Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol 17, 1992

ON THE HOMOGENEITY OF INFINITE PRODUCTS

FONS van ENGELEN

ABSTRACT. We show that if X is a zero-dimensional metric space that is either first category (in itself) or contains a dense complete subspace, then the countable infinite product X^ω is homogeneous.

1. INTRODUCTION

Question 387 of *Open Problems in Topology* [6] asks for which zero-dimensional subsets X of $\mathbb{R}$ it is true that X^ω is homogeneous. The main purpose of this note is to provide a proof that X^ω is homogeneous if either X is first category in itself (i.e. $X = \cup_{i<\omega} X_i$ with X_i closed and nowhere dense in X), or X contains a dense complete subspace. In fact it will be shown that for arbitrary metrizable X of covering dimension zero (so no separability required) X^ω is even strongly homogeneous in these cases, i.e. every non-empty clopen subset of X^ω is homeomorphic to X^ω.

After I obtained the above theorem, I learned that similar results had been announced by S. V. Medvedev in [4]. As far as I know his proofs have never been published; therefore I have organized this note in such a way that some of the other results announced in [4] follow as well.

Whether or not X^ω is homogeneous for *all* zero-dimensional (separable) metric spaces X remains an interesting open problem.

2. Preliminaries

All spaces are zero-dimensional (in the sense of dim) *and metrizable.*

The choice of metric will always be irrevelant, obvious, or indicated; all metrics are assumed to be bounded by 1.

For standard definitions and terminology, we refer the reader to [2], [3], and [5]. A space is called *nowhere compact* if no non-empty clopen subset is compact. We call a space *weight-homogeneous* if all non-empty open subspaces have the same weight. We write $X \approx Y$ (resp. $h : X \approx Y$) if X and Y are homeomorphic (resp. if h is a homeomorphism between X and Y). If $\mathcal{U}$ is a family of subsets of a space, then $\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}(U){:}U \in \mathcal{U}\}$.

In our proofs, we will need a homeomorphism extension theorem (Proposition 2.3) which is a generalization of Lemma 3.2.2 of [1] to the non-separable situation.

2.1 Lemma. *Suppose X is a non-compact weight-homogeneous space of weight κ. Then X can be written as the disjoint union of κ non-empty clopen subsets.*

Proof: Since X is non-compact we can write X as a disjoint union $\bigcup_{j<\omega} X_j$ with X_j clopen and non-empty. Now if $\kappa = \omega$ we are done, so assume that $\kappa > \omega$. Let $\mathcal{U}_i$ be a disjoint covering of X by clopen sets of diameter less than $1/i$. Then $|\bigcup_i \mathcal{U}_i| = \kappa$. If some $\mathcal{U}_i$ has cardinality κ, then we are done, so assume $|\mathcal{U}_i| < \kappa$ for each i. Let $\mathcal{V}^j_k$ be a disjoint covering of X_j by clopen sets of diameter less than $1/k$; since $w(X_j) = \kappa > \omega$ we must have $|\mathcal{V}^j_{k_j}| \geq |\mathcal{U}_j|$ for some k_j. Then $\bigcup_{j<\omega} \mathcal{V}^j_{k_j}$ is the required covering of X. □

2.2 Definition: *Let A be a closed nowhere dense subset of the space X, and let B be a closed nowhere dense subset of the space Y. Suppose that $h : A \to B$ is a homeomorphism, $\{U_\alpha : \alpha < \kappa\}$ is a covering of $X - A$ by disjoint non-empty clopen subsets of X, and $\{V_\alpha : \alpha < \kappa\}$ is a covering of $Y - B$ by disjoint non-empty clopen subsets of Y. Then $\{(U_\alpha, V_\alpha) : \alpha <$*

$\kappa\}$ is a KR-covering for $(X - A, Y - B, h)$ if, whenever $D_\alpha \subset U_\alpha$, $R_\alpha \subset V_\alpha$, and $h_\alpha : D_\alpha \to R_\alpha$ is a bijection for each $\alpha < \kappa$, the combination mapping $\tilde{h} = h \cup \bigcup_{\alpha<\kappa} h_\alpha$ is continuous in points of A, and $\tilde{h}^{-1}$ is continuous in points of B.

