Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 17, 1992

METRIZABLE GENERALIZED ORDER SPACES

E.C. MILNER[1] AND SHANGZHI WANG[2]

ABSTRACT. In 1971 D.J. Lutzer [10] proved a metrization theorem for generalized order topological spaces (GO-spaces) which says that, if X is a p-embedded subspace of a linear ordered topological space, then X is metrizable if and only if it has a G_δ-diagonal. After stating this theorem, he raised the question whether there is any larger class of GO-spaces than the p-embedded subspaces of linear ordered topological spaces for which the G_δ-diagonal metrization theorem is true. In this paper we answer this question negatively by proving the following result. If $(X, \leq, \tau)$ is a metrizable GO-space and d is a metric on X which is compatible with the topology τ, then there is a metrizable linear ordered topological space $(Y, \leq_Y, \lambda)$ and a metric d^* compatible with λ such that (i) $(X, \leq)$ is a subordered set of $(Y, \leq_Y)$, (ii) d^* is equivalent to d on X (equal if d is bounded), and (iii) (X, τ) is a p-embedded closed subspace of (Y, λ).

1. INTRODUCTION

Let $(X, \leq)$ be a linearly ordered set. We denote by

$$X(< a) = \{x \in X : x < a\} \text{ and } X(> a) = \{x \in X : x > a\}$$

the open intervals determined by the element $a \in X$, and as usual (a, b) denotes the open interval $X(> a) \cap X(< b)$. We also write $X(\leq a) = X(< a) \cup \{a\}$, and $X(\geq a)$ is similarly

[1]Research supported by NSERC grant #69-0982.
[2]This paper was written while the second author was a Visiting Scholar at the University of Calgary.

defined. The *linear order topology*, λ, on X has for a subbasis the family of intervals

$$\mathcal{B} = \{X\} \cup \{X(< a) : a \in X\} \cup \{X(> a) : a \in X\}.$$

A subspace of a linear ordered topological space (LOTS) is not, in general, a LOTS. For example, $\Re$, the real line with the natural ordering is a LOTS, but the subspace $X = \{0\} \cup \{x : |x| > 1\}$ is not since $\{0\}$ is an open set in the induced topology on X, but not in the linear order topology on X. A topology τ on the linearly ordered set $(X, \leq)$ is called a *generalized order topology* on X, briefly we say $(X, \leq, \tau)$ is a GO-space, if τ extends the order topology and has a base of order-convex sets. An equivalent formulation, and the one we shall use, is that there are two subsets L, R of X such that, if $a \in L$ then a is not the maximum element of X and $X(\leq a)$ is open, and if $a \in R$ then a is not the minimal element of X and $X(\geq a)$ is open, and

$$\mathcal{B} \cup \{X(\leq a) : a \in L\} \cup \{X(\geq a) : a \in R\}$$

is a subbasis for τ. A subspace of a LOTS is a GO-space.

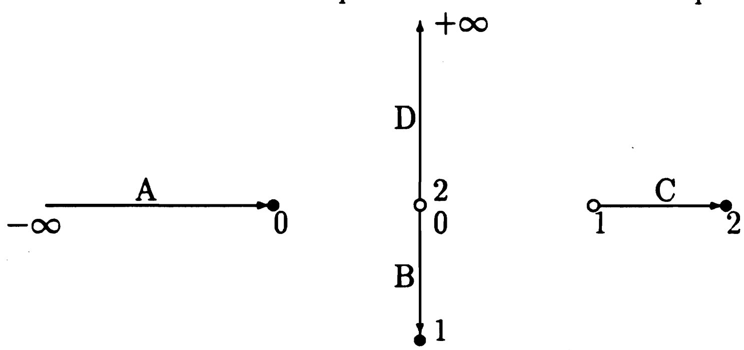

Diagram 1.

A topology τ on a set X is *metrizable* if there is a metric on X giving the same open sets. As an example, consider the metric space $(\Re, d)$ on the real line illustrated in Diagram 1. In the diagram the segments A, B, C, D represent respectively the subintervals of the real line $(-\infty, 0]$, $(0, 1]$, $(1, 2]$, $(2, \infty)$.

The metric is not the usual one for the real line, but the one induced by the distance in the plane. So, for example, the distance between the points ϵ and $2+\epsilon$ is 2ϵ (if $0<\epsilon<1$). Of course, this generalized order space (in which $L=\{0,1,2\}$ and $R=\emptyset$) is equivalent to that induced by the usual metric on $\Re$ by the subspace $(-\infty,0]\cup(1,2]\cup(3,4]\cup(5,\infty)$. In general, the structure of a GO-space is rather more complex.

