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A SPACE X WITH TRIND $X = 1$ EVERY COMPACTIFICATION OF WHICH HAS NO TRIND

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ABSTRACT. In this paper we construct a space X with the properties indicated in the title.

1. INTRODUCTION.

All spaces are assumed to be Tychonoff. Our terminology and notations follow [3].

In this paper we study the compactification theorem for the small transfinite dimension trind. It is well-known that every separable metrizable space X has metrizable compactification αX with ind $\alpha X =$ ind X. However Luxemburg [5] constructed a separable metrizable space X such that trind $\alpha X >$ trind X for every metrizable compactification αX of X. On the other hand it is known (see [3, p. 154]) that every separable metrizable space having trind has a metrizable compactification having trind. Thus it is natural to ask whether all spaces having trind have a compactification having trind .

Applying a technique in van Douwen and Przymusinski [1], and using an example in Pol and Pol [7], van Mill and Przymusinski [6] constructed a space X with ind $X = 1$ every compactification of which has no ind. However, their proof shows that ind $\alpha X \not\leq n$ for every $n < \omega$. Thus we have ind $\alpha X = \infty$, however, trind $\alpha X \geq \omega$. It seems therefore there is a possibility that trind $\alpha X < \infty$ for some compactification αX of X.

In this paper, modifying a method of van Mill and Przymusinski [6], and using an example of Dowker [2], we construct a space X with trind $X = 1$ every compactification of which has no trind.

Our space X is not normal. thus the following problem still remains open. Does there exist a normal (or metrizable) space X having trind such that every compactification of X has no trind ?

2. Preliminaries.

For the sake of completeness we begin with the construction of an example of Dowker [2].

We denote by ω_1 and $\mathfrak{c}$ the first uncountable ordinal number and the continuum cardinal number, respectively. We also regard $\mathfrak{c}$ as the set of all ordinal numbers α with $\alpha < \mathfrak{c}$.

2.1. Construction. Let I be the unit closed interval $[0,1]$. For $x, y \in I$ we set

$$x \sim y \text{ if and only if } x - y \text{ is a rational number.}$$

Obviously, $\sim$ is an equivalent relation on I. Let $\{Q_\alpha : \alpha < \mathfrak{c}\}$ be the set of all equivalent classes with respect to $\sim$. Let us set

$$I_\alpha = I - \cup\{Q_\beta : \alpha < \beta < \omega_1\} \text{ for every } \alpha < \omega_1, \text{ and}$$
$$M = \cup\{\{\alpha\} \times I_\alpha : \alpha < \omega_1\}.$$

Suppose that M is the subspace of $\omega_1 \times I$. This space M is an example of Dowker [2].

Let us set

$$A = (\omega_1 \times \{0\}) \cap M \text{ and } B = (\omega_1 \times \{1\}) \cap M.$$

Let $\pi_\alpha : M^{\mathfrak{c}} = \Pi_{\alpha<\mathfrak{c}} M_\alpha \to M_\alpha$ be the α-th projection, where M_α is a copy of M for every $\alpha < \mathfrak{c}$. Let us set

$$A_\alpha = \pi_\alpha^{-1}(A) \text{ and } B_\alpha = \pi_\alpha^{-1}(B)$$

for every $\alpha < \mathfrak{c}$.

For two elements $\alpha = (\alpha_i)_{i<n}$ and $\beta = (\beta_i)_{i<n}$ of the Cartesian product ω_1^n of n copies of ω_1 we set

$$\alpha < \beta \text{ if and only if } \alpha_i < \beta_i \text{ for every } i < n.$$

A subset C of ω_1^n is *cofinal* in ω_1^n if for every $\alpha \in \omega_1^n$ there exists $\beta \in C$ such that $\alpha < \beta$.

A collection $\{(A_i, B_i) : i < n\}$ of pairs of disjoint closed subsets of a space X is called *essential* if for every collection $\{L_i : i < n\}$, where L_i is a partition in X between A_i and B_i, we have $\bigcap\{L_i : i < n\} \neq \emptyset$.

Dowker [2] proved that ind M = locdim M = locInd $M = 0$ and dim M = Ind $M = 1$. To prove our main theorem we need the following lemma. From this lemma it follows that dim $M^n \geq n$ for every $n < \omega$. The proof is essentially due to Dowker.