2.3 Theorem: *Let X and Y be nowhere compact weight-homogeneous spaces of weight κ. Let A and B be closed nowhere dense subsets of X and Y, respectively, and suppose that $h : A \to B$ is a homeomorphism. then there exists a KR-covering $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ for $(X - A, Y - B, h)$.*

Proof: The case where A and B are empty is trivial so assume $A \neq \emptyset \neq B$. Let $\mathcal{P}$ be a disjoint covering of $X - A$ by clopen subsets of X such that, for each $P \in \mathcal{P}$, $\mathrm{diam}(P) < d(P, A)$; then $\mathcal{P} = \bigcup_{i<\omega} \mathcal{P}_i$ where

$$\mathcal{P}_i = \{P \in \mathcal{P} : d(P, A) \in (2^{-i-1}, 2^{-i}]\}.$$

Similarly, we let $\mathcal{Q}$ be a disjoint covering $Y - B$ by clopen subsets of Y such that, for each $Q \in \mathcal{Q}$, diam $(Q) < d(Q, B)$; also define

$$\mathcal{Q}_i = \{Q \in \mathcal{Q} : d(Q, B) \in (2^{-i-1}, 2^{-i}]\}.$$

With each $P \in \mathcal{P}$ we will associate $Q_P \in \mathcal{Q}$ and $a_P \in A$, and with each $Q \in \mathcal{Q}$ we will associate $P_Q \in \mathcal{P}$ and $b_Q \in B$, such that, putting $b_P = h(a_P)$ and $a_Q = h^{-1}(b_Q)$, the following hold:

(i) $d(P, a_P) < 2d(P, A)$ and $Q_P \subset B(b_P, d(P, A))$;
(ii) $d(Q, b_Q) < 2d(Q, B)$ and $P_Q \subset B(a_Q, d(Q, B))$;
(iii) for each $i \in \omega$, the families $\{Q_P : P \in \mathcal{P}_i\}$ and $\{P_Q : Q \in \mathcal{Q}_i\}$ are discrete.

We will describe the construction of the sets Q_P and the points a_P; the construction of the P_Q and b_Q is analogous.

Fix $i \in \omega$. Let $\mathcal{W}$ be a disjoint clopen covering of Y with $\mathrm{mesh}(\mathcal{W}) < 2^{-i-2}$. With each $W \in \mathcal{W}$ such that $W \cap B \neq \emptyset$ we associate $Q_W \in \mathcal{Q}$ as follows. Take $b \in B$ and $\delta > 0$ such that $B(b, \delta) \subset W$. Since B is nowhere dense in Y there exists $Q_W \in \mathcal{Q}$ such that $Q_W \cap B(b, \frac{1}{2}\delta) \neq \emptyset$. Since diam

$(Q_W) < d(Q_W, B) < \frac{1}{2}\delta$ we have $Q_W \subset B(b,\delta) \subset W$. Now take $P \in \mathcal{P}_i$. Find $a_P \in A$ such that $d(P, a_P) < 2d(P, A)$ (so the first part of (i) is satisfied) and let $b_P = h(a_p)$. Then $b_P \in W$ for some $W \in \mathcal{W}$, and we can define $Q_P = Q_W$. To verify the second part of (i), note that $d(b_P, Q_W) < \text{diam}(W) < 2^{-i-2}$ and $\text{diam}(Q_W) < \text{diam}(W) < 2^{-i-2}$ since $Q_W \subset W$, hence $Q_P = Q_W \subset B(b_P, 2^{-i-1}) \subset B(b_P, d(P, A))$ since $d(P, A) > 2^{-i-1}$. Clearly, (iii) is satisfied because each $W \in \mathcal{W}$ contains at most one element of $\{Q_P : P \in \mathcal{P}_i\}$ (although this element Q_W can be Q_P for many different $P \in \mathcal{P}_i$).

Since each $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ is non-compact and weight-homogeneous, we can find, using Lemma 2.1, disjoint clopen families $\mathcal{U}_P$ and $\mathcal{V}_Q$ of size κ such that $P = \bigcup \mathcal{U}_P$ and $Q = \bigcup \mathcal{V}_Q$. Put

$$\mathcal{U} = \bigcup\{\mathcal{U}_P : P \in \mathcal{P}\}, \ \mathcal{V} = \bigcup\{\mathcal{V}_Q : Q \in \mathcal{Q}\},$$

and well-order $\mathcal{U}$ and $\mathcal{V}$ in type κ. Inductively, we will define bijections $\kappa \to \mathcal{U}, \kappa \to \mathcal{V}$, with the image of α denoted by U_α resp V_α, together with points $a_\alpha \in A$ and $b_\alpha = h(a_\alpha) \in B$.

