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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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ON SOME CHARACTERIZATION OF DARBOUX RETRACTS

RYSZARD JERZY PAWLAK*

ABSTRACT. In this paper, a characterization of Darboux retracts is given. The paper ends with the considerations concerning sufficient conditions for Darboux retracts of given spaces to Borel sets.

The obtaining of many interesting results concerning retracts of topological spaces caused that this notion went through a number of generalizations. Among other things, since the sixties there have been studied situations where as "retractions" discontinuous maps connected with the notion of connectedness were considered ([1], [2], [5], [7]). In 1980 paper [8] appeared in which the authors considered Darboux retracts. This paper started with an example of "neither closed nor locally connected Darboux retract" . Further considerations led to formulating many interesting properties of Darboux retracts of metric spaces. This subject is important because it is closely related to the problems of extending Darboux maps. The relationship between those two questions is reflected by the following two theorems.

Theorem A. *Let X, Y be an arbitrary topological spaces and let $A \subset X$. If A is a Darboux retract of the space X and $f : A \to Y$ is a Darboux function, then f possesses a Darboux extension to the space X (i.e. there exists a Darboux function $f^* : X \to Y$ such that $f^*_{|A} = f$).*

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Theorem B. *Let X be an arbitrary topological space. Then a subset $A \subset X$ is a Darboux retract of the space X if and only if each Darboux function $f : A \to A$ possesses a Darboux extension to the space X.*

The problems of extensions of functions connected with the notion of connectedness have lately been investigated rather intensively by many authors ([4], [9]).

In this paper we shall give a characterization of Darboux retracts. Making use of this characterization, we can show the existence of non-Borel Darboux retracts. The paper ends with considerations concerning sufficient condition for Darboux retracts of given spaces to be Borel sets.

Throughout the paper, we shall consider T_1-spaces only (i.e. topological spaces in which every one-element set is closed). The notions of Darboux function, a Darboux retract and a Darboux retraction are accepted as in [8] (that is, f is a Darboux function if the image of every connected set is connected and a subspace A of a space X is said to be a Darboux retract of X if there exists a Darboux function - called a retraction - $r : X \to A$ such that $r(x) = x$ for each $x \in A$).

Also in the remaining cases we use the classical definitions and notations (as, for example, those from [3] or [6]). In particular, sets A and B of the space X are called functionally separated if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. We say that a set A is a set of type F_σ^* (which is written down as $A \in F_\sigma^*$) if $A = \bigcup_{n=1}^\infty F_n$ where F_n $(n = 1, 2, \dots)$ are closed sets functionally separated from each closed set disjoint from F_n. We say that a set A is of type G_δ^* (written down as $A \in G_\delta^*$) if $X \setminus A \in F_\sigma^*$.

If A is a Darboux retract of the space X, then it is easily noticed that each component of X can contain at most one component of the set A. This last property will be written down in short as $A \Subset X$.

If A is a subset of the topological space X, then the notation $c(A) \leq \mathfrak{c}$ means that each component of the set A has cardinality not greater than the continuum. Let $x_0 \in X$. By $S(X, x_0)$ we denote the component of X to which x_0 belongs.

At last, for the set $A \subset X$ and for $x_0 \in \bar{A}$, let us adopt the notation:

$E_A(x_0)$ is the set of all those $a \in A$ for which:

1^0 if $x_0 \in \bar{C} \setminus C$, then $a \in \bar{C}$ for each connected set $C \subset A$;

2^0 if $A \cap S(X, x_0) \neq \emptyset$, then $a \in S(X, x_0)$.

Lemma. *Let $r : X \to A$ be a Darboux retraction. Then if $x \in \bar{C}$ where C is a connected subset of A, then $r(x) \in \bar{C}$.*

Proof: Suppose to the contrary that $r(x) \notin \bar{C}$. Since r is a Darboux map, therefore $r(C \cup \{x\})$ is a connected set. On the other hand, however, $r(C) = C$, which, in view of the closedness of $\{x\}$ and our supposition, means the disconnectedness of $r(C \cup \{x\})$. The contradiction obtained ends the proof of the lemma.

Theorem 1. *Let $A \neq \emptyset$ be a subset of a space X, such that $c(A) \leq \mathfrak{c}$ and $\bar{A} \in G^*_\delta$. Then A is a Darboux retract of the space X if and only if $A \Subset X$ and, to any $x \in \bar{A} \setminus A$, we may assign an element $z_x \in E_A(x)$ in the way that $(C \cap A) \cup \{z_x : x \in C \setminus A\}$ is a connected set for any connected set $C \subset \bar{A}$.*

Proof: Necessity. Of course, $A \Subset X$. Let $r : X \to A$ be a Darboux retraction. For any $x \in \bar{A} \setminus A$, let us adopt $z_x = r(x)$. Then, in virtue of the Lemma, $z_x \in E_A(x)$. Let further C be any connected subset of $\bar{A}$. Then

$$(C \cap A) \cup \{z_x : x \in C \setminus A\} = r(C),$$

which proves that $(C \cap A) \cup \{z_x : x \in C \setminus A\}$ is really a connected set.

