathcal{A}\} \neq \emptyset$.

  (iii) $\bigcap_{n\in\omega} B_n = \emptyset$ or $\bigcap_{n\in\omega} B_n$ is compact and $\{B_n : n \in \omega\}$ is an outer base at $\bigcap_{n\in\omega} B_n$.

Thus a regular space X is a monotonic p-space iff there is a sequence $(\mathcal{B}_n)_{n\in\omega}$ of bases such that for each closurewise decreasing sequence $(B_n)_{n\in\omega}$ with $B_n \in \mathcal{B}_n$ for each $n \in \omega$, (ii) holds.

(e) Every compact space is a monotonic p-space.

Proof: (a) is a deep theorem of Arhangel'skiĭ. It was proved in [A 63] using his original elegant definition. (b) is obvious. (c) is proved in [W 72]. The equivalence of (d)(ii) and (d)(iii) is stated in 2.10 of [ČN]. The equivalence of (d)(i) and (d)(ii) is stated in [M 77, footnote 11]. (e) follows immediately from Definition 1.5.

1.7 Remark: There are natural equivalent definitions of "monotonic p-space" and "base of countable order" in terms of sieves, and of "sieve-completeness" as a space with an (mc)-sequence of bases (see footnote 4 in [M 77] and also [ČN]). The condition given in 1.6(d) is essentially the original definition of monotonic p-spaces in [W 71] where such spaces were called β_b-spaces. The original definition of spaces having a base of countable order was given in [A 63]; the equivalence of that definition to the formulation of Definition 1.5 was proved in [WoW]. The concept of sieve-completeness was isolated in [WW 71], where it is called condition $\mathcal{K}$. In [W 71], sieve-complete spaces are called λ_b-spaces. We use here the convenient terminology of [CČN], where many results of Wicke and Worrell are summarized and simplified.

2. A TECHNICAL LEMMA

The following lemma generalizes Lemma 2.2 of [M 81] which involves metric X.

2.0 Lemma. *Suppose $f : X \to Y$ is a tri-quotient map with t-assignment $U \mapsto U^*$ from a regular monotonic p-space X onto a regular space Y. Let U be open in X, and $y \in U^*$ be such that $f^{-1}y$ is a sieve-complete subspace of X. Let $(\mathcal{B}_n)_{n\in\omega}$ be an (mp)-sequence of bases for X. Then there exists a decreasing sequence $(U_n)_{n\in\omega}$ of open subsets of U such that for all n:*

(0) Each U_n is the union of finitely many elements of $\mathcal{B}_n$, each of which has nonempty intersection with $f^{-1}y$,

(i) $y \in U_n^*$,

(ii) $cl(U_{n+1}) \subseteq U_n \subseteq U$,

(iii) $cl(U_{n+1}^*) \subseteq U_n^*$,

(iv) *Suppose $\mathcal{G}$ is a filter base which meshes with $\{U_n : n \in \omega\}$. Then $\bigcap\{cl_X(G) : g \in \mathcal{G}\} \neq \emptyset$.*

(v) *$\bigcap_{n\in\omega} U_n$ is nonempty and compact with outer base $\{U_n : n \in \omega\}$. If, in addition, $\{y\}$ is a G_δ-subset of Y, then the $(U_n)_{n\in\omega}$ may be constructed so that $\bigcap_{n\in\omega} U_n \subseteq f^{-1}y$.*

(vi) *$\bigcap_{n\in\omega} U_n^*$ is nonempty and compact with outer base $\{U_n^* : n \in \omega\}$.*

Proof: By assumption and Fact 1.4(b), there exists a complete sieve
$\mathcal{S} = (\langle \mathcal{A}_n, D_n, \pi_n \rangle)_{n\in\omega}$ for $f^{-1}y$ in X. We fix such a sieve $\mathcal{S}$ throughout this proof. Fix open sets H_n as follows: If $\{y\}$ is a

G_δ–subset of Y, then we require that $\{y\} = \bigcap_{n\in\omega} H_n$. If not, then let $H_n = Y$ for every $n \in \omega$.

We are going to construct the sequence $(U_n)_{n\in\omega}$. Since $f^{-1}y \cap U$ is covered by $\mathcal{A}_0$ and $y \in U^*$, by Definition 1.0(iv) there is a finite $\mathcal{A}'_0 \subseteq \mathcal{A}_0$ such that $y \in (\bigcup \mathcal{A}'_0)^*$. Let V_0 denote $\bigcup \mathcal{A}'_0$. Then $f^{-1}y \cap V_0$ is covered by

$\mathcal{B}'_0 = \{B \in \mathcal{B}_0 : \ B \cap f^{-1}y \neq \emptyset \ \& \ \exists A \in \mathcal{A}'_0$ such that $cl(B) \subseteq A \cap U\}$.

Hence there exists a finite $\mathcal{F}_0 \subseteq \mathcal{B}'_0$ such that $y \in (\bigcup \mathcal{F}_0)^*$. Let $U_0 = \bigcup \mathcal{F}_0$.