First assume that α is even. Put $U_\alpha = \min(\mathcal{U} - \{U_\beta : \beta < \alpha\})$, and take $P \in \mathcal{P}$ containing U_α. Since Q_P contains κ elements of $\mathcal{V}$ we can choose $V_\alpha \subset Q_p$ such that $V_\alpha \in \mathcal{V} - \{V_\beta : \beta < \alpha\}$. Define $a_\alpha = a_P$ and $b_\alpha = h(a_\alpha) = b_P$. If α is odd we similarly put $V_\alpha = \min(\mathcal{V} - \{V_\alpha : \alpha < \beta\})$, take $Q \in \mathcal{Q}$ containing V_α, and choose $U_\alpha \subset P_Q$ such that $U_\alpha \in \mathcal{U} - \{U_\beta : \beta < \alpha\}$; also $b_\alpha = b_Q$ and $a_\alpha = h^{-1}(b_\alpha) = a_Q$. Clearly, $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ and $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ are coverings of $X - A$ and $Y - B$ by non-empty disjoint clopen subsets of X and Y, respectively. We claim that $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ is the required KR-covering.

First we note that

(iv) if α is even then $d(U_\alpha, a_\alpha) < 3d(U_\alpha, A)$ and $d(V_\alpha, b_\alpha) < d(U_\alpha, A)$;

(v) if α is odd then $d(V_\alpha, b_\alpha) < 3d(V_\alpha, B)$ and $d(U_\alpha, a_\alpha) < d(V_\alpha, B)$.

Indeed, if α is even and $U_\alpha \subset P$ then

$$d(U_\alpha, a_\alpha) = d(U_\alpha, a_P) \leq d(P, a_P) + \text{ diam } (P).$$

But $d(P, a_P) < 2d(P, A)$ by (i) and $\text{diam}(P) < d(P, A)$, so

$$d(U_\alpha, a_\alpha) < 3d(P, A) \leq 3d(U_\alpha, A).$$

Since $V_\alpha \subset Q_P$ we have, again by (i), that $V_\alpha \subset B(b_P, d(P, A))$ so

$$d(V_\alpha, b_\alpha) = d(V_\alpha, b_P) \leq d(P, A) \leq d(U_\alpha, A),$$

establishing (iv). The proof of (v) is similar.

Now assume that, for each $\alpha < \kappa, D_\alpha \subseteq U_\alpha$, $R_\alpha \subset V_\alpha$, and $h_\alpha : D_\alpha \to R_\alpha$ is a bijection, and put $\tilde{h} = h \cup \bigcup_{\alpha<\kappa} h_\alpha$. Let $a \in A$, and take a sequence $(x_n)_n$ converging to a. Since $\tilde{h}|A : A \approx B$ we may assume that $x_n \in U_{\alpha_n}$ for some $\alpha_n < \kappa$, for all $n < \omega$. Let $P_n \in \mathcal{P}$ and $Q_n \in \mathcal{Q}$ be such that $U_{\alpha_n} \subset P_n$ and $V_{\alpha_n} \subset Q_n$. Then

(vi) $\text{diam}(U_{\alpha_n}) < d(U_{\alpha_n}, A) \to 0$ and
$\text{diam}(V_{\alpha_n}) < d(V_{\alpha_n}, B) \to 0$.

Indeed, $\text{diam}(U_{\alpha_n}) < \text{diam}(P_n) < d(P_n, A) \leq d(U_{\alpha_n}, A) \leq d(x_n, a) \to 0$, and similarly $\text{diam}(V_{\alpha_n}) < d(V_{\alpha_n}, B)$. Furthermore, $d(V_{\alpha_n}, B) \leq d(Q_n, B) +$ $\text{diam}(Q_n)$ so since $\text{diam}(Q_n) < d(Q_n, B)$ it suffices to show that $d(Q_n, B) \to 0$. Assume towards a contradiction that for some i the set $I = \{n < \omega : d(Q_n, B) \in (2^{-i-1}, 2^{-i}]\} = \{n < \omega : Q_n \in \mathcal{Q}_i\}$ is infinite. Now if α_n is even the $Q_n = Q_{P_n} \subset B(b_{P_n}, d(P_n, A))$ by (i), whence $d(Q_n, B) < d(P_n, A)$. But $d(P_n, A) \to 0$ so it must be the case that α_n is odd for infinitely many $n \in I$. For these n we have that $P_n = P_{Q_n}$, and we find that

$$\begin{aligned} a &\in (\bigcup\{P_n : n \in I, \alpha_n \text{ odd }\})^- \subset (\bigcup\{P_Q : Q \in \mathcal{Q}_i\})^- \\ &= \bigcup\{P_Q : Q \in \mathcal{Q}_i\} \subset X - A, \end{aligned}$$

with closures being taken in X, and the equality being true because of (iii). We have a clear contradiction, and (vi) has been proved.