During the last twenty years or so several papers have been written on the theory of LOTS and GO-spaces, and in particular about the metrization problem for such spaces. The first result in this direction was by V.V. Fedorčuk [7] who proved that a LOTS with a σ-locally countable base is metrizable. Then G. Creede [4] proved that a semi-stratifiable LOTS is metrizable. Shortly afterwards, D.J. Lutzer [9] generalized Creede's result by showing that a LOTS is metrizable if and only if it has a G_δ-diagonal, in other words if $\Delta=\{(x,x):x\in X\}$ is a G_δ-set in the product space $X\times X$; of course, any metric space has a G_δ-diagonal. Also, M.J. Faber [5] used some classical theorems of R.H. Bing to obtain metrization theorems for LOTS.

D.J.Lutzer [10] was the first to consider subspaces of LOTS, i.e.GO-spaces, and he established the following sufficient condition for a subspace of a LOTS to be metrizable.

Theorem 1.1. *Let $(Y,\leq,\lambda)$ be a LOTS and let τ be the relative topology on a p-embedded subspace X. If (X,τ) has a G_δ-diagonal, then (X,τ) is metrizable.*

Recall that the space X is a p-embedded subspace of Y if there is a sequence $\langle\mathcal{U}(n):n<\omega\rangle$ of covers of X by open subsets of Y such that, for each $x\in X$,

$$\bigcap_{n<\omega} St(x,\mathcal{U}(n))\subseteq X,$$

where $St(x,\mathcal{U}(n))=\bigcup\{U\in\mathcal{U}(n):x\in U\}$.

M.J. Faber [5], [6], J.M. van Wouwe [12], [13], and H. Bennett & D.J. Lutzer [1] obtained various necessary and sufficient conditions for a GO-space to be metrizable, and H. Bennett in

[2] used some of these results to give another proof of an observation of S. Purisch ([11] Propositions 2.4 and 2.5) that there is a metric ρ on the GO-space $(X, \leq, \tau)$ which is compatible with the topology τ and respects the order in the sense that

$$x \leq y \leq z \Rightarrow \rho(x,y) \leq \rho(x,z). \tag{1}$$

(Note that the metric on $\Re$ described in diagram 1 does not respect the order.) More recently, H. Bennett [3] improved Lutzer's theorem by proving that a LOTS with an S_δ-diagonal is metrizable.

In this paper we settle a question raised by D.J. Lutzer in [10]. After the statement of Theorem 1.1 in [10], Lutzer remarked that he did not know of any class of GO-spaces larger than the p-embedded subspaces of LOTS for which the G_δ-metrization theorem is true. We show that there is no larger class. In other words, if $(X, \leq, \tau)$ is a metrizable GO-space, there is some LOTS Y such that X is a p-embedded induced subspace. In fact, there is a metrizable LOTS Y. We prove the following theorem.

Theorem 1.2. *If $(X, \leq_X, \tau)$ is a metrizable generalized order space with metric d, then there is a metrizable LOTS $(Y, \leq_Y, \lambda)$ with metric d^* such that (i) $\leq_X = \leq_Y |X \times X$, (ii) d^* is equivalent to d on X (equal to d on X if d is bounded), and (iii) X is a p-embedded closed subspace of Y.*

As a corollary of Theorems 1.1 and 1.2 we have a necessary and sufficient condition for a GO-space to be metrizable.

Theorem 1.3. *A GO-space is metrizable if and only if it is a p-embedded closed subspace of a metrizable LOTS.*

2. Arc-connected extension of a metric space

In order to prove Theorem 1.2 we need a result about arc-connected metric spaces. A topological space (X, τ) is *arc-connected* if for any two distinct points $a, b \in X$ there is a homeomorphic map $f : [0,1] \to X$ such that $f(0) = a$

and $f(1) = b$. The following theorem shows that a metric space can be isometrically embedded in an arc-connected metric space. In fact, for our application we shall require the result for pseudo-metric spaces, i.e. when the metric $d : X \times X \Rightarrow \Re$ is non-negative, symmetric and satisfies the triangle inequality, but we do not insist that $d(x, y) = 0 \Rightarrow x = y$. Of course, if (X, d) is a pseudo-metric space and we define an equivalence relation $\sim$ on X by $x \sim y \iff d(x, y) = 0$, then $X/\sim$ is a metric space with the induced metric. Theorem 2.1 is proven in ([8, page 81) for bounded metric spaces (which is the essential content). We give the details of the proof since we require the result for pseudo-metrics and we continue to use the notation introduced in the proof.

Theorem 2.1. *If (X, d) is a (pseudo-) metric space, then there is an arc-connected (pseudo-) metric space (X^*, d^*) such that (X, d) can be isometrically embedded into (X^*, d^*).*

Proof: Let $<$ be a linear ordering of X. For distinct elements $a, a' \in X$ with $a < a'$ we introduce a copy of the open unit interval $I(a, a') = \{x_\lambda(a, a') : 0 < \lambda < 1\}$; we also define $x_0(a, a') = a$ and $x_1(a, a') = a'$. We assume that $I(a, a') \cap I(b, b') = \emptyset$ if $(a, a') \neq (b, b')$, and define $X^* = X \cup \bigcup\{I(a, b) : a, a' \in X, a < a'\}$. We define a (pseudo-)metric d^* on X^* by setting, for $x = x_\lambda(a, a')$ and $y = x_\mu(b, b')$,

$$d^*(x, y) = \begin{cases} |\lambda - \mu| d(a, a') & \text{if } (b, b') = (a, a') \\ \lambda'\mu' d(a, b) + \lambda'\mu d(a, b') + \lambda\mu' d(a', b) & \\ +\lambda\mu d(a', b') & \text{if} (b, b') \neq (a, a') \end{cases}$$

where we have written $\lambda' = 1 - \lambda$, $\mu' = 1 - \mu$.