For $n > 0$, suppose U_k, $\mathcal{A}'_k$ and $\mathcal{F}_k$ are defined for all $k < n$ and satisfy:

$(a)_k$ $\mathcal{A}'_k$ and $\mathcal{F}_k$ are finite, $\mathcal{A}'_k \subseteq \mathcal{A}_k$, $\mathcal{F}_k \subseteq \mathcal{B}_k$, and $\bigcup \mathcal{F}_k = U_k$,

$(b)_k$ If $B \in \mathcal{F}_k$, then $B \cap f^{-1}y \neq \emptyset$,

$(c)_k$ If $k > 0$ and $B \in \mathcal{F}_k$, then there exist $C_B \in \mathcal{F}_{k-1}$, $A \in \mathcal{A}'_k$ and an open $W_k \subseteq H_k$ such that $cl(B) \subseteq C_B \cap A \cap f^{-1}W_k$ and $y \in W_k \subseteq cl(W_k) \subseteq (\bigcup \mathcal{F}_{k-1})^*$.

$(d)_k$ $y \in (\bigcup \mathcal{F}_k)^*$ $(= U_k^*)$.

From $(c)_{n-1}$ we know that $f^{-1}y \cap U_{n-1}$ is covered by $\mathcal{A}'_{n-1}$; hence it is covered by $\mathcal{E} = \{A_d \in \mathcal{A}_n : \ A_{\pi_{n-1}(d)} \in \mathcal{A}'_{n-1}\}$. So there exists a finite $\mathcal{A}'_n \subseteq \mathcal{E}$ such that $y \in (\bigcup \mathcal{A}'_n)^*$. Let V_n denote $\bigcup \mathcal{A}'_n$, and choose an open subset W_n of H_n such that $y \in W_n \subseteq cl(W_n) \subseteq (\bigcup \mathcal{F}_{n-1})^*$. Let

$\mathcal{B}'_n = \{B \in \mathcal{B}_n : \ B \cap f^{-1}y \neq \emptyset$ and $cl(B) \subseteq C \cap A \cap f^{-1}W_n$ for some $A \in \mathcal{A}'_n$, $C \in \mathcal{F}_{n-1}\}$.

Then $\mathcal{B}'_n$ covers $f^{-1}y \cap V_n$, and by tri-quotiency of f, there is a finite $\mathcal{F}_n \subseteq \mathcal{B}'_n$ such that $y \in (\bigcup \mathcal{F}_n)^*$. Set $U_n = \bigcup \mathcal{F}_n$.

This concludes the construction of the sequence $(U_n)_{n\in\omega}$. We show that it satisfies (0)–(vi).

Point (0) is evident from the construction and (b), and (i) follows immediately from (d). It is also easy to see how (ii) and (iii) follow from (c): Fix n, and let

$C_n = \bigcup \{C_B : \ B \in \mathcal{F}_{n+1}\}$.

Then $cl(U_{n+1}) = cl(\bigcup \mathcal{F}_{n+1}) \subseteq C_n \subseteq \bigcup \mathcal{F}_n = U_n$, thus (ii) holds.

Similarily, $cl(W_n) \subseteq (\bigcup \mathcal{F}_n)^* = (U_n)^*$, and $U_{n+1} = \bigcup \mathcal{F}_{n+1} \subseteq f^{-1}W_n$. By the latter and point (i) of Definition 1.0(a), $(U_{n+1})^* \subseteq f[U_{n+1}] \subseteq W_n$, and therefore $cl_Y((U_{n+1})^*) \subseteq U_n^*$, i.e., (iii) holds.

To prove (iv), suppose $\mathcal{G}$ is a filter base which meshes with $\{U_n : n \in \omega\}$. Then $\bigcap\{cl_X(G) : G \in \mathcal{G}\} \neq \emptyset$. For $n \in \omega$ denote:

$\mathcal{M}_n = \{B \in \mathcal{F}_n : \{B\}$ meshes with $\mathcal{G}\}$.

$\mathcal{M}_n$ is nonempty; otherwise we could find a $G \in \mathcal{G}$ such that $G \cap B = \emptyset$ for every $B \in \mathcal{F}_n$. The latter contradicts our assumption on $\mathcal{G}$ (since $U_n = \bigcup \mathcal{F}_n$).

If $B \in \mathcal{M}_{n+1}$, then by $(c)_{n+1}$ there is a $C \in \mathcal{M}_n$ such that $cl(B) \subseteq C$. Since each $\mathcal{M}_n$ is finite, by König's Lemma there exists a closurewise decreasing sequence $(B_n)_{n\in\omega}$ such that $B_n \in \mathcal{M}_n$ for every $n \in \omega$. Moreover, by (b), the family $\mathcal{H} = \{B_k \cap f^{-1}y : k \in \omega\}$ is a filter base.