Put $a_n = a_{\alpha_n}$ and $b_n = b_{\alpha_n} = h(a_n)$. We find that

$$\begin{aligned} d(a_n, a) &\le d(a_n, x_n) + d(x_n, a) \\ &\le d(a_n, U_{\alpha_n}) + \operatorname{diam}(U_{\alpha_n}) + d(x_n, a), \end{aligned}$$

and by (iv) and (v)

$$d(a_n, U_{\alpha_n}) < \max\{3d(U_{\alpha_n}, A), d(V_{\alpha_n}, B)\},$$

so $d(a_n, a) \to 0$ by (vi). Then also $b_n = h(a_n) \to h(a) = b$, whence

$$\begin{aligned} d(\tilde{h}(x_n), \tilde{h}(a)) &= d(\tilde{h}(x_n), b) \le d(\tilde{h}(x_n), b_n) + d(b_n, b) \\ &\le d(V_{\alpha_n}, b_n) + \operatorname{diam}(V_{\alpha_n}) + d(b_n, b) \to 0 \end{aligned}$$

using (vi) and the fact that by (iv) and (v),

$$d(V_{\alpha_n}, b_n) < \max\{3d(V_{\alpha_n}, B), d(U_{\alpha_n}, A)\}.$$

Thus, $\tilde{h}(x_n) \to \tilde{h}(a)$ proving that $\tilde{h}$ is continuous in points of A. The proof that $\tilde{h}^{-1}$ is continuous in points of B is completely analogous. □

2.4 Definition: *An associated decomposition of the space X is a pair $(G, (X_i)_i)$ consisting of a dense complete subspace G of X (with complete metric ρ) and a sequence $\{X_i : i < \omega\}$ of closed subsets of X such that*

(i) $X - G = \bigcup_{i<\omega} X_i$ *and*
(ii) $X - X_i = \bigcup \mathcal{U}_i$, *where $\mathcal{U}_i$ is a disjoint family of clopen subsets of X such that ρ-mesh$(\mathcal{U}_i|G) \to 0$*

Note that the sets X_i in the above definition will automatically be nowhere dense.

2.5 Proposition: *If X has a dense complete subspace G, then X has an associated decomposition $(G, (X_i)_i)$.*

Proof: For each i let $\mathcal{V}_i$ be a disjoint covering of G by (relatively) open sets of ρ-diameter less that $1/i$. It is well-known that $\mathcal{V}_i$ can be extended to a disjoint open family (but not necessarily covering) $\tilde{\mathcal{U}}_i$ in X. Since G is a G_δ in X we can write $X - G = \bigcup_{i<\omega} Y_i$ with Y_i closed in X. Now put $X_i =$

$Y_i \cup (X - \bigcup \tilde{\mathcal{U}}_i)$, and define $\mathcal{U}_i$ to be a refinement of $\tilde{\mathcal{U}}_i|(X - X_i)$ by disjoint clopen subsets of X. □

3. Main Lemmas

3.1 Lemma: *Let X and Y be weight-homogeneous spaces of weight κ such that $X = \bigcup_{i<\omega} X_i$ and $Y = \bigcup_{i<\omega} Y_i$ with X_i (resp. Y_i) closed and nowhere dense in X (resp. Y). Suppose furthermore that every non-empty clopen subset of X (resp. Y) contains a closed nowhere dense copy of each X_i (resp. Y_i). Then $X \approx Y$.*

Proof: Since we are going to use Theorem 2.3, first note that X and Y (and hence all clopen subspaces) are nowhere compact. Without loss of generality we may assume that $X_i = \emptyset$ (resp. $Y_i = \emptyset$) if i is odd (resp. even).

We will define, for each $n < \omega$, closed nowhere dense subsets A_n and B_n of X and Y, a homeomorphism $h_n : A_n \approx B_n$, and KR-coverings $\{(U_\alpha^n, V_\alpha^n) : \alpha < \kappa\}$ for $(X - A_n, Y - B_n, h_n)$ such that the following conditions are satisfied for all $n < \omega$.