Sufficiency. Let $\{A_t\}_{t \in T}$ be a family of all components of the set A and let, for any $t \in T$, X_t be the component of the space X, containing A_t. Besides, let $\hat{X} = X \setminus \bigcup_{t \in T} X_t$ and let us adopt the notation $A^*_t = \bar{A} \cap X_t$.

Fix $t \in T$. Define first the map $r_t : X_t \to A_t$. For $x \in A_t^*$ let

$$r_t(x) = \begin{cases} x & \text{when } x \in A_t, \\ z_x & \text{when } x \in A_t^* \setminus A_t. \end{cases}$$

If $A_t^* = X_t$, the defining of r_t would be finished. So, let us consider the case when A_t^* is a proper subset of X_t.

Of course $X \setminus \bar{A} = \bigcup_{n=1}^{\infty} F_n$, where $F_n = \bar{F}_n$ $(n = 1, 2, \dots)$, and, for any n, there exists a continuous function $f_n : X \to [0,1]$ such that $f_n(\bar{A}) = \{0\}$ and $f_n(F_n) = \{1\}$. Put $K_n^t = F_n \cap X_t$. Since $X_t \setminus A_t^* = \bigcup_{n=1}^{\infty} K_n^t$, therefore, for infinitely many n, $K_n^t \neq \emptyset$ (if $\{n : K_n^t \neq \emptyset\}$ were finite, $X_t \setminus A_t^*$ would be a closed set and, in consequence, $X_t = A_t^* \cup (X_t \setminus A_t^*)$ would be a disconnected set). So, for simplicity of notation, assume that, for all n, $K_n^t \neq \emptyset$. Put $f_n^t = f_{n|X_t} : X_t \to [0,1]$ $(n = 1, 2, \dots)$. Then f_n^t $(n = 1, 2, \dots)$ is a continuous function with the property that $f_n^t(A_t^*) = \{0\}$ and $f_n^t(K_n^t) = \{1\}$. Let us adopt $g_t = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n^t : X_t \to [0,1]$. Then g_t is a continuous function such that $g_t(A_t^*) = \{0\}$ and $g_t(x) > 0$ for $x \notin A_t^*$.

Consequently, define in $X_t \setminus A_t^*$ the equivalence relation $\vdash\!\dashv$ in the following way:

$$x \vdash\!\dashv y \text{ if and only if } g_t(x) - g_t(y) \in Q$$

where Q denotes the set of rational numbers. This relation partitions the set $X_t \setminus A_t^*$ into disjoint equivalence classes. In view of the connectedness of X_t, we may infer that the number of these classes is continuum. Let P_t be the family of all those equivalence classes and let $\varphi_t : P_t \xrightarrow{onto} A_t$. Let then

$$r_t(x) = \varphi_t([x]) \quad \text{for } x \in X_t \setminus A_t^*,$$

where $[x]$ stands for the equivalence class of the relation $\vdash\!\dashv$ to which x belongs. Then r_t is defined on the entire set X_t.

Moreover, let a_0 be any fixed element of the set A. Then, for $x \in \hat{X}$, let us adopt $\hat{r}(x) = a_0$. Now, define $r : X \to A$ in the following manner: $r = \hat{r} \underset{t \in T}{\nabla} r_t$ (where ∇ denote a combination of maps, see [3], p.99).

Of course, then $r : X \to A$ and $r(x) = x$ for any $x \in A$. Consequently, in order to finish the proof of this theorem, it suffices to demonstrate that r is a Darboux function.

So, let C be any connected set in X. Then C is contained in some component L of the space X. If $L \cap A = \emptyset$, then $r(C) = \{a_0\}$. Thus we may assume that there exists $t_0 \in T$ such that $L = X_{t_o}$. Evidently, if $C \subset A_{t_o}$, then $r(C) = C$. Whereas if $C \subset A^*_{t_0}$, then $r(C) = (C \cap A_{t_0}) \cup \{z_x : x \in C \setminus A_{t_0}\}$; therefore, also in this case, $r(C)$ is a connected set.

So, suppose that $C \setminus A^*_{t_0} \neq \emptyset$. Then there may occur the cases:

1^0 $C \subset p \in P_{t_0}$. Then $r(C) = r_{t_0}(C)$ is a one- element set.

2^0 C cuts at least two equivalence classes belonging to P_{t_0} (this situation comprises the case when $C \setminus A^*_{t_0} \neq \emptyset \neq C \cap A^*_{t_0}$).

We shall show that

$$\forall_{q \in P_{t_0}}, q \cap C \neq \emptyset. \tag{1}$$

Indeed, let $q \in P_{t_0}$. Fix $x_1 \in q$ and let $g_{t_0}(x_1) = \beta$. In view of the assumption on C, there exist two distinct real numbers $\alpha_1, \alpha_2 \in g_{t_0}(C)$. Let ξ be a rational number such that $\beta + \xi \in (\alpha_1, \alpha_2)$. In view of the continuity of g_{t_0} and connectedness of C, we may infer that there exists $c \in C$ such that $g_{t_0}(c) = \beta + \xi$. It is easy to see that $c \in q$, which ends the proof of (1).

