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LINEARLY ORDERED ZERO-DIMENSIONAL COMPACT SPACES AS REMAINDERS

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ABSTRACT. A concept of a GDA-space over a linearly ordered set is introduced and applied to a direct method of constructing a compactification whose remainder is a fixed zero-dimensional linearly ordered compact space.

INTRODUCTION

It is well known that, for any Tikhonov space Y, there exists a Tikhonov space X with $\beta X\backslash X$ homeomorphic to Y (cf.[1: 4.18]). However, the general problem of finding an internal characterization of spaces which have compactifications whose remainders are homeomorphic to a fixed Tikhonov space Y is difficult. Various authors discovered conditions on a locally compact space X which guarantee that members of a certain class of compact spaces are remainders of X (cf. for instance [2-4,7,8]). The fact that Y is a remainder of a locally compact space X is usually proved by using a theorem of Magill, i.e. by showing that Y is a continuous image of $\beta X\backslash X$ (cf. [6; Thm. 2.1]). There are not too many methods of adding Y to X in order to compactify X. In the present paper we introduce a concept of a generalized double-arrow space (abbr. a GDA-space) over a linearly ordered set and observe that all GDA-spaces over the same set are homeomorphic. It occurs that a zero-dimensional compact Hausdorff space is linearly ordered if and only if it is a GDA-space over some set. This property leads us to describing those locally compact spaces X which have remainders homeomorphic to a fixed linearly ordered zero-dimensional compact space Y, and to giving a direct method of constructing a compactification αX of X with $\alpha X \backslash X = Y$. Our results generalize those obtained by Hatzenbuhler and Mattson in [3-4]. All the spaces considered here are assumed to be completely regular and Hausdorff.

GENERALIZED DOUBLE-ARROW SPACES

In what follows, S denotes a linearly ordered set with a minimal element p and a maximal element q where $p \neq q$.

1. Definition. *Suppose that Y is a compact Hausdorff space which has a collection $\{U_s : s \in S\}$ of clopen sets satisfying the following conditions:*

I. $U_p = \emptyset$ *and* $U_q = Y$;

II. $U_s \subset U_t$ *for* $s < t$ $(s, t \in S)$;

III. *the family* $\{U_t \backslash U_s : s, t \in S \ \& \ s < t\}$ *forms an open base for* Y.

Then we shall call Y a generalized double arrow-space over S (abbr. a GDA(S)-space)

2. Theorem. *Any linearly ordered zero-dimensional compact space X is a generalized double-arrow space over some set.*

Proof: Let $X = [a, b]$ and let $p \notin X$. Say that $p < x$ for any $x \in X$. Considering the set $T = \{x \in X : x$ has an immediate successor$\} \cup \{p, b\}$ with the order inherited from that of X, and putting $U_t = \{x \in X : x \leq t\}$ for $t \in T$, we see that X is a GDA(T)-space. □

From now on, Y will be a fixed GDA(S)-space, and $\{U_s : s \in S\}$- a collection of clopen sets in Y fulfilling conditions (I)-(III) of 1.

By a lower section of S we shall mean a nonempty proper subset D of S such that $s < t$ for any $s \in D$ and $t \in S \backslash D$. Denote by $L(S)$ the space of all lower sections of S, equipped with the topology induced by the linear order in $L(S)$ given by inclusion, i.e. $D_1 < D_2$ if and only if $D_1 \subset D_2$. Clearly, the sets $\{D \in L(S) : D < [p,s]\}$ with $s \in S$ witness that $L(S)$ is a GDA(S)-space.

3. Theorem. *Any* GDA(S)*-space is homeomorphic to* $L(S)$.

Proof: For $y \in Y$, put $f(y) = \{s \in S : y \notin U_s\}$. In the light of (I) and (II), $f(y) \in L(S)$. We shall show that the function $f : Y \to L(S)$ is a homeomorphism.

Let $y, z \in Y$ and $y \neq z$. Since Y is Hausdorff, it follows from (III) that there exist $s, t \in S$ with $s < t$, $y \in U_t \backslash U_s$ and $z \notin U_t \backslash U_s$. Then $s \in f(y)$, $t \notin f(y)$, but either $t \in f(z)$ or $s \notin f(z)$; so $f(y) \neq f(z)$.