(i) $X_n \subset A_n \subset A_{n+1}$ and $Y_n \subset B_n \subset B_{n+1}$;
(ii) $h_{n+1}|A_n = h_n$;
(iii) for each $\alpha < \kappa$, $h_{n+1}[U_\alpha^n \cap A_{n+1}] = V_\alpha^n \cap B_{n+1}$;
(iv) for each $\alpha < \kappa$, there exists $\beta < \kappa$ such that $U_\alpha^{n+1} \subset U_\beta^n$ and $V_\alpha^{n+1} \subset V_\beta^n$.

Supposing this can be done, it follows form (i) and (ii) that $h = \bigcup_{n<\omega} h_n$ is a well-defined bijection form X to Y. To show that h is a homeomorphism it suffices to show, by the definition of KR-covering, that for each $n < \omega$, $h[U_\alpha^n] = V_\alpha^n$. We only show that $h[U_\alpha^n] \subset V_\alpha^n$. Take $x \in U_\alpha^n$, then for some $k \geq n$, $x \in A_{k+1} - A_k$ and hence $x \in U_\beta^k$ for some $\beta < \kappa$. By (iv) we must have $U_\beta^k \subset U_\alpha^n$, whence also $V_\beta^k \subset V_\alpha^n$. Thus, by (iii), $h(x) = h_{k+1}(x) \in h_{k+1}[U_\beta^k \cap A_{k+1}] \subset V_\beta^k \subset V_\alpha^n$, and we are done.

For the construction it will be convenient to take $h_{-1} = A_{-1} = B_{-1} = \emptyset$, and to let $\{(U_\alpha^{-1}, V_\alpha^{-1}) : \alpha < \kappa\}$ be a KR-covering for $(X, Y, \emptyset)$. Now let $n = -1$ or $n < \omega$ and assume that A_n, B_n, h_n and $\{(U^n_\alpha, V^n_\alpha) : \alpha < \kappa\}$ have been constructed. For each $\alpha < \kappa$, if n is odd put $A^n_\alpha = U^n_\alpha \cap X_{n+1}$ and let B^n_α be a closed nowhere dense copy of A^n_α in V^n_α (note that V^n_α is non-empty by the definition of *KR*-covering!); and if n is even put $B^n_\alpha = V^n_\alpha \cap Y_{n+1}$ and let A^n_α be a closed nowhere dense copy of B^n_α in U^n_α. In both cases put $h^n_\alpha : A^n_\alpha \approx B^n_\alpha$. Apply Theorem 2.3 to obtain *KR*-coverings $\{(U^n_{\alpha,\beta}, V^n_{\alpha,\beta}) : \beta < \kappa\}$ for $(U^n_\alpha - A^n_\alpha, V^n_\alpha - B^n_\alpha, h^n_\alpha)$. Define

$$A_{n+1} = A_n \cup \bigcup_{\alpha<\kappa} A^n_\alpha, \; B_{n+1} = B_n \cup \bigcup_{\alpha<\kappa} B^n_\alpha, \text{ and}$$