Let $\mathcal{E}_n = \{A \in \mathcal{A}'_n : \exists k \ B_k \subseteq A\}$. Then $\mathcal{E}_n$ is finite and nonempty by $(c)_n$. If $A_d \in \mathcal{E}_{n+1}$, then $A_{\pi_n(d)} \in \mathcal{E}_n$, so again by König's Lemma, there exists a thread $(A_{d_n})_{n\in\omega}$ of $\mathcal{S}$ such that A_{d_n} contains some B_{k_n}. By the choice of the B_k's, the filter base $\mathcal{H}$ meshes with $\{A_{d_n} \cap f^{-1}y : n \in \omega\}$. Since the sieve $\mathcal{S}$ is complete for $f^{-1}y$, the family $\mathcal{H}$ clusters at some point $x \in f^{-1}y$. The sequence $(B_k)_{k\in\omega}$ is closurewise decreasing, therefore $x \in \bigcap_{k\in\omega} B_k$. Now by Definition 1.5, since $G \cap B_k \neq \emptyset$ for $G \in \mathcal{G}$, also $\bigcap\{cl(G \cap B_k) : G \in \mathcal{G}, \ k \in \omega\} \neq \emptyset$, and thus $\bigcap\{cl_X(G) : G \in \mathcal{G}\} \neq \emptyset$.

The first part of (v) follows from (iv), see Fact 1.6(d). For the second part, if $\{y\}$ is a G_δ–subset of Y, then $\bigcap_{n\in\omega} U_n \subseteq f^{-1}(\bigcap_{n\in\omega} W_n) \subseteq \bigcap_{n\in\omega} H_n = f^{-1}y$.

To prove (vi), note that (iv) is preserved by continuous mappings. By 1.0(i) any filter base $\mathcal{G}$ which meshes with $\{U_n^* : n \in \omega\}$ also meshes with $\{f[U_n] : n \in \omega\}$. Hence $\bigcap\{cl(G) : G \in \mathcal{G}\} \neq \emptyset$. By the equivalence of d(ii) and d(iii) in 1.6, (vi) follows.

This completes the proof of Lemma 2.0.

3. The Theorems

3.0 Theorem. *Let $f : X \to Y$ be a tri-quotient map from a regular space X onto a regular space Y such that for each $y \in Y$, the fibre $f^{-1}y$ is a sieve-complete subspace of X.*

(a) *If X is a monotonic p-space, then Y is of pointwise countable type.*

(b) *If X has a base of countable order, then Y is first countable.*

Proof: (a) The function f satisfies the assumptions of Lemma 2.0. For $y \in Y$, let $(U_n)_{n\in\omega}$ be a sequence such that (i)–(vi) of 2.0 are satisfied. Then $\bigcap_{n\in\omega} U_n^*$ is a compact subset of Y of countable character that contains y.

(b) Suppose $(\mathcal{B}_n)_{n\in\omega}$ is a sequence of bases witnessing that X has a base of countable order. Note that this is automatically an (mp)-sequence of bases. Let $y \in Y$, and let $(U_n)_{n\in\omega}$ be a sequence such that (0)–(vi) of 2.0 are satisfied. Denote $C_y = \bigcap_{n\in\omega} U_n$. It suffices to show that $C_y \subseteq f^{-1}y$. If this inclusion holds, then $f[C_y] = \{y\}$ and $\{U_n^* : n \in \omega\}$ is a base at y in Y, and we are done.

So let $z \in C_y$. By points (0) and (ii) of 2.0 and Kõnig's Lemma, there is a closurewise decreasing sequence $(B_n)_{n\in\omega}$ such that $z \in B_n \subseteq U_n$, $B_n \in \mathcal{B}_n$ and $B_n \cap f^{-1}y \neq \emptyset$ for every $n \in \omega$. It follows from Definition 1.5 that $\{B_n : n \in \omega\}$ is a base at z in X. If $z \notin f^{-1}y$, then because $f^{-1}y$ is closed, there is some $k \in \omega$ such that $B_k \cap f^{-1}y = \emptyset$. This contradicts the choice of B_k, and thus concludes the proof of Theorem 3.0.

3.1 Theorem. *Every tri-quotient map from a regular monotonic p-space X onto a countable regular space Y, with each fibre $f^{-1}y$ a sieve-complete subspace of X, is inductively perfect.*

Proof: Theorem 1.4 of [M 81] differs from the present theorem only in that X is assumed metrizable, and the fibres of f are assumed completely metrizable. The only place in [M 81] where these assumptions are actually used is the proof of Michael's Lemma 2.2, which is superceded by our Lemma 2.0. The remainder of the proof goes through verbatim, noting that since Y is countable and regular, $\{y\}$ is a G_δ–subset of Y for each $y \in Y$, and thus we get $\bigcap_{n\in\omega} U_n \subseteq f^{-1}y$ from 2.0(v).

3.2 Theorem: *Let $f : X \to Y$ be a tri-quotient map from a regular monotonic p–space X onto a regular space Y such that for each $y \in Y$, the fibre $f^{-1}y$ is a sieve-complete subspace of X. Then f is countable-compact-covering.*

Proof: Let f, X and Y be as in the assumptions, and let Z be a compact, countable subspace of Y. We show that there exists a compact cover of Z. Let $W = f^{-1}Z$. The space W is a closed subspace of X, and thus a regular, monotonic p–space. Therefore, the restriction of f to W satisfies the assumptions of 3.1. It follows that there is a subspace $W' \subseteq W$ that is mapped by f onto Z and such that $f|W'$ is perfect. Since inverse images of compact spaces under perfect maps are compact, W' is a compact cover of Z.