In virtue of (1), we may deduce that $r(C) = r_{t_0}(C) = A_{t_0}$, which concludes the proof of the theorem.

By using the above theorem, it is not difficult to construct an example of a non-Borel Darboux retract even if we shall consider normal connected spaces. On the other hand, there exist spaces which possess only Darboux retracts being Borel sets, although they need not be closed sets. An example of a space with this property can be $X = \{(x, y) \in \mathbb{R}^2 : x = 0 \wedge -1 \leq y \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1 \ \wedge \ y = \sin \frac{1}{x}\}$ with the topology generated by the natural topology of the plane. This fact follows, among other things, from the theorem below which will be preceded by the notation:

> Let A and B be any subsets of X. We shall write $A \in \mathbb{R}_B$ if, for any distinct points $x, y \in \bar{A}$ of which at least one belongs to B, there exists a connected set $C \subset A$ such that $x \in \bar{C}$ and $y \notin \bar{C}$.

Theorem 2. *Let X be a connected topological space which possesses an open and dense set V such that $X \setminus V$ is a connected set. Moreover, let the following conditions be satisfied:*

(i) *$C \cap V \in \mathbb{R}_V$ for any connected set $C \subset X$;*
(ii) *$C \in \mathbb{R}_{X\setminus V}$ for any connected set $C \subset X \setminus V$;*
(iii) *if $C \cap V \neq \emptyset \neq C \setminus V$, then $X \setminus V \subset \overline{C \cap V}$ for any connected set $C \subset X$.*

Then each Darboux retract of the space X is a Borel set. If we additionally assume that X is perfectly normal ([3], p. 68), then each of its Darboux retracts is a set of types F_σ and G_δ.

Proof: Suppose that A is a Darboux retract of the space X, and $r : X \to A$ is a Darboux retraction. Of course, A is a connected set. First, note that

(2) $A \cap V$ is a closed subset of the subspace V.

Indeed, let x be an arbitrary element of the closure of the set $A \cap V$ in the subspace V. Then $x \in \overline{A \cap V}$ (that is, x belongs to the closure of the set $A \cap V$ in the space X). Suppose that $x \notin A \cap V$. Then $r(x) \neq x$. On the basis of condition (iii), we may infer that $r(x) \in \overline{A \cap V}$, which, in view of (i) means that there exists a connected set $C \subset A \cap V$ such that $x \in \bar{C}$ and $r(x) \notin \bar{C}$. However, this fact contradicts our Lemma. The contradiction obtained proves that $x \in A \cap V$, which completes the proof of (2).

Now, we shall show that

(3) $A \setminus V$ is a closed subset of the subspace $X \setminus V$.

This fact is obvious when $A \setminus V = \emptyset$. So, we shall further assume that $A \setminus V \neq \emptyset$. Let us consider the cases:

1^0 $A \subset X \setminus V$. Let $x \in \bar{A}$ and suppose that $x \notin A$. Then

$r(x) \neq x$, thus, by condition (ii) there exists a connected set $C \subset A$ such that $x \in \bar{C}$ and $r(x) \notin \bar{C}$, which again leads to a contradiction with the Lemma.

2^0 $A \cap V \neq \emptyset$. Note that, in this case

$$r(X \setminus V) \subset X \setminus V. \tag{4}$$

Indeed, in view of (iii), we have $X \setminus V \subset \overline{A \cap V}$. So, suppose that inclusion (4) is not true. Consequently, there exists an $x \in X \setminus V$ such that $r(x) \in V$. Evidently, $r(x) \neq x$. In virtue of (i), we may deduce that there exists a connected set $C \subset A \cap V$ such that $x \in \bar{C}$ and $r(x) \notin \bar{C}$, which leads to a contradiction with the Lemma. The contradiction obtained proves (4).

In view of (4) we may conclude that $r(X \setminus V) = A \setminus V$, thus $A \setminus V$ is a connected set. So, let x be any element of the closure of the set $A \setminus V$ in the subspace $X \setminus V$. Then $x \in \overline{A \setminus V}$. Suppose that $x \notin A \setminus V$. Proceeding further similarly as above, we come (on the basis of assumption (ii)) to a contradiction with the Lemma. The contradiction obtained ends the proof of (3).

From (2) i (3) one can easily deduce the assertion of the theorem.

PROBLEM. It remains open to characterize those spaces which possess Borel Darboux retracts. However, this question seems rather difficult, therefore it would be interesting to give partial solutions, for instance, sufficient conditions (other than those in the above Theorem) for Darboux retracts to be Borel sets. In connection with this, let us finally note down a simple proposition being a corollary from Theorem 1 and methods applied in the proof of Theorem 2.

PROPOSITION. Let X be a connected T_6-space and let A be a set of cardinality continuum, such that $A \in \mathbb{R}_X$. Then A is Darboux retract of the space X if and only if A is a closed set.

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Institute of Mathematics
Łódź University
Banacha 22, 90-238 Łódź, Poland
e-mail RPAWLAK@Plunlo51