Let $D \in L(S)$. Since Y is compact, there exists $y \in \bigcap\{U_t \backslash U_s : t \in S\backslash D \ \&\ s \in D\}$. For this y, we have $f(y) = D$; so $f(Y) = L(S)$.

Let $y \in Y$ be such that $\{p\} \neq f(y) \neq S\backslash\{q\}$. Consider any $D_1, D_2 \in L(S)$ with $D_1 < f(y) < D_2$. Take $t_0 \in D_2 \backslash f(y)$ and $s_0 \in f(y) \backslash D_1$. Then $y \in U_{t_0} \backslash U_{s_0}$. If $z \in U_{t_0} \backslash U_{s_0}$, then $s_0 \in f(z)$, while $t_0 \notin f(z)$. This implies that $D_1 < f(z) < D_2$, so f is continuous at y. Arguing similarly, we can prove that f is continuous at $f^{-1}(\{p\})$ and at $f^{-1}(S\backslash\{q\})$. □

As an immediate, consequence of 2 and 3, we obtain the following

4. Corollary. *A compact zero-dimensional space* X *is linearly ordered if and only if there exists a collection* $\mathcal{U}$ *of its clopen subsets such that* $\mathcal{U}$ *is linearly ordered by inclusion and the family* $\{U \backslash V : U, V \in \mathcal{U}\}$ *forms an open base for* X

5. Remarks. (a) Suppose that S is infinite and any $s \in S\backslash\{p,q\}$ has an immediate successor. Observe that if S, equipped with the order topology, is compact, then the GDA(S)-space is homeomorphic to S. Indeed: if $\{p\}$ is nonisolated in S, it suffices to put $U_s = [p,s)$ for $s \in S\backslash\{q\}$; if p is isolated in S, let $s_0 = \sup\{s \in S$: any $t \in [p,s]$ is isolated in $S\}$, $U_s = [p,s)$ for $s < s_0$, and $U_s = [p,s]$ for $s \geq s_0$ $(s \in S)$. Of course, if S is finite, then $|L(S)| = |S| - 1$.

(b) Suppose that S with the order topology is both compact and connected. Put $S_0 = (S\backslash\{q\}) \times \{0\}$ and $S_1 = (S\backslash\{p\}) \times \{1\}$. Let the space $Y_0 = S_0 \cup S_1$ be endowed with the topology whose base consists of all sets of the form $([s,t) \times \{0\}) \cup ((s,t] \times \{1\})$ where $s, t \in S$ and $s < t$. Defining $U_s = ([p,s) \times \{0\}) \cup ((p,s] \times \{1\}$ for $s \in S$, we show that Y_0 is the GDA(S)-space.

SPACES FOR WHICH THE GDA(S)-SPACE IS A REMAINDER

6. Theorem. *The* GDA(S)*-space with* $|S| \geq 3$ *is a remainder of a locally compact space* X *if and only if there exist collections* $\{V_s : s \in S\backslash\{p,q\}\}$ *and* $\{W_s : s \in S\backslash\{p,q\}\}$ *of noncompact closed subsets of* X*, such that*
(6.1) $V_s \cap W_s = \emptyset$ *for any* $s \in S\backslash\{p,q\}$;
(6.2) $\mathrm{cl}_X[X\backslash(V_s \cup W_s)]$ *is compact for any* $s \in S\backslash\{p,q\}$;
(6.3) $\mathrm{cl}_X(V_s\backslash V_t)$ *is compact whenever* $s, t \in S\backslash\{p,q\}$ *and* $s < t$;
(6.4) $\mathrm{cl}_X(V_t\backslash V_s)$ *is noncompact whenever* $s, t \in S\backslash\{p,q\}$ *and* $s < t$.