2.2. Lemma. *For every finite subset Λ of $\mathfrak{c}$ the collection $\{(A_\alpha, \beta_\alpha) : \alpha \in \Lambda\}$ is essential in $M^{\mathfrak{c}}$.*

Proof: It suffices to prove that $\{(A_i, B_i) : i < n\}$ is essential in M^n for every $n < \omega$. We suppose on the contrary. Then there exist a closed subset L_i of M^n and open subsets U_i and V_i of M^n for every $i < n$ such that $M^n - L_i = U_i \cup V_i$, $U_i \cap V_i = \emptyset$, $A_i \subset U_i, B_i \subset V_i$ and $\bigcap\{L_i : i < n\} = \emptyset$. For every $\alpha = (\alpha_i)_{i<n} \in \omega_1^n$ let us set

$T_\alpha = \Pi_{i<n}[\alpha_i, \omega_1)$

$U_i' = \{x \in I^n : \{x\} \times T_\alpha \subset U_i$ for some $\alpha \in \omega_1^n\}$ and

$V_i' = \{x \in I^n : \{x\} \times T_\alpha \subset V_i$ for some $\alpha \in \omega_1^n\}$.

Claim 1. *U_i' and V_i' are open in I^n for every $i < n$.*

It suffices to prove that for every $x \in U_i'$ there exist $\epsilon > 0$ and $\beta \in \omega_1^n$ such that $(B(x, \epsilon) \times T_\beta) \cap M^n \subset U_i$, where $B(x, \epsilon)$ is the ϵ-neighborhood of x in I^n. We suppose on the contrary. Then for every $j < \omega$ there exist $x_j \in I^n$ and $\gamma_j \in \omega_1^n$ such that $(x_j, \gamma_j) \in M^n - U_i$, $x_j \to x$ and $\alpha < \gamma_j < \gamma_{j+1}$. Let $\gamma = \lim_{j\to\omega} \gamma_j$. Then we have $(x, \gamma) \in M^n - U_i$. On the other hand, since $\alpha < \gamma$, we have $(x, \gamma) \in \{x\} \times T_\alpha \subset U_i$. This is a contradiction. Hence U_i' is open in I^n. Similarly, V_i' is open in I^n.

For every $i < n$ let us set
$A_i' = \{(x_j) \in I^n : x_i = 0\}$ and
$B_i' = \{(x_j) \in I^n : x_i = 1\}$.

Claim 2. *$A_i' \subset U_i'$ and $B_i' \subset V_i'$ for every $i < n$.*

Let $p : M^n \to I^n$ be the projection. Then for every $x \in A_i'$ we have $(\{x\} \times T_0) \cap M^n = p^{-1}(x) \subset p^{-1}(A_i') = A_i \subset U_i$. Since $\cup\{I_\alpha : \alpha < \omega_1\} = I$, there exists $\alpha \in \omega_1^n$ such that $\{x\} \times T_\alpha \subset M^n$. Thus we have $\{x\} \times T_\alpha \subset U_i'$. Similarly, we have $B_i' \subset V_i'$.

Claim 3. $\cup\{U_i' \cup V_i' : i < n\} = I^n$.

We suppose on the contrary. Then we take a point $x \in I^n - \cup\{U_i' \cup V_i' : i < n\}$. Then
$(*)$ $\{\alpha \in \omega_1^n : (x, \alpha) \in p^{-1}(x) - W\}$ is cofinal in ω_1^n, where $U = U_i$ or V_i, $i < n$.
We enumerate $\{U_i : i < n\} \cup \{V_i : i < n\}$ as $\{W_k : k < 2n\}$. Let us set
$D_0 = p^{-1}(x) - W_0$,
$D_k = D_{k-1} - W_k$ for every $1 \leq k < 2n$, and
$D_k' = \{\alpha \in \omega_1^n : (x, \alpha) \in D_k\}$ for every $k < 2n$.
Obviously, $D_k = \{x\} \times D_k'$ for every $k < 2n$. We show, inductively, D_k' is cofinal in ω_1^n for every $k < 2n$. By $(*)$, D_0' is cofinal in ω_1^n. Let $k < 2n - 1$ and assume that D_k' is cofinal in ω_1^n. Suppose that D_{k+1}' is not cofinal in ω_1^n. Then we can take $\alpha \in \omega_1^n$ such that $\{x\} \times (T_\alpha \cap D_k') \subset W_{k+1}$. On the other hand, by $(*)$, $\{\alpha \in \omega_1^n : (x, \alpha) \in p^{-1}(x) - W_{k+1}\}$ is cofinal in ω_1^n. Thus for every $j < \omega$ we can take $(x, \gamma_j) \in p^{-1}(x) - W_{k+1}$ and $\delta_j \in T_\alpha \cap D_k'$ such that $\gamma_j < \delta_j < \gamma_{j+1}$. Let $\beta = \lim_{j \to \omega} \gamma_j = \lim_{j \to \omega} \delta_j$. Since $p^{-1}(x) - W_{k+1}$ and $\{x\} \times (T_\alpha \cap D_k')$ are closed in M^n, we have