$$h_{n+1} = h_n \cup \bigcup_{\alpha<\kappa} h^n_\alpha.$$

It is clear that A_{n+1} and B_{n+1} are closed and nowhere dense sets containing A_n and X_{n+1} resp. B_n and Y_{n+1}, and using the fact that $\{(U^n_\alpha, V^n_\alpha) : \alpha < \kappa\}$ is a *KR*-covering for $(X - A_n, Y - B_n, h_n)$ it is easily verified that h_{n+1} is a homeomorphism. Furthermore, $h_{n+1}[U^n_\alpha \cap A_{n+1}] = h_{n+1}[A^n_\alpha] = h^n_\alpha[A^n_\alpha] = B^n_\alpha = V^n_\alpha \cap B_{n+1}$ proving (iii), and (iv) is also readily obtained. Since $\{(U^n_{\alpha,\beta}, V^n_{\alpha,\beta}) : \alpha, \beta < \kappa\}$ can be reindexed as $\{(U^{n+1}_\alpha, V^{n+1}_\alpha) : \alpha < \kappa\}$, it remains to show that the former family is a *KR*-covering for $(X - A_{n+1}, Y - B_{n+1}, h_{n+1})$. So suppose that $D_{\alpha,\beta} \subset U^n_{\alpha,\beta}$, $R_{\alpha,\beta} \subset V^n_{\alpha,\beta}$, and $h_{\alpha,\beta} : D_{\alpha,\beta} \to R_{\alpha,\beta}$ is a bijection for each $\alpha < \kappa$; put $\tilde{h}_\alpha = \bigcup_{\beta<\kappa} h_{\alpha,\beta}$ and $\tilde{h} = h_{n+1} \cup \bigcup_{\alpha<\kappa} \tilde{h}_\alpha$. Since $\{(U^n_{\alpha,\beta}, V^n_{\alpha,\beta}) : \beta < \kappa\}$ is a *KR*-covering for $(U^n_\alpha - A^n_\alpha, V^n_\alpha - B^n_\alpha, h^n_\alpha)$ and $\tilde{h}$ is $h^n_\alpha \cup \tilde{h}_\alpha$ on U^n_α, $\tilde{h}$ (resp. $\tilde{h}^{-1}$) is continuous in points of A^n_α (resp. B^n_α). Finally note that $\tilde{h} = h_n \cup \bigcup_{\alpha<\kappa}(h^n_\alpha \cup \tilde{h}_\alpha)$, so since $h^n_\alpha \cup \tilde{h}_\alpha$ is a bijection between subsets of U^n_α and V^n_α, and $\{(U^n_\alpha, V^n_\alpha) : \alpha < \kappa\}$ is a *KR*-covering for $(X - A_n, Y - B_n, h_n)$ we find that $\tilde{h}$ is continuous in points of A_n, and $\tilde{h}^{-1}$ is continuous in points of B_n. □

3.2 Lemma: *Let X and Y be nowhere compact spaces with associated decompositions $(G, (X_i)_i)$ and $(H, (Y_i)_i)$. Suppose furthermore that every non-empty clopen subset of X (resp.*

Y) contains a closed nowhere dense copy of each X_i (resp. Y_i). Then $X \approx Y$.

Proof: Fix complete metrics ρ_1 on G and ρ_2 on H. Let $\mathcal{U}_i$ (resp. $\mathcal{V}_i$) be a disjoint family of clopen subsets of X (resp. Y) as in Definition 2.4, such that ρ_1-mesh$(\mathcal{U}_i|G) \to 0$ (resp. ρ_2-mesh$(\mathcal{V}_i|H) \to 0$). Without loss of generality we may assume that for each $n < \omega, X_{2n} = X_{2n+1}$, $\mathcal{U}_{2n} = \mathcal{U}_{2n+1}$, $Y_0 = \emptyset$, $\mathcal{V}_0 = \{Y\}, Y_{2n+1} = Y_{2n+2}$, and $\mathcal{V}_{2n+1} = \mathcal{V}_{2n+2}$. Also put $\mathcal{V}_{-1} = \mathcal{V}_0$. Now construct A_n, B_n, h_n and $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ exactly as in the proof of Lemma 3.1, but subject to the further condition

(*) for each $\alpha < \kappa, \{U_\alpha^n : \alpha < \kappa\}$ refines $\mathcal{U}_n$ and $\{V_\alpha^n : \alpha < \kappa\}$ refines $\mathcal{V}_n$.

This is accomplished as follows. If the construction has just been carried out for $n+1$ as above, and n is odd, then $X_{n+1} \subset A_{n+1}$ so $X - A_{n+1} \subset \bigcup \mathcal{U}_{n+1}$, and we can replace each U_α^{n+1} by a family $\mathcal{U}_\alpha^{n+1}$ of non-empty disjoint clopen subsets of X which refines $\mathcal{U}_{n+1}$. Because of nowhere compactness and weight-homogeneity we can apply Lemma 2.1 and replace the corresponding V_α^{n+1} by a family $\mathcal{V}_\alpha^{n+1}$ of non-empty disjoint clopen subsets of Y such that $|\mathcal{U}_\alpha^{n+1}| = |\mathcal{V}_\alpha^{n+1}|$ (in face, we could choose this cardinality to be κ). Reindexing $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha^{n+1}$ and $\bigcup_{\alpha<\kappa} \mathcal{V}_\alpha^{n+1}$, the new $\{(U_\alpha^{n+1}, V_\alpha^{n+1}) : \alpha < \kappa\}$ still satisfies the "old" requirements. But also, by definition $\{U_\alpha^{n+1} : \alpha < \kappa\}$ refines $\mathcal{U}_{n+1}$, and by (iv) and the inductive hypothesis $\{V_\alpha^{n+1} : \alpha < \kappa\}$ refines $\mathcal{V}_n = \mathcal{V}_{n+1}$. If n is even a similar argument applies.