4. Examples

Example 1

Our first example shows that the image of a metrizable (and hence monotonic p-space with base of countable order) under an open (and hence tri-quotient) map with sieve-complete fibres may not have a base of countable order; in fact, it need not even be a monotonic p-space. This example was first published in [WW 67] in different terminology. We sketch it here for the convenience of the reader.

Let S be the collection of all points (x, y) in the Euclidean plane such that exactly one of the numbers x, y is rational. Let τ be the topology on S generated by the sets that are open in S treated as a subspace of the Euclidean plane, and all singletons $\{(x, y)\}$ such that x is irrational. Then τ is regular and has a σ-locally finite base. To see the latter, enumerate the rationals

$(q_n)_{n\in\omega}$, and choose for every $n \in \omega$ a locally finite family $\mathcal{D}_n$ of open disks of radius 2^{-n} in the Euclidean plane such that $\bigcup \mathcal{D}_n = \boldsymbol{R}^2 \backslash (\boldsymbol{R} \times \{q_n\})$.

Let $\mathcal{B}_n = \{D \cap (\boldsymbol{Q} \times \boldsymbol{R}) \cap S : \ D \in \mathcal{D}_n\} \cup \{\{(x, q_n)\} : \ x \in \boldsymbol{R} \backslash \boldsymbol{Q}\}$. Then $\mathcal{B}_n$ is a locally finite collection of open subsets of S, and $\bigcup_{n\in\omega} \mathcal{B}_n$ is a σ-locally finite base for τ.

It follows from the Nagata-Smirnov Theorem that (S, τ) is metrizable.

Let M be the Michael Line, i.e., the set of all reals equipped with the topology generated by the usual metric topology and the set of all singletons $\{x\}$, where x is irrational. Define $f : S \to M$ by $f(x, y) = x$. It is easy to see that f is an open continuous surjection. Moreover, if x is rational, then $f^{-1}x$ is homeomorphic to the set of irrationals, which is completely metrizable. If x is irrational, then $f^{-1}x$ is countable and discrete. Thus in both cases, $f^{-1}x$ is sieve-complete.

However, M is paracompact and non-metrizable (see e.g., [P, Example 2.3] for a proof), and thus by Fact 1.6(a) does not have a base of countable order. Since M is a submetrizable space, it has a G_δ-diagonal. By Fact 1.6(c), M is not even a monotonic p-space.

Example 2

This example shows that in Theorem 3.0(b) the assumption that X has a base of countable order can not be weakened to the assumption that X is a first countable monotonic p-space, not even if we assume that the map f is perfect. The example is described, e.g., in [E, Example 3.1.26]. We put it here in the context of our results.

Let X be the Alexandroff Double Circle $C_1 \cup C_2$, and let Y be the one-point compactification of the discrete space of cardinality $2^{\aleph_0}$. Let f map C_2 bijectively onto the isolated points of Y, and C_1 onto the accumulation point of Y. Then X and Y are compact (hence monotonic p-spaces by 1.6(e)), f is perfect (hence tri-quotient with sieve-complete fibres), X is first countable, but $\chi(Y) = 2^{\aleph_0}$.

Example 3

This example and the next show that in Theorem 3.0, the assumption that f is tri-quotient cannot be weakened to "bi-quotient," even if X is a complete metric space. (Note that the latter implies that all the fibres are complete in the given metric, rather than individually sieve-complete.)

The class of bi-quotient images of metric spaces has a neat combinatorial characterization; these are the *bi-sequential spaces* (see [M 72] for details). Finding suitable examples would be a straightforward matter if the following question had a positive answer:

4.0 Question: Is every bi-sequential space the continuous image of a metrizable space under a map f with sieve-complete fibres?

Unfortunately, neither the proof of Theorem 3.D.2 of [M 72] nor the techniques of section 5 of [M 71] shed much light on this matter.

Here we present two ad hoc examples relevant to Theorem 3.0. The fibres in both examples are discrete, which naturally leads to the following variation on the theme of 4.0.

4.1 Question: Is every bi-sequential space the continuous image of a metrizable space under a map f with discrete fibres?

Example 3 shows that neither in part (a) nor in part (b) of Theorem 3.0 does it suffice to assume that f is bi-quotient rather than tri-quotient, even if X is metric.

Let σ denote the metric topology on $Z = \boldsymbol{R}^2$, and let L denote the x–axis. Let (Z, τ) be a "bow-tie space," where at each $x \in Z \backslash L$ the neighborhoods of x are those of σ, and at each $x \in L$ there is a countable decreasing base $\mathcal{B}_x = \{U_n(x) : n \in \omega\}$ consisting of bow-tie shaped sets with x as their "knot." Different shapes of bow-ties have been fashionable with different authors; the reader may consult [E 3.1.I] for one elegant style. We take here a neutral approach and just list the properties of the bases $\mathcal{B}_x$ and τ that we will use.