It is easy to check that d^* is unambiguously defined. For example, using the second line of the definition to compute the distance $d^*(x_\lambda(a, a'), a) = d^*(x_\lambda(a, a'), x_1(c, a))$, where $c \neq a$, we get (since $\mu = 1, \mu' = 0$)

$$\lambda' d(a, a) + \lambda d(a', a) = \lambda d(a, a'),$$

and this is the same as the value that we obtain using the first line.

Note that, if $b \in X$, then

$$d^*(x_\lambda(a,a'),b) = \lambda' d(a,b) + \lambda d(a',b).$$

Also, if $(a,a') \neq (b,b')$ then

$$\begin{aligned} d^*(x_\lambda(a,a'),x_\mu(b,b')) &= \lambda' d^*(a,x_\mu(b,b')) + \lambda d^*(a',x_\mu(b,b') \\ &= \mu d^*(x_\lambda(a,a'),b') + \mu' d^*(x_\lambda(a,a'),b). \end{aligned} \tag{2}$$

To show that d^* is a (pseudo-) metric is a little tedious. It is obvious that d^* is symmetric. Also, if d is a metric, then $d^*(x,y) = 0 \iff x = y$. We have to check that the triangle inequality holds.

Case 1: If $x = x_\lambda(a,a')$, $y = x_\mu(a,a')$, $z = x_\nu(a,a')$, it is obvious that $d^*(x,z) \leq d^*(x,y) + d^*(y,z)$.

Case 2: Let $x = x_\lambda(a,a')$, $y = x_\mu(b,b')$, $z = x_\nu(c,c')$, where $(a,a'),(b,b')$ and (c,c') are all different. We have

$$\begin{aligned} d^*(x,y) &= (\lambda'\mu' d(a,b) + \lambda'\mu d(a,b') + \lambda\mu' d(a',b) \\ &\quad + \lambda\mu d(a',b'))(\nu + \nu') \\ &\leq \lambda'\mu'\nu'(d(a,c) + d(b,c)) + \lambda'\mu'\nu(d(a,c') + d(b,c')) \\ &\quad + \lambda'\mu\nu'(d(a,c) + d(b',c)) + \lambda'\mu\nu(d(a,c') + d(b',c')) \\ &\quad + \lambda\mu'\nu'(d(a',c) + d(b,c)) + \lambda\mu'\nu(d(a',c') + d(b,c')) \\ &\quad + \lambda\mu\nu'(d(a',c) + d(b',c)) + \lambda\mu\nu(d(a',c') + d(b',c')) \\ &= (\lambda'\nu' d(a,c) + \lambda\nu' d(a',c) + \lambda'\nu d(a,c') + \lambda\nu d(a',c')) \\ &\quad + (\mu'\nu' d(b,c) + \mu\nu' d(b',c) + \mu'\nu d(b,c') + \mu\nu d(b',c')) \\ &= d^*(x,z) + d^*(y,z). \end{aligned}$$

Case 3: Let $x = x_\lambda(a,a')$, $y = x_\mu(a,a')$, $z = x_\nu(c,c')$, where (a,a') and (c,c') are different. We need to verify that the following two inequalities hold:

$$d^*(x,y) \leq d^*(x,z) + d^*(y,z) \tag{3}$$

$$d^*(y,z) \leq d^*(x,y) + d^*(x,z). \tag{4}$$

First we show that (3) and (4) hold in the special case when $\lambda = 0, \mu = 1$, i.e. when $x = a, y = a'$. For these special values we have

$$\begin{aligned} d(a,a') &\leq \nu(d(a,c') + d(a',c')) + \nu'(d(a,c) + d(a',c)) \\ &= d^*(a,z) + d^*(a',z), \end{aligned}$$

and

$$\begin{aligned} d^*(a',z) &= \nu' d(a',c) + \nu d(a',c') \\ &\leq \nu'(d(a,a') + d(a,c)) + \nu(d(a,a') + d(a,c')) \\ &= d(a,a') + d^*(a,z). \end{aligned}$$