Proof: Necessity. Suppose that there exists a compactification αX of X with $\alpha X\backslash X = Y$. For $s \in S\backslash\{p,q\}$, take a continuous function $f_s : \alpha X \to [0,1]$ such that $f_s(Y\backslash U_s) = \{0\}$ and $f_s(U_s) = \{1\}$. Put $W_s = f_s^{-1}([0,\frac{1}{3}]) \cap X$ and $V_s = f_s^{-1}([\frac{2}{3},1]) \cap X$. Then $\mathrm{cl}_{\alpha X}[X\backslash(V_s \cup W_s)] \subseteq f_s^{-1}([\frac{1}{3},\frac{2}{3}]) \subseteq X$; hence (6.2) holds. Fix $s, t \in S\backslash\{p,q\}$ with $s < t$. Suppose that $\mathrm{cl}_X(V_s\backslash V_t)$ is not compact. There exists $y_0 \in \mathrm{cl}_{\alpha X}(V_s\backslash V_t) \cap Y$. Then $y_0 \in f_s^{-1}([\frac{2}{3},1])$, so $y_0 \in U_s$. On the other hand, $y_0 \in f_t^{-1}([0,\frac{2}{3}])$, so $y_0 \in Y\backslash U_t$. But this contradicts the fact that $U_s \subset U_t$. Thus we have (6.3). Further, there exists $y_1 \in U_t\backslash U_s$. If U is any open neighbourhood of y_1 in αX such that $U \subseteq f_t^{-1}((\frac{2}{3},1]) \cap f_s^{-1}([0,\frac{1}{3}))$, then $U \cap (V_t\backslash V_s) \neq \emptyset$; hence $y_1 \in \mathrm{cl}_{\alpha X}(V_t\backslash V_s)$ and we obtain (6.4).

Sufficiency. Before constructing the required compactification of X, let us denote $V_p = \emptyset$, $V_q = X$ and prove the following:
(6.5) $bd_X(V_s)$ is compact for any $s \in S$:
(6.6) $[\mathrm{int}_X(V_t)]\backslash(V_s \cup K) \neq \emptyset$ for any compact subset K of X and any pair $s, t \in S$ with $s < t$;

(6.7) for any compact subsets K_i of X and any $s_i, t_i \in S$ with $i = 1,2$ and $s_1 \leq s_2 < t_1 \leq t_2$, there exists a compact set $K \subseteq X$ such that $[\text{int}_X(V_{t_1})]\backslash(V_{s_2} \cup K) \subseteq [[\text{int}_X(V_{t_1})]\backslash(V_{s_1} \cup K_1)] \cap [[\text{int}_X(V_{t_2})]\backslash(V_{s_2} \cup K_2)]$;
(6.8) for any $s_i, s_i \in S$ with $i = 1,2$ and $s_1 < t_1 \leq s_2 < t_2$, there exists a compact set $K \subseteq X$ such that $[(\text{int}_X(V_{t_1}))\backslash(V_{s_1} \cup K)] \cap [(\text{int}_X(V_{t_2}))\backslash V_{s_2}] = \emptyset$.

To check (6.5), suppose that $x \in [bd_X(V_s)]\backslash \text{cl}_X[X\backslash(V_s \cup W_s)]$ ($s \in S\backslash\{p,q\}$). There exists a neighbourhood G_0 of x with $G_0 \subseteq V_s \cup W_s$. Since $x \in bd_X(V_s)$, for any neighbourhood G of x, we have $(G \cap G_0) \cap W_s \neq \emptyset$ and $G \cap V_s \neq \emptyset$; however, this is impossible by (6.1). Consequently, $bd_X(V_s) \subseteq \text{cl}_X[X\backslash(V_s \cup W_s)]$, so $bd_X(V_s)$ is compact by (6.2).

Suppose that there are $s,t \in S$ with $s < t$ and a compact subset K of X, such that $\text{int}_X(V_t) \subseteq V_s \cup K$. Then $V_t \backslash V_s \subseteq K \cup bd_X(V_t)$. This, together with (6.5), contradicts (6.4). Hence (6.6) holds.

To show (6.7), observe that $A = [(\text{int}_X(V_{t_1}))\backslash V_{s_2}] \backslash [(\text{int}_X(V_{t_1} \cap V_{t_2}))\backslash(V_{s_1} \cup V_{s_2} \cup K_1 \cup K_2)] \subseteq [V_{t_1} \backslash \text{int}_X(V_{t_2})] \cup (V_{s_1} \backslash V_{s_2}) \cup K_1 \cup K_2 \subseteq (V_{t_1} \backslash V_{t_2}) \cup bd_X(V_{t_2}) \cup (V_{s_1} \backslash V_{s_2}) \cup K_1 \cup K_2$. Using (6.3) and (6.5), we deduce that $\text{cl}_X(A)$ is compact; thus (6.7) is satisfied.

Property (6.8) follows from (6.3) and from the inclusion $[(\text{int}_X(V_{t_1}))\backslash V_{s_1}] \cap [(\text{int}_X(V_{t_2}))\backslash V_{s_2}] \subseteq V_{t_1} \backslash V_{s_2}$.