(A) Suppose $y \in L$ and $y \in B \in \mathcal{B}_x$. Let W be a σ-open neighborhood of y, and let ℓ_y be the vertical line through y. If $B \cap \ell_y \cap W = \{y\}$, then $y = x$.
(B) The topologies induced by τ on L and $Z \backslash L$ coincide with those induced by σ.
(C) (Z, τ) is a regular space.

Let $I \subset L$ be the closed unit interval, and let Y be the quotient space obtained from Z by contracting I into one point i^*. Let $h : Z \to Y$ be the quotient map.

4.2 Lemma. *The space Y is not of pointwise countable type.*

This was first observed by Čoban in [Č, footnote on p. 89]. A proof of 4.2 seems never to have been published. Therefore, we sketch one for the convenience of the reader.

Proof: Assume by way of contradiction that C is a compact subspace of Y such that $i^* \in C$ and that $\mathcal{V}$ is a countable base for C in Y. Then $D = h^{-1}C$ has a countable base $\mathcal{U}$ in Z. Let $U \in \mathcal{U}$. For each $x \in D \cap L$ we choose a bow-tie $B(x) \in \mathcal{B}_x$ such that $B(x) \subseteq U$, and for each $x \in D \backslash L$ an open set $W(x)$ in the metric topology such that $W(x) \subseteq U \backslash L$. Since D is compact, there exists a finite set $X(U) \subseteq D$ such that

$$D \subseteq U' = \bigcup\{W(x) : \ x \ \in X(U) \backslash L\} \cup \bigcup\{B(x) : \ x \ \in X(U) \cap L\} \subseteq U.$$

For each $U \in \mathcal{U}$ choose such $X(U)$ and U'. Then $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is a countable base for D in Z. Moreover, the set $X(\mathcal{U}) = \bigcup\{X(U) : U \in \mathcal{U}\} \cap I$ is countable.

4.3 Claim. *Let $x \in I$, and let $B \in \mathcal{B}_x$. Then there exists a neighborhood V_x of x in the metric topology on the plane such that $V_x \cap D \backslash B = \emptyset$.*

Proof: If not, let $\{V_n : n \in \omega\}$ be a countable decreasing base at x for the metric topology on Z. For all $n \in \omega$, pick

$x_n \in V_n \cap D\backslash B$. By compactness of $D\backslash B$, the set $\{x_n : n \in \omega\}$ clusters at some $y \in D\backslash B$. Since $cl_\tau(V_n) = cl_\sigma(V_n)$, the equality $x = y$ must hold, which is impossible, since $x \in B$.

4.4 Claim. *Let $y \in I$, and let $B \in \mathcal{B}_y$. If there exists a neighborhood V_y of y in the metric topology on the plane such that $V_y \cap D\backslash B = \emptyset$, then $y \in X(\mathcal{U})$.*

Proof: Fix y, V_y, B as above, and let W be an open neighborhood of y in the metric topology on Z such that $cl(W) \subset V_y$. Then $D \subset B \cup (Z\backslash cl(W))$; therefore we can find $U \in \mathcal{U}$ such that $U' \subseteq B \cup (Z\backslash cl(W))$. Since the sets $W(x)$ in the definition of U' were chosen disjoint from L, there is some $x \in X(U)$ such that $y \in B(x)$. Since $B(x) \subseteq B \cup Z\backslash cl(W)$, we have $B(x) \cap W \cap \ell_y \subseteq B \cap W \cap \ell_y = \{y\}$. By (A), $B(x) \in \mathcal{B}_y$, and thus $y \in X(\mathcal{U})$.

Now Claims 4.3 and 4.4 imply that $I \subseteq X(\mathcal{U})$, which is impossible, since I is uncountable.

We have thus proved lemma 4.2.

Now the following theorem shows that in Theorem 3.0 it is necessary to assume that f is tri-quotient.

4.5 Theorem. *Y is the image of a complete metric space X under a bi-quotient map f.*

Proof: The following lemma follows easily from results of Vedenisov [V]. We give an alternative proof using sieves.

4.6 Lemma. *Suppose (M, τ) is a completely metrizable space, and ϱ is a complete metric on M that induces τ. Suppose there is a closurewise decreasing sequence $(U_n)_{n\in\omega}$ of nonempty open subsets of M such that $\bigcap_{n\in\omega} U_n = \emptyset$. Let x^* be a point not in M. Define a T_1-topology τ' on $M \cup \{x^*\}$ by letting the subspace topology on M induced by τ' agree with τ, and $\{U_n \cup \{x^*\} : n \in \omega\}$ be a neighborhood base at x^*. Then $(M \cup \{x^*\}, \tau')$ is completely metrizable.*

Proof: First note that τ' is regular. Regularity at $x \in M$ follows from regularity of τ and the fact that $\{x^*\} \notin cl_\sigma\{x\}$. Regularity at x^* is implied by the assumption that $(U_n)_{n\in\omega}$ is closurewise decreasing. Also, since for any σ-locally finite base $\mathcal{B}$ for τ the family $\mathcal{B} \cup \{B \cap U_n : B \in \mathcal{B}, n \in \omega\}$ is a σ-locally finite base for τ', the space $(M \cup \{x^*\}, \tau')$ is metrizable. By Fact 1.4(a), in order to show that τ' is induced by a complete metric, it suffices now to show that $(M \cup \{x^*\}, \tau')$ is sieve-complete for some sieve $\mathcal{S}$. For $n \in \omega$, let $\mathcal{A}_n$ be the collection of all open subsets of $(M \cup \{x^*\}, \tau')$ such that:

- *either* $A \subseteq M \backslash cl_\tau(U_{n-1})$ and $diam_\varrho A \leq 2^{-n}$,
- *or* $A = U_n$.