(3) and (4) follow from these special cases. For (3) we may assume that $\lambda < \mu$. Then, since $\mu - \lambda \leq \min\{\mu + \lambda, \mu' + \lambda'\}$, it follows that

$$\begin{aligned} d^*(x,y) &= (\mu - \lambda)d(a,a') \leq (\mu - \lambda)(d^*(a,z) + d^*(a',z)) \\ &\leq (\mu' + \lambda')d^*(a,z) + (\mu + \lambda)d^*(a',z) \\ &= (\mu' + \lambda')(\nu' d(a,c) + \nu d(a,c')) + (\mu + \lambda)(\nu' d(a',c) \\ &\quad + \nu d(a',c')) \\ &= (\lambda'\nu' d(a,c) + \lambda\nu' d(a',c) + \lambda'\nu d(a,c') + \lambda\nu d(a',c')) \\ &\quad + (\mu'\nu' d(a,c) + \mu\nu' d(a',c) + \mu'\nu d(a,c') + \mu\nu d(a',c')) \\ &= d^*(x,z) + d^*(y,z). \end{aligned}$$

This proves (3). We prove (4) under the same assumption that $\lambda < \mu$ (the case when $\mu < \lambda$ is similar). By (2), we have

$$\begin{aligned} d^*(y,z) &= \mu' d^*(a,z) + \mu d^*(a',z) \\ &= (\mu - \lambda)d^*(a',z) + \lambda d^*(a',z) + \mu' d^*(a,z) \\ &\leq (\mu - \lambda)(d^*(a,a') + d^*(a,z)) + \lambda d^*(a',z) + \mu' d^*(a,z) \\ &= (\mu - \lambda)d^*(a,a') + \lambda' d^*(a,z) + \lambda d^*(a',z) \\ &= d^*(x,y) + d^*(x,z). \end{aligned}$$

Clearly the space (X^*, d^*) is an arc-connected isometric extension of (X,d). For example, if $x = x_\lambda(a,a')$, $y = x_\mu(b,b')$, where $(a,a') \neq (b,b')$ and $a < b$, then there is a homeomorphic map $f : [0,1] \to \{x_\nu(a,a') : \nu \leq \lambda\} \cup I(a,b) \cup \{x_\nu(b,b') : \nu \leq \mu\}$ with $f(0) = x$, $f(1) = y$. $\square$

We call the (pseudo-)metric space (X^*, d^*) constructed in the theorem the *arc-connected extension* of (X, d). It should be noted that the linear ordering imposed upon X in the proof was no more than a notational convenience, the construction of (X^*, d^*) does not depend upon this ordering. In the case when (X, d) is a pseudo-metric space, then so also is (X^*, d^*). But in this case it is clear from our definitions that, if $a, a', b, b' \in X$, $a \neq a'$, $b \neq b'$, then:

(1) If $d(a,a') = 0$ and $x, y \in I(a,a')$, then $d^*(x,y) = 0$.
(2) If $d(a,a') = d(a,b) = d(a',b') = 0$, $x \in I(a,a')$, $y \in I(b,b')$, then $d^*(x,y) = 0$.
(3) If $d(a,a') = 0$, $x \in I(a,a')$, $y \in I(b,b')$ and $d^*(a,y) > 0$, then $d^*(x,y) > 0$.

Corollary 2.2. *If (X^*, d^*) is the arc-connected extension of the pseudo-metric space (X, d), and if $d(a,a') \neq 0$ and $x \in I(a,a')$, then there is $r > 0$ such that $B^*(x,r) = \{y \in X^* : d^*(x,y) < r\} \subseteq I(a,a')$.*

Proof: Let $x = x_\lambda(a,a')$, where $0 < \lambda < 1$. Choose r so that $0 < r < r' < \min\{\lambda d(a,a'), \lambda' d(a,a')\}$. Then $d^*(a,x) > r$, $d^*(a',x) > r$. Also, if $y = x_\mu(b,b')$, where $(b,b') \neq (a,a')$, then

$$\begin{aligned} d^*(x,y) &= \lambda'\mu' d(a,b) + \lambda'\mu d(a,b') + \lambda\mu' d(a',b) + \lambda\mu d(a',b') \\ &\geq (\mu'(d(a,b) + d(a',b)) + \mu(d(a,b') \\ &\quad + d(a',b'))r'/d(a,a') \geq r' > r, \end{aligned}$$

and the result follows. $\square$

From Corollary 2.2 we immediately obtain the following fact.

Corollary 2.3. *Let (X, d) be a pseudo-metric space with arc-connected extension (X^*, d^*). Let $X' \subseteq X$ be a set such that $\{a, a'\} \cap X' \neq \emptyset$ whenever $a \neq a'$ and $d(a,a') = 0$, and let $\hat{X} = \bigcup\{I(a,a') : a \neq a' \in X, d(a,a') = 0\}$. Then $d^*(x,y) > 0$*

for $x \neq y$ *and* $x, y \in X^{**} = X^{*} \backslash (X' \cup \hat{X})$, *i.e the subspace* X^{**} *is a metric space.*

We conclude this section with the observation that the arc-connected extension of a metric space reflects completeness.

Theorem 2.4. *A metric space is complete if and only if its arc-connected extension is complete.*

Proof: Let (X^*, d^*) be the arc-connected extension of the metric space (X, d). Suppose X^* is complete. Then, if (a_n) is a Cauchy sequence in X, there is $x \in X^*$ such that a_n converges to x. By Corollary 2.2 it follows that $x \in X$, and so X is complete.