$$\begin{aligned}(x,\ \beta) &\in (p^{-1}(x) - W_{k+1}) \cap (\{x\} \times (T_\alpha \cap D'_k)\\ &\subset (p^{-1}(x) - W_{k+1}) \cap W_{k+1}\\ &= \emptyset.\end{aligned}$$

This is a contradiction. Hence D'_k is cofinal in ω_1^n for every $k < 2n$. This implies that $D'_{2n-1} \neq \emptyset$. However, we have

$$D_{2n-1} = p^{-1}(x) - \cup\{W_k : k < 2n\} = \emptyset.$$

This is a contradiction. Hence we have $\cup\{U'_i \cup V'_i : i < n\} = I^n$.

Let us set $L'_i = I^n - (U'_i \cup V'_i)$ for every $i < n$. Then, by Claims 1 and 2, L'_i is a partition in I^n between A'_i and B'_i. Since $\{(A'_i,\ B'_i) : i < n\}$ is essential in I^n, we have $\cap\{L'_i : i < n\} \neq \emptyset$. However, by Claim 3, we have $\cap\{L'_i : i < n\} = \emptyset$. This is a contradiction. Hence $\{(A_i,\ B_i) : i < n\}$ is essential in M^n. This completes the proof of Lemma 2.2.

3. An example.

3.1. Example. *There exists a space X having trind every compactification of which has no trind .*

3.2. Construction. Let M be an example of Dowker described in section 2, and C be the standard Cantor set in I. We take a countable dense subset D of C. Let $\varphi : C \to \mathfrak{c}$ be a bijection. For $\alpha < \mathfrak{c}$ and $n < \omega$ let us set

$$F(\alpha,\ n) = \pi_\alpha^{-1}((\omega_1 \times [1/n,\ 1]) \cap M),$$

where $\pi_\alpha : M^{\mathfrak{c}} = \Pi_{\alpha<\mathfrak{c}} M_\alpha \to M_\alpha$ is the α-th projection, and M_α is a copy of M. For every $t \in C$ we take a countable subset $Q(t) = \{q_n(t) : n < \omega\}$ of $D - \{t\}$ such that $q_n(t) \to t$. Let us set

$$X = (D \times M^{\mathfrak{c}}) \cup (C \times \{\theta\}),$$

where $\theta \notin M^{\mathfrak{c}}$.

For every $x = (t,\ y) \in D \times M^{\mathfrak{c}}$ let us set

$$\mathcal{B}(x) = \{\{t\} \times U : U \text{ is an open neighborhood of } y \text{ in } M^{\mathfrak{c}}\}.$$

For every $x = (t, \theta) \in C \times \{\theta\}$ let us set

$$V_n(t) = ((B(t, 1/n) \times (M^c \cup \{\theta\})) - ((\{t\} \times M^c) \cup (Q(t) \times F(\varphi(t), n)))) \cap X$$

for every $n < \omega$, and

$$\mathcal{B}(x) = \{V_n(t) : n < \omega\}.$$

We give X the topology induced by $\{\mathcal{B}(x) : x \in X\}$ as a neighborhood base system.