It is not hard to see that we can construct a sequence of index sets $(D_n)_{n\in\omega}$ and a sequence of functions $(\pi_n)_{n\in\omega}$ such that $\mathcal{S} = (\langle \mathcal{A}_n, D_n, \pi_n \rangle)_{n\in\omega}$ is a sieve in the sense of Definition 1.2. (The indexing may involve some repetitions in the sense of Remark 1.3.)

Now suppose that $t = (A_{d_n})_{n\in\omega}$ is a thread in $\mathcal{S}$. If $A_{d_n} = U_n$ for all $n \in \omega$, then $\mathcal{T} = \{A_{d_n} : n \in \omega\}$ is a base at x^*, and every filter that meshes with $\mathcal{T}$ clusters at x^*. If not, then with the exception of perhaps the first few terms, t is a thread in the prototypical complete sieve for (M, τ), so every filter that meshes with $\mathcal{T}$ clusters in M.

This concludes the proof of Lemma 4.6.

Now we are ready to construct the space X. Let X be the direct sum of the family of spaces $\{X_i : i \in \mathbf{R} \cup \{\infty\}\}$, where:

- X_∞ is $L \times \{\infty\}$ with the usual topology of the real line,
- For $i \in \mathbf{R}$, $X_i = ((Z\backslash L) \cup \{(i,0)\}) \times \{i\}$ with the topology that of a subspace of $Z \times \{i\}$.

Lemma 4.6 applies, and thus each X_i is completely metrizable. Letting ϱ_i be a complete metric bounded by 1 that induces the topology on X_i, and setting $\varrho(x,y) = 1$ whenever $x \in X_i$, $y \in X_j$, and $i \neq j$, we convince ourselves that the topology on X is induced by the complete metric $\bigcup\{\varrho_i : i \in \mathbf{R} \cup \{\infty\}\} \cup \varrho$.

Define $g : X \to Z$ by $g(x,y,i) = (x,y)$. Since for each $i \in \boldsymbol{R} \cup \{\infty\}$, the restriction $g|X_i$ is a homeomorphic embedding of the clopen subspace X_i of X into Z, the map g is a continuous surjection. Let $f = h \circ g : X \to Y$.

It remains to show that f is bi-quotient. Since h is perfect, and the composition of bi-quotient maps is bi-quotient (see [M 72 Proposition 3.D.3]), we really need only to show that g is bi-quotient.

So let $(x,y) \in Z$, and $\mathcal{W}$ be an open cover of $g^{-1}(x,y)$. If $y \neq 0$, then let $W \in \mathcal{W}$ be such that $(x,y,0) \in W$. Find an open $U \subset X$ such that $(x,y,0) \in U \subseteq W \cap X_0 \backslash \{(0,0,0)\}$. Then U is mapped onto an open neighborhood of (x,y), hence (x,y) is an interior point of $g[W]$. Now suppose $y = 0$. Find $W_0, W_1 \in \mathcal{W}$ such that $(x,0,\infty) \in W_0$ and $(x,0,x) \in W_1$. By Property (B), $g[W_0]$ contains an open (in the subspace topology on L) neighborhood $V_0 \subseteq L$, and $g[W_1]$ contains a relatively open neighborhood $V_1 \subseteq (Z \backslash L) \cup \{(x,0)\}$ of $(x,0)$. Choose open $U_0, U_1 \subseteq Z$ such that $U_0 \cap L = V_0$ and $U_1 \cap ((Z \backslash L) \cup \{(x,0)\}) = V_1$. Let $U = U_0 \cup U_1$. Then $(x,0) \in U \subseteq g[W_0 \cup W_1]$, and we are done.

Example 4

In view of the previous example, the present one is somewhat redundant for the purpose of justifying the assumptions of our theorems. However, the gist of Example 3 is a strong negation of compactness of Y. The present example shows that in Theorem 3.0(b) it does not suffice to assume that f is bi-quotient, rather than tri-quotient, even if X is a complete metric space and Y is compact. Moreover, the mechanism for showing that the map is bi-quotient is radically different from the one used in Example 3. This sheds some light on the difficulties one encounters when trying to give a positive answer to Question 4.1.