Now suppose that X is complete. Let (y_n) be a Cauchy sequence in X^*, $y_n = x_{\lambda_n}(a_n, b_n)$. We need to show that some subsequence of (y_n) converges. Suppose $\liminf \lambda_n = 0$; we can assume that $\lambda_n \to 0$. Since $d^*(a_n, y_n) \to 0$ it follows from the triangle inequality that (a_n) is also Cauchy and so converges to some $a \in X$. Since $d^*(a_n, a)$ and $d^*(a_n, y_n)$ both converge to 0, it follows that $y_n \to a$. A similar argument applies if $\limsup \lambda_n = 1$. Thus we may assume that (some subsequence) $\lambda_n \to p$ where $0 < p < 1$. By Corollary 2.2 it follows that the pairs (a_n, b_n) are eventually constant, say equal to (a, b). Then $y_n \to x_p(a, b)$. $\square$

3. Proof of Theorem 1.2

Let $(X, \leq, \tau)$ be a metrizable GO-space. We may assume that the metric d on X which is compatible with τ is bounded. Let $L = \{x \in X : X(\leq x) \textit{ is open}\} \backslash \{\max X\}$, $R = \{x \in X : X(\geq x) \textit{ is open}\} \backslash \{\min X\}$.

If the element $x \in X$ has an immediate successor in the ordering on X, we denote its successor by x^+; similarly if there is an immediate predecessor we denote it by x^-. If $x \in L$ has no immediate successor in $(X, \leq)$, then we extend the order by introducing a new element x^+ which is the immediate successor of x in the extended order. Similarly, for each element of R

which has no immediate predecessor in the order on X, we introduce one which we denote by x^-. Let $(X', \leq)$ be the extended ordered set which includes these additional elements x^+ or x^- for appropriate elements $x \in L \cup R$. Thus each element of L has an immediate successor and each element of R has an immediate predecessor in this extended order.

We define a symmetric non-negative real function $d' : X' \times X' \to \Re$ as follows: for $x, y \in X'$,

$$d'(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in X; \\ \inf\limits_{v \in X(>a)} \sup\limits_{a<u<v} d(x,u) & \text{if } x \in X, a \in L, \\ & y = a^+ \notin X; \\ \inf\limits_{v \in X(<a)} \sup\limits_{v<u<a} d(x,u) & \text{if } x \in X, a \in R, \\ & y = a^- \notin X; \\ \inf\limits_{\substack{v \in X(>a) \\ w \in X(>b)}} \sup\limits_{\substack{a<u<v \\ b<t<w}} d(u,t) & \text{if } a, b \in L, \\ & x = a^+ \notin X, \\ & y = b^+ \notin X. \end{cases}$$

There are similar definitions for $d'(a^+, b^-)$ and $d'(a^-, b^-)$ obtained by modifying the last line of the above in an obvious manner.

We first observe that

$$(5) \qquad d'(x,y) > 0 \text{ if } x \in X, y \in X' \backslash X.$$

We only prove this for the case when $y = a^+$ for some $a \in L$ which has no immediate successor in X; the case when $y = a^-$ for some $a \in R$ is similar. Suppose $x < a$. Since $X(\leq a)$ is open and the metric d is compatible with τ, there is $r > 0$ such that $B_X(x,r) = \{y \in X : d(x,y) < r\} \subseteq X(\leq a)$. Thus $d(x,u) \geq r$ for all $u \in X(>a)$ and since $X(>a) \neq \emptyset$ it follows that $d'(x,y) \geq r$. Now suppose that $x > a$. Since $X(>a)$ has no first element in the ordering of X, there is some $v \in X$ such that $a < v < x$. Then $X(>v)$ is an open neighbourhood of x and so there is some $r > 0$ such that $B_X(x,r) \subseteq X(>v)$. This implies that $d'(x,y) \geq r$ and (5) follows.

We now verify that d' is a pseudo-metric on X'. Since d' is symmetric by definition, we need only check that the triangle inequality

$$d'(x,z) \leq d'(x,y) + d'(y,z), \tag{6}$$

holds for distinct $x, y, z \in X'$. There are several different cases that need to be considered, but these are all rather similar, and to avoid trivial repetition when we consider a point, say x, in $X' \backslash X$ we assume $x = a^+$ for some $a \in L$.

Case 1. $x, z \in X$, $y \in X' \backslash X$.

Assume $y = a^+$ for some $a \in L$. Then, for $a < u < v$, $u, v \in X$, we have

$$\begin{aligned} d'(x,z) &= d(x,z) \leq d(x,u) + d(u,z) \\ &\leq \sup\{d(x,u) : a < u < v\} + \sup\{d(u,z) : a < u < v\} \end{aligned}$$

and hence (6) holds.

Case 2. $x, y \in X$, $z \in X' \backslash X$.

