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# BETWEEN APOSYNDETIC AND INDECOMPOSABLE CONTINUA

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ABSTRACT. We define a continuum $M$ to have Property $C$ (or to be a $C$-continuum) provided it is true that if $x, y$ and $z$ are three points of $M$, then $M$ contains a continuum, $K$, containing $x$ and containing only one of the points $y$ and $z$. In this note we show that this property defines a class of continua lying between the aposyndetic and the indecomposable continua, and identify some of the early theorems of Jones and Whyburn which can and cannot be extended to this class of continua.

## INTRODUCTION.

By a continuum, we shall mean a compact connected metric space. F. Burton Jones, [1], introduced the notion of aposyndetic continua (not necessarily compact) and studied them as a class of continua, (which included the semi-locally connected continua of Whyburn, [2] ), " lying between" the locally connected and indecomposable ones. The author, [3], introduced a class of continua, called $C$-continua, which he identified as being a generalization of the arcwise connected continua. The purpose of this note is to identify these continua as also being a class "lying between" the aposyndetic and the indecomposable ones, and to establish some properties of aposyndetic continua which are and are not shared by the $C$-continua. We repeat the pertinent definitions here for completeness. By a closed connected neighborhood of a point we mean a continuum containing the point in its interior.

## Definitions

A continuum $M$ is aposyndetic at a point $x$ of $M$ provided it is true that if $y$ is any other point of $M$, then some closed connected neighborhood of $x$ misses $y$.

A continuum $M$ is aposyndetic provided it is true that if $x$ and $y$ are any two points of $M$, then some closed connected neighborhood of $x$ misses $y$.

A continuum $M$ is semi-aposyndetic provided it is true that if $x$ and $y$ are any two points of $M$, then some closed connected neighborhood of one of them misses the other.

A continuum $M$ has property $C$ (or is a $C$-continuum) provided it is true that if $x$, $y$, and $z$ are three points of $M$, there is a subcontinuum, $K$, of $M$ containing $x$ and only one of the points $y$ and $z$.

A continuum $M$ is semi-locally-connected provided it is true that each point of $M$ belongs to arbitrarily small neighborhoods whose complements have only a finite number of components.

**Theorem 1.** *If the continuum $M$ is aposyndetic, then $M$ has property $C$.*

*Proof:* Suppose $M$ is aposyndetic and that $x, y$, and $z$ are three points of $M$. If $y$ separates $x$ from $z$ in $M$, say $M\setminus\{y\} = H_x \cup K_z$, then $H_x \cup \{y\}$ is a continuum containing $x$ and $y$ but not $z$. Suppose $y$ does not separate $x$ from $z$ in $M$, say $M\setminus\{y\} = H \cup K'$ ($K'$ may be empty) where $H$ and $K'$ are mutually separated and $H$ contains both $x$ and $z$.

Now, for each point, $a$, of $H$ there is a closed connected neighborhood, $U_a$, of a which misses $y$. Then $U_a$ is contained in $H$. We note that $H \cup \{y\}$ is a subcontinuum of $M$. Let $H'$ be the set of all points of $H$ which can be joined to $x$ by a finite chain of closed connected neighborhoods contained in $H$. If $z$ is in $H'$, some subcontinuum of $H$ contains $x$ and $z$, so suppose $z$ is not in $H'$.

Now, suppose that $y$ is a limit point of $H'$. There is a closed connected neighborhood, $N$, of $y$ which misses $z$. $N$ contains a point, $h$, of $H'$. $h$ can be joined to $x$ by a finite chain $U_1, U_2, \ldots U_k$ of closed connected neighborhoods contained in $H'$. Then $U_1 \cup U_2 \cup \ldots \cup U_k \cup N$ is a subcontinuum of $M$ containing $x$ and $y$ but not $z$.

Suppose, then that $y$ a is not a limit point of $H'$. Suppose that $H'$ has a boundary point, p. Since $H \cup \{y\}$ is closed, $p$ is in $H$. Now, some closed connected neighborhood, $N$, of $p$ misses $y$, and $N$ is contained in $H$. Since $N$ intersects $H', p$ and each point of $N$ can be joined to $x$ by a finite chain of closed connected neighborhoods, so $N$ is a subset of $H'$, contradicting $p$ being a boundary point. Thus $H$ has no boundary points. Then $H'$ and $(H \cup \{y\}) \backslash H'$ are mutually separated, contradicting the connectedness of $H \cup \{y\}$.

Thus, either $z$ is in $H'$ or $y$ is a limit point of $H'$. In either case, $M$ contains a subcontinuum $K$ containing $x$ and only one of the points $y$ and $z$.

## EXAMPLES.

Although the countable harmonic fan would provide an example of a $C$-continuum which is not aposyndetic, the example $M$ of L. E. Rogers [6, p.494], is more interesting. We describe Rogers' example as follows. Let $T$ be an equilateral triangle in the plane with vertices $A, B, C$. Let $C_1$ be a cantor fan with vertex $A$, having side $AB$ as one of its edges, whose end points lie on a line segment perpendicular to side $AB$ at the point $B$ which, except for point $B$, lies outside the triangle $T$. Similarly, let $C_2$ be a cantor fan with vertex at $B$, having side $BC$ as one edge, whose end points lie on a line segment perpendicular to side $BC$ at the point $C$ and lying, except for point $C$, outside of $T$. Finally, let $C_3$ be a cantor fan with vertex at $C$, edge side $CA$, end points on a line segment perpendicular to $CA$ at $A$ and lying, except for $A$, outside $T$. The continuum $M$ is the union of $C_1, C_2$, and $C_3$. This continuum is not aposyndetic at any point and has property $C$ hereditarily.