The space Y in our example is $A(\kappa)$, the one-point compactification of the discrete space of size κ. If κ is uncountable, then $A(\kappa)$ is not first countable. Michael has shown in [M 72, Example 10.15] that $A(\kappa)$ is the bi-quotient image of a metric space iff κ does not exceed the first measurable cardinal. The map one gets by retracing his chain of evidence is different from ours though, and his proof does not give any information about sieve-completeness of the fibres.

4.7 Theorem. *Let $A(\kappa)$ be the one-point compactification of the discrete space of size κ. If κ is smaller than the first measurable cardinal, then there exist a complete metric space X and a bi-quotient map $f : X \to A(\kappa)$ from X onto $A(\kappa)$.*

Proof of 4.7: Let κ be as in the assumptions, and let $\mathcal{E}$ be the set ${}^{\kappa}\omega$ of all functions $e : \kappa \to \omega$. For $e \in \mathcal{E}$, let M_e be the set $(\kappa + 1) \times \{e\}$ equipped with the metric d, where:
$d((\alpha, e), (\alpha, e)) = 0$ for $\alpha \leq \kappa$,
$d((\alpha, e), (\kappa, e)) = d((\kappa, e), (\alpha, e)) = \frac{1}{e(\alpha)+1}$ for $\alpha < \kappa$, and
$d((\alpha, e), (\beta, e)) = d((\beta, e), (\alpha, e)) = \frac{1}{e(\alpha)+1} + \frac{1}{e(\beta)+1}$ for $\alpha < \beta < \kappa$.

Let X be the union of the spaces M_e; make it a complete metric space by setting

$d(\alpha, e), (\beta, e')) = 1$ for $e \neq e'$.

Let the underlying set of $A(\kappa)$ be $\kappa \cup \{x^*\}$, where x^* is the accumulation point. Define a map $f : X \to A(\kappa)$ by:

$f((\kappa, e)) = x^*$, and

$f((\alpha, e)) = \alpha$ for $\alpha < \kappa$.

It is clear that f is a continuous surjection. So all that remains to show is that f is bi-quotient. Since condition $(!)_y$ of Definition 1.0(b) is obvious for the isolated points $y \in A(\kappa)$, we may concentrate on the case $y = x^*$.

Let $\mathcal{W}$ be an open cover of $f^{-1}x^*$, i.e., an open cover of the set $\{(\kappa, e) : e \in \mathcal{E}\}$. Assume by way of contradiction that for every finite subcover $\mathcal{F}$ of $\mathcal{W}$, the point x^* is not in the interior of $f[\bigcup \mathcal{F}]$. In other words, the following holds:

(*) For every finite subset $\mathcal{F}$ of $\mathcal{W}$, there exists an infinite $A_{\mathcal{F}} \subseteq \kappa$ such that $(\alpha, e) \notin \bigcup \mathcal{F}$ whenever $\alpha \in A_{\mathcal{F}}$ and $e \in \mathcal{E}$.

Since (*) will continue to hold for every refinement of $\mathcal{W}$, we may without loss of generality assume that for every $e \in \mathcal{E}$ there exists exactly one $W_e \in \mathcal{W}$ such that $(\kappa, e) \in W_e \subseteq M_e$. For W_e, let $B_e = \{\alpha < \kappa : \ (\alpha, e) \in W_e\}$, and denote $\mathcal{B} = \{B_e : \ e \in \mathcal{E}\}$.

Let $\mathcal{I}$ be the ideal on κ generated by $\mathcal{B}$, i.e., let

$\mathcal{I} = \{C \subseteq \kappa :$ there is a finite $\mathcal{B}' \subset \mathcal{B}$ such that $C \subseteq \bigcup \mathcal{B}'\}$.

An ideal of subsets of κ will be called *σ-complete* if it is closed under countable unions.

4.8 Lemma. *There exists a $D \subseteq \kappa$ such that the σ-ideal $\mathcal{J}$ of subsets of D generated by the family*

$\mathcal{K} = \{K \subseteq D : \ K \in \mathcal{I}$ or K is finite$\}$

is a maximal, proper σ-complete ideal on D.

The above lemma immediately implies that the cardinality of D is at least that of the least measurable cardinal (see [J, Corollary to Lemma 27.4]). Since $|D| \leq |\kappa|$, we have reached a contradiction with the assumption on κ. So all that remains to be done for the proof of 4.7 is the

Proof of Lemma 4.8: We split this into several steps. Let $\mathcal{I}'$ be the ideal on κ generated by $\mathcal{I}$ and the family of finite subsets of κ.

4.9 Claim. *$\mathcal{I}'$ is a proper ideal on κ.*

Proof: Suppose not. Then $\kappa \in \mathcal{I}$, i.e., $\kappa = H \cup \bigcup \mathcal{B}'$ for some finite set $H \subset \kappa$ and a finite subfamily $\mathcal{B}' \subseteq \mathcal{B}$. Let $\mathcal{F} = \{W_e : \ B_e \in \mathcal{B}'\}$. By the definition of the B_e's, if $\alpha \in \kappa \backslash H$, then $(\alpha, e) \in W_e$ for some $W_e \in \mathcal{F}$. This however means that the set $A_{\mathcal{F}}$ mentioned in (*) must be contained in H, which is a contradiction, since the latter is finite, but the former was supposed to be infinite.