Assume $z = a^+$ for some $a \in L$. For $a < u < v$, $u, v \in X$ we have

$$d(x,u) \leq d(x,y) + d(y,u)$$

and so, taking the supremum of both sides for $u < v$, we have

$$\sup\{d(x,u) : a < u < v\} \leq d(x,y) + \sup\{d(y,u) : a < u < v\}.$$

Finally taking the infimum of both sides of this for $v > a$ we get (6).

Case 3. $x \in X$; $y, z \in X' \backslash X$.

Assume $y = a^+$, $z = b^+$ for some $a, b \in L$. For $a < u < v$, $b < u' < v'$ and $u, v, u', v' \in X$ we have

$$\begin{aligned} d(x,u') &\leq d(x,u) + d(u,u') \\ &\leq \sup\{d(x,u) : a < u < v\} \\ &\quad + \sup\{d(u,u') : a < u < v\}, \end{aligned}$$

and hence

$$\begin{aligned} \sup\{d(x,u') : b < u' < v'\} &\leq \sup\{d(x,u) : a < u < v\} \\ &\quad + \sup\{d(u,u') : a < u < v, b < u' < v'\}. \end{aligned}$$

Taking the infimum over $v > a$ and $v' > b$, (6) follows.

Case 4. $x, z \in X'\backslash X$, $y \in X$. This is similar to Case 3.

Case 5. $x, y, z \in X'\backslash X$.

Assume $x = a^+$, $y = b^+$, $z = c^+$ for some $a, b, c \in L$. Let $a < u < v$, $b < u' < v'$, $c < u'' < v''$. We have

$$\begin{aligned} d(u,u'') &\leq d(u,u') + d(u',u'') \\ &\leq \sup\{d(u,u') : a < u < v, b < u' < v'\} \\ &\quad + \sup\{d(u',u'') : b < u' < v', c < u'' < v''\}, \end{aligned}$$

and therefore,

$$\begin{aligned} &\sup\{d(u,u'') : a < u < v, c < u'' < v''\} \\ &\qquad \leq \sup\{d(u,u') : a < u < v, b < u' < v'\} \\ &\qquad\quad + \sup\{d(u',u'') : b < u' < v', c < u'' < v''\}. \end{aligned}$$

Taking the infimums of the terms on the left and right sides of this inequality gives (6).

This proves that d' is a pseudo-metric on X'. Unfortunately, it need not be a metric. To see this consider again the example illustrated in Diagram 1. In that example, $L = \{0,1,2\}$, $R = \emptyset$, and we have to adjoin the additional points 0^+, 1^+ and 2^+. The distance between the distinct points 0^+ and 2^+ is $d'(0^+,2^+) = \inf_{0<\epsilon<1} \sup\{d(\xi, 2+\eta) : 0 < \xi < \epsilon, 0 < \eta < \epsilon\} = 0$. However, by (5), the set $Z = \{x \in X' : (\exists y \neq x) d'(x,y) = 0\} \subseteq X'\backslash X$.

By Theorem 2.1 there is an arc-connected extension (X^*, d^*) of the pseudo-metric space (X', d'). Also, by Corollary 2.3 the subspace X^{**} is a metric space, where $X^{**} = X^*\backslash\hat{X}$ and $\hat{X} = \bigcup\{I(a,a') \cup \{a,a'\} : a \neq a' \in X', d'(a,a') = 0\}$. Here we use the same notation as in the proof of Theorem 2.1 so that $I(a,a') = \{x_\lambda(a,a') : 0 < \lambda < 1\}$ for points $a, a' \in X'$ with $a < a'$.

We now show that (5) extends to the following:

$$d^*(x,y) > 0 \text{ if } x = x_\lambda(a,a'), a \in X, 0 \leq \lambda < 1 \text{ and } y \in X^*\backslash\{x\}. \tag{7}$$

For, let $y = x_\mu(b, b')$, where $b, b' \in X'$ and $0 \le \mu \le 1$. If $(b, b') = (a, a')$, then $\mu \ne \lambda$ and $d^*(x, y) = |\lambda - \mu| d'(a, a') > 0$ by (5). Also, if $(b, b') \ne (a, a')$, then $d^*(x, y) \ge \lambda'(\mu' d(a, b) + \mu d(a, b')) > 0$ again by (5).

It follows from (7) that $Y = X \cup L^* \cup R^*$ is disjoint from $\hat{X}$, where $L^* = \bigcup\{I(x, x^+) : x \in L\}$, $R^* = \bigcup\{I(x^-, x) : x \in R\}$. Hence the restriction of X^{**} to Y is also a metric space.