3.3. Lemma. *The space X is a Tychonoff space.*

Proof: It is easy to see that every point $x \in D \times M^c$ and any closed subset F of X with $x \notin F$ are completely separated. Thus it suffices to show that for every $x = (t, \theta) \in C \times \{\theta\}$ and any $V_n(t)$ there exists a continuous mapping $f : X \to I$ such that $f(x) = 1$ and $f(X - V_n(t)) \subset \{0\}$. Let $g : M^c \to I$ be the mapping defined by

$$g(y) = \max \{1 - n \cdot p_{\varphi(t)}(y), 0\},$$

where $p_\alpha : M^c \to I_\alpha = I$ is the projection. Obviously, g is continuous. We take a continuous mapping $h : C \to I$ such that $h(t) = 1$ and $h(C - B(t, 1/n)) \subset \{0\}$. Let $f : X \to I$ be the mapping defined by

$$f(s,y) = \begin{cases} h(s)g(y) & \text{if } s \in Q(t) \text{ and } y \neq \theta \\ h(s) & \text{if } s \notin Q(t) \cup \{t\} \\ h(s) & \text{if } s \in Q(t) \text{ and } y = \theta \\ 0 & \text{if } s = t \text{ and } y \neq \theta \\ 1 & \text{if } s = t \text{ and } y = \theta \end{cases}$$

Then f has the all required properties. Hence X is a Tychonoff space.

The proof of the following fact is straightforward, so we omit the proof.

3.4. Fact. *trind* $X = 1$

The following lemma which is proved by Fedorcuk [4] is needed to prove Theorem 3.6.

3.5. Lemma. *Every compact space having trind has trInd.*

3.6. Theorem. *Every compactification αX of the space X constructed in Construction 3.2 has no trind .*

Proof: Since every space having trInd is weakly infinite-dimensional, by Lemma 3.5, it suffices to prove that αX is not weakly infinite-dimensional. for every $t \in C$ we take open subsets U and W in αX such that $V_1(t) = U \cap X$ and $(t, \theta) \in W \subset \mathrm{Cl}_{\alpha X} W \subset U$. Then we have $V_n(t) \subset W$ for some $n < \omega$. Since $q_n(t) \to t$, we can take $y_t \in Q(t) \cap B(t, 1/n)$. Let $\psi : C \to D$ be the mapping defined by

$\psi(t) = y_t$ for every $t \in C$.

Since

$\mathrm{Cl}_{\alpha X}(\{\psi(t)\} \times A_{\varphi(t)}) \subset \mathrm{Cl}_{\alpha X} V_n(t) \subset \mathrm{Cl}_{\alpha X} W \subset U$ and
$\mathrm{Cl}_{\alpha X}(\{\psi(t)\} \times B_{\varphi(t)}) \subset \alpha X - U$,

we have

$\mathrm{Cl}_{\alpha X}(\{\psi(t)\} \times A_{\varphi(t)}) \cap \mathrm{Cl}_{\alpha X}(\{\psi(t)\} \times B_{\varphi(t)}) = \emptyset$.

Since D is countable, we can take $t \in D$ such that $\psi^{-1}(t)$ is infinite. Let $\{t_i : i < \omega\}$ be a countably infinite subset of $\psi^{-1}(t)$. Assume that αX is weakly infinite-dimensional. Then $\mathrm{Cl}_{\alpha X}(\{t\} \times M^c)$ is also weakly infinite-dimensional. Thus there exists a partition L_i in $\mathrm{Cl}_{\alpha X}(\{t\} \times M^c)$ between $\mathrm{Cl}_{\alpha X}(\{t\} \times A_{\varphi(t_i)})$ and $\mathrm{Cl}_{\alpha X}(\{t\} \times B_{\varphi(t_i)})$ for every $i < \omega$ such that $\cap\{L_i : i < n\} = \emptyset$ for some $n < \omega$. Let us set

$$L'_i = p(L_i \cap (\{t\} \times M^c)) \text{ for every } i < \omega,$$

where $p : D \times M^c \to M^c$ is the projection. Then we have $\cap\{L'_i : i < n\} = \emptyset$. Obviously, L'_i is a partition in M^c between $A_{\varphi(t_i)}$ and $B_{\varphi(t_i)}$. Thus by Lemma 2.2, we have $\cap\{L'_i : i < n\} \neq \emptyset$. This is a contradiction. Theorem 3.6 has been proved.

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