4.10 Claim. *Let $(D_n)_{n\in\omega}$ be a sequence of pairwise disjoint subsets of κ. Then for some $n_0 \in \omega$, $\bigcup_{n>n_0} D_n \in \mathcal{I}$. In particular, all but finitely many of the D_n's are in $\mathcal{I}$.*

Proof: Let $(D_n)_{n\in\omega}$ be as above, and let $e \in \mathcal{E}$ be such that $e(\alpha) = n$ whenever $\alpha \in D_n$. Let n_0 be such that W_e contains an open ball of radius $\geq \frac{1}{n_0+1}$ centered at (κ, e). It is now evident from the definition of the metric on X that $(\alpha, e) \in W_e$ whenever $e(\alpha) > n_0$. Now it follows from our choice of e that $D_n \subseteq B_e \in \mathcal{I}$ for $n > n_0$.

This concludes the proof of 4.10.

4.11 Corollary. *There exist an $N \in \omega$ and pairwise disjoint sets $D_0, \ldots, D_N$ of κ such that for every $n \leq N$, $D_n \notin \mathcal{I}'$, and it is impossible to find disjoint sets $E_n^0, E_n^1 \notin \mathcal{I}'$ such that $E_n^0 \cup E_n^1 = D_n$.*

Proof: By 4.10 and a standard argument involving König's Lemma. See e.g. [CN, Lemma 2.10] for a very similar proof.

Now let D be any of the sets we get from 4.11. The family $\mathcal{J} = \mathcal{I}' \cap \mathcal{P}(D)$ is a proper maximal ideal on D, hence D must be infinite. Since $\mathcal{J}$ is maximal, in order to show that $\mathcal{J}$ is also σ-complete, it suffices to show that D is not a union of countably many elements of $\mathcal{J}$. So suppose by way of contradiction that $D = \bigcup_{n\in\omega} D_n$, where $D_n \in \mathcal{I}'$. We may assume without loss of generality that the sets D_n are pairwise disjoint. Now it follows from 4.10 that $\bigcup_{n>n_0} D_n \in \mathcal{I}$ for some n_0. But $\mathcal{I} \subseteq \mathcal{I}'$, hence D is the union of finitely many elements of $\mathcal{I}'$, and thus D itself is an element of $\mathcal{I}'$. This contradicts the choice of D, so we are done.

Example 5

Our last example shows that in Theorems 3.1 and 3.2 it does not suffice to assume that f is bi-quotient, even if X and Y are metric and every fibre $f^{-1}y$ is compact.

4.12 Claim. *There exists a subset X of the unit square I^2 with the following properties:*

(i) *For every $x \in I$, the vertical section of X at x is of the form $I \backslash U_x$, where U_x is an open interval of length $\frac{1}{3}$,*

(ii) *The projection $p_1 : X \to I$ of X onto the first coordinate is not countable-compact-covering.*

Proof: Michael constructed a space $X' \subseteq I^2$ with the following properties (see [M 59, Example 4.1]):

(i)' For every $x \in I$, the vertical section of X' at x is of the form $I \backslash \{y_x\}$, where $y_x \in I$.

(ii)' The projection $p_1 : X' \to I$ of X onto the first coordinate is not compact-covering.

Now we start out with X' as above, pick for each $x \in I$ an open interval U_x of length $\frac{1}{3}$ such that $y_x \in U_x$, and let $X = \bigcup_{x \in I} \{x\} \times (I \backslash U_x)$.

Then X satisfies (i), and $X \subseteq X'$. Since the property of "not being compact-covering" is inherited by any restriction of a map to a part of its domain as long as the range remains the same, the map $p_1|X$ is not compact-covering. Now it follows from the result of [CJ] that $p_1|X$ is not even countable-compact-covering, i.e., (ii) holds.

4.13 Claim. *Let $X \subset I^2$ be such that (i) of 4.12 holds. Then the projection $p_1 : X \to I$ is a bi-quotient map.*

Proof: Let $y \in I$, and let $\mathcal{W}$ be an open cover of $p_1^{-1}y$. By compactness, there exists a finite subcover $\mathcal{F} \subseteq \mathcal{W}$ of $p_1^{-1}y$. Let $\varepsilon > 0$ be such that $(I \cap (y - \varepsilon, y + \varepsilon)) \times (I \backslash U_y) \subseteq \bigcup \mathcal{F}$. If $x \neq y$, then there is still some $z \in I$ such that both (x, z) and (y, z) belong to X (since $U_x \cup U_y \neq I$). It follows that $I \cap (y - \varepsilon, y + \varepsilon) \subseteq p_1[\bigcup \mathcal{F}]$, which shows that $p_1|X$ is bi-quotient.

The relevance of this example to Theorem 3.2 is clear. To see its relation to 3.1, let E be a compact countable subset of I without a compact cover under $p_1|X$. Then $Z = X \cap p_1^{-1}E$ is metrizable, hence a monotonic p-space, and $p_1|Z$ is bi–quotient.

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