We define a linear ordering $\le_Y$ of Y as follows:

$$x \le_Y y \iff \begin{cases} x \le y & \text{when } x, y \in X \\ x \le a & \text{when } x \in X, a \in L \text{ and } y \in I(a, a^+) \\ x \le a & \text{when } x \in X, a \in R \text{ and } y \in I(a^-, a) \\ a \le y & \text{when } y \in X, a \in L \text{ and } x \in I(a, a^+) \\ a \le y & \text{when } y \in X, a \in R \text{ and } x \in I(a^-, a) \\ a \le b & \text{when } a, b \in L \cup R, x \in I(a, a^+) \text{ or} \\ & I(a^-, a), \text{and } y \in I(b, b^+) \text{ or } I(b^-, b) \\ \lambda < \mu & \text{when } a \in L, x = x_\lambda(a, a^+), \\ & y = x_\mu(a, a^+) \text{ or } a \in R, x = x_\lambda(a^-, a), \\ & y = x_\mu(a^-, a) \end{cases}$$

It is easy to check that $\le_Y$ is a linear order which extends the order on X, and also that, for $a \in L$ and $b \in R$, $I(a, a^+)$ and $I(b^-, b)$ are intervals in $(Y, \le_Y)$. (As observed by the referee, the order on Y is more easily visualized if we identify Y with $[X \times \{0\}] \cup [L \times (0, 1)] \cup [R \times (-1, 0)]$, and then $\le_Y$ is just the order inherited from the lexicographic order on $X \times (-1, 1)$.)

To complete the proof of the theorem we need to show two things: (A) the metric d^* is compatible with the linear order topology on Y; (B) X is a p-embedded, closed subspace of Y.

Proof of (A): We first show that the linear order topology on Y is contained in the metric topology defined by the metric d^*. Let $z \in Y$ and let J be an open interval in the order topology on Y which contains z. We have to show that there is $r > 0$ such that the open ball $B_Y(z, r) = \{y \in Y : d^*_Y(y, z) < r\}$ is contained in J.

If $z \in L^*$, then there is $a \in X$ such that $z \in I(a, a^+)$. By

Corollary 2.2 there is $r' > 0$ such that $B_Y(z, r') \subseteq I(a, a^+)$ and hence there is $r > 0$ such that $B_Y(z, r) \subseteq J$. Similarly if $z \in R^*$. Thus we may assume that $z \in X$. We need to consider several different cases.

Suppose that $z \in L \backslash R$. Since J is an open interval of Y, we may assume that $J \cap Y(> z) \subseteq I(z, z^+)$ so that $J \cap X \subseteq X(\leq z)$ is an open neighbourhood of z in X. Since $z \notin R$, $\{z\}$ is not open in X and so there is some element $b \in X(< z)$ such that $(b, z) \cap X \subseteq J$. Thus we may assume that $J = (b, c)$, where $b \in X$ and $b < z < c \in I(z, z^+)$. Since the metric d is compatible with the topology τ on X, there is $r_1 > 0$ such that $B_X(z, r_1) \subseteq J \cap X$. Also, there is $r_2 > 0$ such that $B_Y(z, r_2) \cap Y(\geq z) \subseteq J$. We claim that $B_Y(z, r) \subseteq J$, where $r = \min\{r_1, r_2\}$. Since $B_Y(z, r) \cap X = B_X(z, r)$ we need only show that $B_Y(z, r) \backslash (X \cup I(z, z^+)) \subseteq J$.

Let $y \in B_Y(z, r) \backslash (X \cup I(z, z^+))$. We consider only the case when $y = x_\lambda(a, a^+)$ for some $a \in L$ and $0 < \lambda < 1$; the other case when $y = x_\lambda(a^-, a)$ for some $a \in R$ is similar. Clearly $a < z$ since $B_Y(z, r) \cap Y(\geq z) \subseteq I(z, z^+)$. Suppose that $a < b$. It follows from the definition of d^* (see Proof of Theorem 2.1) that $d^*(y, z) = \lambda' d'(a, z) + \lambda d'(a^+, z) \geq \min\{d'(a, z), d'(a^+, z)\}$. Now $d'(a, z) = d(a, z) \geq r$ since $B_X(z, r) \subseteq \{x \in X : b < x \leq z)$. Also,

$$d'(z, a^+) = \inf_{v \in X(>a)} \sup\{d(z, u) : a < u < v\} \geq r.$$

This is true since if $b < v$, then $\sup\{d(z, u) : a < u < v\} \geq d(z, b) \geq r$, and if $a < v \leq b$, $d(z, u) \geq r$ for all $u \in X$ such that $a < u < v$. Thus $d^*(y, z) \geq r$. This is a contradiction and hence $b \leq a$. It follows that $y \in J$ since $b \leq a <_Y y \leq_Y z$ and b, z are elements in the interval J of Y.

The case $z \in R \backslash L$ is similar. The case $z \in L \cap R$ is simpler since, in this case, $\{z\}$ is open in X, and we may assume that $J \subseteq (z^-, z^+)$ and so there is $r > 0$ such that $B_Y(z, r) \subseteq J$.

Finally, suppose that $z \in X \backslash (L \cup R)$. Since neither $X(\geq z)$ nor $X(\leq z)$ is open, it follows that there are $b, c \in J \cap X$ such that $b < z < c$. Thus we may assume that $J = (b, c)$. Since

$(b,c) \cap X$ is an open neighbourhood of z in X, there is $r > 0$ such that $B_X(z,r) \subseteq J$. Then by a similar argument to the one above it follows that $B_Y(z,r) \subseteq J$.