At last, we are in a position to construct a compactification αX of X with $\alpha X \backslash X = Y$. We may assume that $X \cap Y = \emptyset$. Put $\alpha X = X \cup Y$ and denote by $\mathcal{B}$ the collection containing the original topology of X as well as all the sets $(U_t \backslash U_s) \cup [(\text{int}_X(V_t))\backslash(V_s \cup K)]$ where K is a compact subset of X, $s,t \in S$ and $s < t$. It follows from (I)-(III) and (6.7) that $\mathcal{B}$ forms a base for some topology in αX. Let us consider αX with the topology induced by $\mathcal{B}$. Property (6.6) implies that X is a dense subspace of αX. By (6.8) and the local compactness of X, the space αX is Hausdorff.

Take any $t_i, s_i \in S$ and any compact subsets K_i of X, such that $s_i < t_i$ for $i = 0, \ldots, n+1$, $p < t_0 \leq t_1 \leq \ldots \leq t_n < t_{n+1} = q$, $s_0 = p$ and $\bigcup_{i=0}^{n+1}(U_{t_i} \backslash U_{s_i}) = Y$. In order to prove that αX is compact, it suffices to check that the set $D = X \setminus \bigcup_{i=0}^{n+1} [(\mathrm{int}_X(V_{t_i})) \setminus (V_{s_i} \cup K_i)]$ is compact in X.

We have $D = \bigcap_{i=0}^{n+1} [(X \setminus \mathrm{int}_X(V_{t_i})) \cup V_{s_i} \cup K_i] \subset E \cup \bigcup_{i=0}^{n+1} K_i$ where $E = \bigcap_{i=1}^{n} [(V_{s_{n+1}} \setminus (\mathrm{int}_X(V_{t_i}) \cup \mathrm{int}_X(V_{t_0}))) \cup ((V_{S_{n+1}} \cap V_{s_i}) \setminus \mathrm{int}_X(V_{t_0}))]$. Put $E_{i,1} = V_{s_{n+1}} \setminus [\mathrm{int}_X(V_{t_i}) \cup \mathrm{int}_X(V_{t_0})]$ and $E_{i,2} = (V_{s_{n+1}} \cap V_{s_i}) \setminus \mathrm{int}_X(V_{t_0})$ for $i = 1, \ldots, n$. Then $E = \bigcup \{\bigcap_{i=1}^{n} E_{i,f(i)} : f$ maps $\{1, \ldots, n\}$ into $\{1, 2\}\}$. Conditions (6.3) and (6.5) imply that the sets $\bigcap_{i=1}^{n} E_{i,1}$ and $\bigcap_{i=1}^{n} E_{i,2}$ are compact in X because, by (II), there exist $i, j \in \{1, \ldots, n\}$ with $s_{n+1} \leq t_i$ and $s_j \leq t_0$. Fix $f : \{1 \ldots, n\} \xrightarrow{onto} \{1, 2\}$. The compactness of D will be evident if we show that the set $E_f = \bigcap_{i=1}^{n} E_{i,f(i)}$ is compact.

Observe that if there exist $i, j \in \{1, \ldots, n\}$ with $f(i) = 1$, $f(j) = 2$ and $i > j$, then, by (6.3) and (6.5), E_f is compact since $E_{i,1} \cap E_{j,2} \subset V_{s_j} \setminus \mathrm{int}_X(V_{t_i})$. Put $i_0 = \max f^{-1}(1)$ and suppose that $\{1, \ldots, i_0\} = f^{-1}(1)$. Then $i_0 < n$ and, by (II), there exists $j_0 \in \{i_0+1, \ldots, n+1\}$ with $s_{j_0} \leq t_{i_0}$. The inclusion $E_f \subseteq V_{s_{j_0}} \setminus \mathrm{int}_X(V_{t_{i_0}})$, taken together with (6.3) and (6.5), implies that E_f is compact.