The author, [3], described an example (due to B. Fitzpatrick, Jr.) of a $C$-continuum which was not arcwise connected, but the example was not planar. A planer example can be obtained using a continuum $H$ constructed by G. T. Whyburn [2, p.735]. We describe Whyburn's example as follows. Let $C$ be the unit circle in the plane with center at the origin. Let $C_1, C_2, \ldots$ be a sequence of circles concentric with $C$, where $C_i$ has radius $1 + 3/(i+2)$. Let $S_1, S_2$, and $S_3$ be three spirals each starting on $C_1$ and "spiraling down" onto the circle $C$ such that for each positive integer $i, S_1 \cap C_i$ is degenerate and $S_1 \cap C_{i+1}$ is 120 degrees from $S_1 \cap C_i$. Then $S_2$ and $S_3$ are obtained from $S_1$ by rotations of 120 and 240 degrees, respectively. The continuum $H$ is the union of $S_1$, $S_2, S_3, C$, and the circles $C_i$, $i = 1, 2, 3, \ldots$. This continuum is a semi-locally connected continuum (and hence is a $C$-continuum) which is not arcwise connected.

We observe that if one were to circumscribe the triangle $T$ in Rogers' continuum $M$ about the outside circle $C_1$ of Whyburn's continuum $H$, one would obtain an example of a planar $C$-continuum which is neither aposyndetic nor arcwise connected.

Clearly, an indecomposable continuum cannot have property $C$, so the $C$-continua lie "strictly between" the aposyndetic and the indecomposable ones.

## Additional Theorems.

The following theorem, proved in [3], is included here as an example of extending a well-known result for aposyndetic continua to the $C$-continua.

**Theorem 2.** *If the $C$-continuum $M$ is irreducible between two of its points, then $M$ is an arc.*

While the $C$-continua do not have to be aposyndetic, the following generalization of a theorem of G. R. Gordh ([4], theorem 7) shows that many of them are semi-aposyndetic. We repeat one of his definitions here.

**Definition (Gordh).** *The continuum $M$ is aposyndetic toward a point $p$ provided that whenever $r$ does not cut $p$ from $q$, then some closed connected neighborhood of $q$ misses $r$.*

**Theorem 3.** *If $M$ is a $C$-continuum and is aposyndetic toward some point, then $M$ is semi-aposyndetic.*

*Proof:* Suppose that the $C$-continuum $M$ is aposyndetic toward the point $p$. Gordh [4], observed that if $q$ is any other point of $M$, some closed connected neighborhood of $p$ misses $q$. Suppose that $r$ and $q$ are two points of $M\backslash\{p\}$. Then $M$ contains a subcontinuum containing $p$ and only one of the points $r$ and $q$, say $q$. Then $r$ does not cut $p$ from $q$ so some closed connected neighborhood of $q$ misses $r$. Thus, $M$ is semi-aposyndetic.

Jones [7], showed that each homogeneous planar aposyndetic continuum is a simple closed curve. The next theorem shows that this result can also be extended to the class of $C$-continua.

**Theorem 4.** *Each homogeneous planar $C$-continuum is a simple closed curve.*

*Proof:* Jones, [5], proved that a homogeneous nonseparating plane continuum must be indecomposable. He later, [8], proved the following general decomposition theorem: Suppose that $M$ is a decomposable, homogeneous continuum. Then $\{L_x : x \in M\}$ is a nondegenerate collection of homogeneous continua forming a continuous decomposition of $M$ such that

(1) the decomposition space $N$ is a homogeneous, aposyndetic continuum, and
(2) if $x$ is a point of $M$ and if $K$ is a subcontinuum of $M$ containing a point in $L_x$ and a point not in $L_x$, then $K$ contains $L_x$.

Now suppose that $M$ is a homogeneous planar $C$-continuum. Then $M$ is decomposable so it admits the above decomposition. Since $M$ is planar, J. T. Rogers, [9], observed that it follows from the earlier Jones' results that the decomposition space

$N$ is a simple closed curve and that each element of the decomposition space is a homogeneous, indecomposable, acyclic continuum.

Now suppose that some element, $L$ of $N$ is nondegenerate. Then $L$ contains three points, $x, y$, and $z$ such that it is irreducible between each two of them. Since $M$ is a $C$-continuum, $M$ contains a subcontinuum, $K$ containing $x$ and containing only one of the points $y$ and $z$. Then $K$ is not a subset of $L$, so $K$ contains both a point of $L$ and a point not in $L$. Then, by part (2) of Jones' decomposition theorem, $K$ contains $L$, resulting in a contradiction. It now follows that $M = N$ is a simple closed curve.

## REMARK.

Jones' generalization of the Torhorst theorem to aposyndetic continua ([1], theorem 11), will not extend to the class of $C$-continua. The countable harmonic fan in the plane is a $C$-continuum, is the boundary of its complementary domain, but is neither locally connected nor aposyndetic.

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