We now prove the converse, that the metric topology on Y is contained in the linear order topology on Y. We have to show that, for any $z \in Y$ and $r > 0$, there are $b, c \in B_Y(z,r)$ such that $b <_Y z <_Y c$ and $x \in B_Y(z,r)$ whenever $b <_Y x <_Y c$.

If $z \in Y \backslash X$, say $z \in I(a,a^+)$ for some $a \in L$, the result is obvious since, by Corollary 2.2 there is r_1 such that $0 < r_1 < r$ and $B_Y(z,r_1) \subseteq I(a,a^+)$ and $B_Y(z,r_1)$ is an interval in $(Y, \leq_Y)$.

Suppose $z \in X$. We only consider the case when $z \in X \backslash (L \cup R)$; the other cases are similar. Since $x \notin L \cup R$, and since the metric d on X is compatible with the generalized order topology on X, it follows that there are $r > 0$ and $b, c \in X$ such that $b < z < c$ and $\{y \in X : b \leq y \leq c\} \subseteq X \cap B_Y(z, r/2)$ We will show that $d^*(y,z) < r$ holds for all $y \in Y$ such that $b <_Y y <_Y c$. If $y \in X$ this is clear. Suppose $y \in Y \backslash X$, say $y = x_\lambda(a,a^+)$ for some $a \in L$ and $0 < \lambda < 1$. If $a < b$ then we get the contradiction that $y <_Y b$. Therefore, $b \leq a$. Similarly, $a < c$. Hence $d(a,z) \leq r/2$. If $a^+ \in X$, then $b < a^+ \leq c$ and so $d(a^+,z) \leq r/2$; on the other hand, if $a^+ \notin X$ then $d'(a^+,z) \leq \sup\{d(z,u) : u \in X, a < u < z\} \leq r/2$. In any case, $d^*(y,z) = \lambda' d(a,z) + \lambda d'(a^+,z) < r$. This completes the proof of (A).

Proof of (B): Clearly (X, τ) is a subspace of Y and it is closed since the sets $I(a,a^+)$ $(a \in L)$ and $I(a^-,a)$ $(a \in R)$ are open intervals of Y.

For any positive integer n let $\mathcal{U}(n) = \{B_Y(x, \frac{1}{2n}) : x \in X\}$. Then $\mathcal{U}(n)$ is a cover of X by open subsets of Y. Also, for $x \in X$, we have $St(x, \mathcal{U}(n)) \subseteq B_Y(x, \frac{1}{n})$, and so $\bigcap_{n \geq 1} St(x, \mathcal{U}(n)) \subseteq \bigcap B_Y(x, \frac{1}{n}) = \{x\} \subseteq X$. Thus X is a p-embedded closed subset of Y. This completes the proof of the theorem. $\square$

REFERENCES

1. H. Bennett & D.J. Lutzer, Certain hereditary properties and metrizability in generalized ordered spaces, *Fund. Math.* **107** (1980), 71-84.
2. H. Bennett, A metric for metrizable GO-space, *Proc. of the 1984 Topology Conference, Topology Proc.* **9** (1984), 217-225. (MR 87h:54059)
3. H. Bennett, LOTS with S_δ-diagonals, *Proc. of the 1987 Topology Conference, Topology Proc.* **12** (1987), 211-216. (MR 89j:54001)
4. G. Creede, *Semistratifiable spaces and a factorization of a metrization theorem due to Bing,* Dissertation, Arizona State University, Tampe, Ariz. 1968.
5. M.J. Faber, Metrizability of linear ordered spaces, *Topics in Topology, Proc. Colloq. Keszthely* (1972), 257-265. (MR 50#1
6. M.J. Faber, Metrizabilicy in generalized ordered spaces, *Mathematical Centre Tracts* **N53** (1974). (MR 54#6097)
7. V.V. Fedorčuk, Ordered sets and the product of topological spaces, *Vestnik Moskov Univ. Ser. I Mat. Meh.* **21**(1966 66-71.
8. Sze-Tsen Hu, *Theory of Retracts* Wayne State University Press, Berkeley, Calif., 1965.)
9. D.J. Lutzer, A metrization theorem for linear ordered spaces, *Proc. Amer. Math. Soc.* **22** (1969),557-558.
10. D.J. Lutzer, On generalized ordered spaces, *Dissertations Math.* **89** (1971).
11. S. Purisch, The orderability and suborderability of metric spaces, *Trans. Amer. Math. Soc.* **226** (1977), 59-76.
12. J.M. van Wouwe, GO-spaces and generalizations of metrizabilicy, *Mathematical Centre Tracts* **104** (1979). (MR 80m:540
13. J.M. van Wouwe, GO-spaces and (generalized) metrizabilicy, *Mathematical Centre Tracts* **116** (1979). (MR 54#6097)

University of Calgary
Alberta, Canada

The Capital Normal University
100037, Beijing, P. R. of China