Define $M = \{\sigma \in \kappa^\omega : \text{for all } n < \omega, U_{\sigma(n+1)}^{n+1} \subset U_{\sigma(n)}^n\}$. If $\sigma \in M$ then there exists a unique point $x_\sigma \in \bigcap_{n<\omega} U_{\sigma(n)} \cap G$: indeed, ρ_1 is a complete metric on G and, by (*), ρ_1-diam$(U_{\sigma(n)}^n \cap G) \le \rho_1$-mesh$(\mathcal{U}_n) \to 0$. Since each $\{U_\alpha^n : \alpha < \kappa\}$ is disjoint, different σ yield different x_σ. Also, $x_\sigma \in X - A$, and every $x \in X - A$ is some x_σ because of (iv) and the fact that $X - A = \bigcap_{n<\omega} \bigcup \{U_\alpha^n : \alpha < \kappa\}$. Again by (iv), $M = \{\sigma \in \kappa^\omega$: for all $n < \omega, V_{\sigma(n+1)}^{n+1} \subset V_{\sigma(n)}^n\}$ so there is a similar one-to-one correspondence $\sigma \to y_\sigma$ between M and

$Y - B$. If we define $g : X - A \to Y - B$ by $g(x_\sigma) = y_\sigma$ then clearly $g[U^n_\alpha \cap (X - A)] = V^n_\alpha \cap (Y - B)$. Endowing $X - A \subset G$ with ρ_1 and $Y - B \subset H$ with ρ_2, we see from ρ_1-diam$(U^n_\alpha \cap G) \leq \rho_1$-mesh$(\mathcal{U}_n) \to 0$ and ρ_2-diam$(V^n_\alpha \cap G) \leq \rho_2$-mesh$(\mathcal{V}_n) \to 0$ that $\{U^n_\alpha \cap (X - A) : n < \omega,\ \alpha < \kappa\}$ and $\{V^n_\alpha \cap (Y - B) : n < \omega,\ \alpha < \kappa\}$ are bases for $X - A$ and $Y - B$, whence g is a homeomorphism.

Finally, define $h = g \cup \bigcup_{n<\omega} h_n$. Since $\{(U^n_\alpha, V^n_\alpha) : \alpha < \kappa\}$ is a *KR*-covering for $(X - A_n, Y - B_n, h_n)$ and $g : U^n_\alpha \cap (X - A) \approx V^n_\alpha \cap (Y - B)$, the map $g \cup h_n$ is a homeomorphism. This implies that h is a homeomorphism since $X - A$ and $Y - B$ are dense in X and Y □.

4. Main Theorems

4.1 Theorem: *Let X be first category (in itself), such that every non-empty clopen subset of X contains a closed copy of X. Then*

(a) X is strongly homogeneous;

(b) $X \approx \mathbb{Q} \times X \approx (\omega + 1) \times X$; in particular, X contains a closed nowhere dense copy of X.

Proof: Put $X = \bigcup_{i<\omega} X_i$ with X_i closed and nowhere dense X. (a) Let U be a non-empty clopen subset of X. Then $U = \bigcup_{i<\omega} U_i$ where $U_i = U \cap X_i$, and U_i is closed and nowhere dense in X. Now every non-empty clopen subset of U, being also clopen in X, contains a closed copy of X, and hence closed nowhere dense copies of each X_i; and similarly each non-empty clopen subset of X contains closed nowhere dense copies of the U_i. By Lemma 3.1, $U \approx X$.

(b) Write $\mathbb{Q} \times X = \bigcup_{i<\omega} \bigcup_{q\in\mathbb{Q}}(\{q\} \times X_i)$. Then each non-empty clopen subset of X contains closed nowhere dense copies of each $\{q\} \times X_i \approx X_i$, and each non-empty clopen subset of $\mathbb{Q} \times X$ contains a clopen subset of X as a closed subset and thus contains closed nowhere dense copies of each X_i. By Lemma 3.1, $\mathbb{Q} \times X \approx X$. The proof for $(\omega + 1) \times X$ is similar. □.

4.2 Theorem: *Let X be first category (in itself). Then X^ω is strongly homogeneous.*

Proof: Note that X^ω is first category, and that every non-empty clopen subset of X^ω contains a closed copy of X^ω. Now apply Theorem 4.1(a). □

4.3 Theorem: *Let X contain a dense complete subspace, and suppose that every non-empty clopen subset of X contains a closed copy of X. Then*

(a) X is strongly homogeneous;

(b) if X contains more than one point then $X \approx (\omega+1)\times X$; in particular, X contains a closed nowhere dense copy of X.