7. Corollary. *Let $T \subseteq S$. If the* GDA(S)*-space is a remainder of X, then the* GDA($T \cup \{p, q\}$)*-space is a remainder of X. Consequently, the* GDA($T \cup \{p, q\}$)*-space is a continuous image of the* GDA(S)*-space.*

Proof: The first part of the corollary follows from 6. To show that the GDA($T \cup \{p, q\}$)-space is a continuous image of the GDA(S)-space, it suffices to consider any space X with $\beta X \setminus X$ being the GDA(S)-space. □

8. Examples. (a) Let X be the free union of noncompact locally compact spaces X_s with $s \in S$. Putting $V_s = \bigcup_{t \leq s} X_s$ and $W_s = X \setminus V_s$, we see that Y is a remainder of X where Y is a GDA(S)-space.

(b) Let X be an infinite discrete space with $|X| \geq d(Y)$. Take a dense set $D \subseteq Y$ with $|D| \leq d(Y)$ where Y is a

GDA(S)-space. Considering $D \times X$ with the discrete topology and defining $V_s = (U_s \cap D) \times X$ and $W_s = (D \times X) \backslash V_s$, we can construct a compactification αX of X with $\alpha X \backslash X$ homeomorphic to Y.

(c) Denote by C the Cantor set. Then C is the GDA-space over the set $T = \{0,1\} \cup \{a_n : n \in N\}$ with the usual order induced from the real line, where $(a_1, b_1), (a_2, b_2), \ldots$ is the sequence of all components of the set $[0,1] \backslash C$. Let $X = C \backslash \{1\}$. Take any sequence (x_n) of points from the set $\{b_n : n \in N\}$ such that $x_n \to 1$ and $x_n < x_{n+1}$ for $n \in N$. For any $i \in N$, we can inductively define a sequence $(y_n(i))$ of points from $\{a_n : n \in N\}$ such that $x_n < y_n(i) < x_{n+1}$ and $y_n(i) < y_n(j)$ whenever $a_i < a_j (i, j, n \in N)$. Using the sets $V_i = \bigcup_{n \in N}([x_n, y_n(i)] \cap C)$ and $W_i = X \backslash V_i$ for $i \in N$, we can apply Theorem 6 to obtain a compactification of X with C as its remainder.

(d) Let Z be the GDA($[0,1]$)-space, i.e. the interval $[-1,1)$ endowed with the topology whose base consist of all the sets $[a,b) \cup [-b,-a)$ where $0 \leq a < b \leq 1$. Consider the subspace $X = Z \backslash \{0\}$ of Z. For $s \in (0,1)$, define $V_s = \bigcup_{n \in N}([\frac{1}{n+1}, \frac{n+s}{n(n+1)}) \cup [-\frac{n+s}{n(n+1)}, -\frac{1}{n+1}))$ and $W_s = X \backslash V_s$. The families $\{V_s : s \in (0,1)\}$ and $\{W_s : s \in (0,1)\}$ satisfy conditions (6.1)-(6.4); thus Z is a remainder of X.

(e) Juhász, Kunen and Rudin showed in [5] that if CH holds then there exists a first countable, locally countable, locally compact, perfectly normal, hereditarily separable, zero-dimensional topology $\mathcal{T}$ on the real line R which is finer than the Euclidean topology and has the property that, for each $U \in \mathcal{T}$, there exists a usual open set $G \subseteq U$ such that $|U \backslash G| \leq \omega$. Let $X = (R, \mathcal{T})$. For any $s \in (0,1)$, choose a countable compact neighborhood K_s of s in X. Put $V_s = \{x \in R : x \leq s\} \backslash \operatorname{int}_X(K_s)$ and $W_s = \{x \in R : x \geq s\} \backslash \operatorname{int}_X(K_s)$. Now, by applying Theorem 6, we can find a compactification of X whose remainder is the GDA($[0,1]$)-space. Similarly, we can construct a compactification of X which has the Cantor set as its remainder.

It should be mentioned that conditions (6.1)-(6.4) were originally formulated from $S = [0, \omega_1]$ by Hatzenbuhler and Mattson in [4]. However, the authors of [4] proved the existence of αX with $\alpha X \backslash X = [0, \omega_1]$ by the applying Magill's theorem.

Finally, let us notice that if S is infinite and compact with the order topology, then S is a continuous image of $L(S)$. This, along with the Magill theorem, gives at once the following

9. Proposition. *Suppose that S is infinite and compact with the order topology. If the* GDA(S)*-space is a remainder of X, then S is a remainder of X.*

Of course, the requirement that S be infinite cannot be omitted in the above proposition since there are noncompact locally compact spaces that do not have two-point compactifications.

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