Proof: If some non-empty clopen subset of X is compact, then X itself is compact and it easily follows that either X has one point or $X \approx 2^\omega$, in which cases the theorem holds. So assume that X is nowhere compact. Applying Proposition 2.5, fix an associated decomposition $(G,(X_i)_i)$ for X, and let ρ and $\mathcal{U}_i$ be as in Definition 2.4.

(a) Let U be a non-empty clopen subset of X. It is easily verified that $(G\cap U,(X_i\cap U)_i)$ is an associated decomposition for U (endow $G\cap U$ with the restriction of ρ). Put $U_i = X_i \cap U$. Since each every non-empty clopen subset of X contains a closed copy of X, it also contains closed nowhere dense copies of all U_i; and similarly every non-empty clopen subset of U contains closed nowhere dense copies of all X_i. By Lemma 3.2, $X \approx U$.

(b) It is easily verified that $((\omega+1)\times G,((\omega+1)\times X_i)_i)$ is an associated decomposition for $(\omega+1)\times X$ (take e.g. the max-metric on $(\omega+1)\times G$). For an application of Lemma 3.2 it suffices to show that X contains a closed nowhere dense copy of each $(\omega+1)\times X_i$. Let Y be the subspace $(\omega\times X)\cup(\{\omega\}\times X_i)$ of $(\omega+1)\times X$. Since Y clearly contains a closed nowhere dense copy of $(\omega+1)\times X_i$, it suffices to prove that $X\approx Y$. Since every non-empty clopen subset of X or Y contains a closed copy of X, both X and Y are weight-homogeneous, of the same weight. Put $A = X_i$, $B = \{\omega\}\times X_i$, and let $h: A \to B$ be any homeomorphism. Then by Proposition 2.3 there exists a *KR*-covering $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ for $(X - A, Y - B, h)$. However, for each $\alpha < \kappa, U_\alpha \approx X$ since X is strongly homogeneous by (a), and $V_\alpha \approx \omega \times X \approx X$ since X is in addition non-compact. Thus, there exist homeomorphisms $h_\alpha : U_\alpha \to V_\alpha$, and by the definition of *KR*-covering, $\tilde{h} = h \cup \bigcup_{\alpha<\kappa} h_\alpha : X \to Y$ is a homeomorphism. □

4.4 Theorem: *Let X contain a dense complete subspace. Then X^ω is strongly homogeneous.*

Proof: Note that X^ω contains a dense complete subspace and that every non-empty clopen subspace of X^ω contains a closed copy of X^ω. Now apply Theorem 4.3. □

4.5 Corollary: *Let Γ be one of the classes Δ^1_n, Π^1_n, or Σ^1_n. If Det(Γ) and $X \in \Gamma$ then X^ω is strongly homogeneous.*

Proof: By Moschovakis [7], X has the Baire property. Thus, either X contains a dense complete subset or some non-empty open subset of X is first-category (see Kuratowski [3], §11.IV). In the former case, Theorem 4.4 applies, and in the latter case X^ω is first category whence $X^\omega \approx (X^\omega)^\omega$ is strongly homogeneous by Theorem 4.2. □

It follows from Corollary 4.5 that X^ω is homogeneous if X is a (separable) absolute Borel set. This has been proved independently by Gruenhage and Zhou (and possibly others). They also proved corollary 4.5 for Σ^1_1, and the following corollary to Theorem 4.2: $(MA + \neg CH)$ if $|X| \leq \omega_1$ then X^ω is homogeneous. Furthermore, they showed that X^ω is homogeneous if X is zero-dimensional and first-countable and contains a dense set of isolated points.

References

[1] F. van Engelen, *Homogeneous zero-dimensional absolute Borel sets*, CWI Tract 27, 1986.

[2] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[3] K. Kuratowski, *Topology I*, Academic Press, New York, 1966.

[4] S. V. Medvedev, *Characterization of h-homogeneous spaces*, Interim Report of the Prague Top. Symp. 2, 1987.
[5] J. van Mill, *Infinite-dimensional topology: prerequisites and introduction*, North-Holland, Amsterdam, 1989.
[6] J. van Mill and G. M. Reed (eds.), *Open problems in topology*, North-Holland, Amsterdam, 1990.

Erasmus Universiteit Rotterdam
Econometrisch Instituut
Postbus 1738
3000 DR Rotterdam, The Netherlands