Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 17, 1992

Invitied Speaker

PSEUDOCOMPACT TOPOLOGIES ON GROUPS[1]

DIKRAN N. DIKRANJAN and DMITRI B. SHAKHMATOV[2]

A topology $\mathcal{T}$ on a group G is a *group topology* if both group operations of G, $(g,h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$, are $\mathcal{T}$-continuous, i.e. if $(G, \mathcal{T})$ is a topological group. In what follows *all topological groups and group topologies considered are assumed to be Hausdorff.*

In 1944 Halmos [12] asked for a characterization of the Abelian groups admitting a compact group topology. This problem was partially resolved by Kaplansky [15] and completely by Harrison [13] and Hulanicki [14]. We investigate the following "heir" of Halmos' question:

1 Problem. Which infinite groups admit a (necessarily non-discrete) pseudocompact group topology? Or equivalently, what special algebraic properties must pseudocompact groups have? (A topological space is *pseudocompact* if every real-valued continuous function defined on it is bounded.)

Erik van Douwen [9] found some constraints on the cardinality of a pseudocompact group, and he characterized possible cardinalities of pseudocompact groups under a mild additional set-theoretic assumption consistent with ZFC, the Zermelo-Fraenkel axioms of set theory. The simplest of van Douwen's

[1] Most of our results were published with complete proofs in July of 1991 in the preprint [8], and some of them were announced earlier in [7].

[2] The content of this note was presented in the invited lecture of the second author at the 26th Annual Spring Topology Conference in Charlotte, NC (April 2-4, 1992).

restrictions says that the cardinality of every infinite pseudocompact group is at least $\mathfrak{c}$, the cardinality of the continuum. This provides a variety of groups without pseudocompact group topologies. Comfort and Robertson [4] and Comfort and van Mill [2] announced a result in the same spirit: If G is a pseudocompact Abelian group, then either the free rank $r(G)$ of G is at least continuum, or G is a bounded torsion group.

Another source of groups without a pseudocompact group topology is the classical result of Comfort and Ross [6] which says that pseudocompact groups are precompact. (A topological group G is *precompact* if it is (algebraically and topologically isomorphic to) a subgroup of a compact group, or equivalently, if the two-sided uniformity completion of G is a compact group.) J. von Neumann and Wigner [16] showed that the group $SL(2, \mathbb{C})$ of all complex 2×2 matrices having determinant 1 does not admit a precompact group topology, and Gaughan [10] established the same for the group $S(X)$ of permutations of an infinite set X. Therefore none of these groups can be equipped with a pseudocompact group topology. Comfort and Ross [5] proved that each Abelian group admits a precompact group topology, and Hall [11] showed that every free group also admits a precompact group topology. Hence the precompact version of Problem 1 has a positive answer for Abelian groups and free groups, which does not add any new information to the problem itself except the conclusion that these classes of groups might possibly have pseudocompact group topologies.

We completely resolve Problem 1 for the following classes of groups:

- (i) free groups and free Abelian groups, or more generally, free groups in some variety of abstract groups (Theorems 3 and 4),
- (ii) torsion-free Abelian groups, or even Abelian groups G with $|G| = r(G)$ (Theorem 5),
- (iii) torsion Abelian groups (Theorem 6), and
- (iv) divisible Abelian groups (Theorem 7).

We also completely describe Abelian groups which admit a *connected* pseudocompact group topology (Theorem 8).

In 1978 Cater, Erdös and Galvin [1] introduced a purely set-theoretic condition $Ps(\tau, \sigma)$ between infinite cardinals τ and σ, which says that the set $\{0,1\}^\sigma$ of all functions from (a set of cardinality) σ to the two-point set $\{0,1\}$ contains a subset F of cardinality τ whose projection on every countable subproduct $\{0,1\}^A$ is a surjection. Despite its innocently-sounding character, the condition $Ps(\tau, \sigma)$ is not always easy to verify, and many questions related to it remain unsolved. The importance of the condition $Ps(\tau, \sigma)$ for our study is based on a result of Comfort and Robertson [3] saying that $Ps(\tau, \sigma)$ is equivalent to the existence of a pseudocompact group of cardinality τ and weight σ. We introduce $\mathbf{Ps}(\tau)$ as an abbreviation for the statement "$Ps(\tau, \sigma)$ holds for some infinite cardinal σ". It is clear that $\mathbf{Ps}(\tau)$ is equivalent to the existence of a pseudocompact group of cardinality τ. Therefore, $\mathbf{Ps}(|G|)$ is a necessary condition for the existence of a pseudocompact group topology on a group G. We discovered a variety of other necessary conditions of this type, both of algebraic and set-theoretic nature, but the shortage of space allows us to mention only one of them:

2 Theorem. *For a pseudocompact Abelian group G we have $|\{ng : g \in G\}| \leq 2^{2^{r(G)}}$ for some $n \in \mathbb{N}$, in particular, $|G| \leq 2^{2^{r(G)}}$ for a pseudocompact divisible Abelian group G.*

Now we pass to our principal contributions to Problem 1.

3 Theorem. *A free (Abelian) group G admits a pseudocompact group topology if and only if $\boldsymbol{Ps}(|G|)$ holds. Moreover, if $\boldsymbol{Ps}(|G|)$ holds for a free (Abelian) group G, then G has both a zero-dimensional, pseudocompact group topology and a connected, locally connected, pseudocompact group topology.*

We also completely resolve the question of when a free group G in a variety $\mathcal{V}$ of abstract groups admits a pseudocompact group topology. The solution naturally led to introducing a new notion of precompact varieties. We call a variety $\mathcal{V}$ of (abstract) groups *precompact* if $\mathcal{V}$ is generated by finite groups from $\mathcal{V}$, or equivalently, if each free group in the variety $\mathcal{V}$ is residually finite. (A group is *residually finite* if the intersection of all its normal subgroups of finite index is trivial.) Most of the known varieties of groups (such as, for example, the varieties of all groups, Abelian groups, nilpotent groups, polynilpotent groups, soluble groups, metaabelian groups etc.) are precompact. In fact, the only non-precompact varieties the authors are aware of come from the infinite series of the Burnside varieties $\mathcal{B}_p$, for a prime $p > 665$, each of which consists of all groups G such that $g^p = e$ for every $g \in G$. It turns out that precompactness of the variety in question is exactly what one needs to add to Theorem 3 in order to get the solution in the general case:

4 Theorem. *A free group G in a variety $\mathcal{V}$ admits a non-discrete pseudocompact group topology if and only if the variety $\mathcal{V}$ is precompact and* $\boldsymbol{Ps}(|G|)$ *holds.*

Unlike Theorem 3, in general the pseudocompact group topology in Theorem 4 *cannot* be connected: If some free group in a variety $\mathcal{V}$ admits a pseudocompact connected group topology, then $\mathcal{V}$ *must* coincide with either the variety of all groups or the variety of Abelian groups.

5 Theorem. *Let G be either an Abelian group with $|G| = r(G)$ or a torsion-free Abelian group. Then the following conditions are equivalent:*

(i) *G admits a pseudocompact group topology,*
(ii) *G admits a pseudocompact, connected and locally connected group topology, and*
(iii) $\boldsymbol{Ps}(|G|)$.

Since pseudocompact torsion Abelian groups are bounded, the following theorem completely characterizes pseudocompact torsion Abelian groups:

6 Theorem. *Let G be a bounded torsion Abelian group. For each prime number p, let $G_p = \{g \in G : p^k g = 0$ for some $k \in \mathbb{N}\}$ be the p-torsion part of G, and let $G_p = \oplus\{\mathbb{Z}(p^k)^{(\alpha_{k,p})} : 1 \leq k \leq r_p\}$ be the canonical decomposition of G_p, where $r_p \geq 1$ is an integer number and $\alpha_{1,p}, \dots, \alpha_{r_p,p}$ are cardinals with $\alpha_{r_p,p} > 0$. (We set $r_p = 0$ in case $G_p = \{0\}$.) Then the following conditions are equivalent:*

(a) *G has a pseudocompact group topology,*
(b) *each G_p has a pseudocompact group topology, and*
(c) *for every prime number p and any integer k, $1 \leq k \leq r_p$, either the cardinal $\beta_{k,p} = \max\{\alpha_{k+1,p}, \dots, \alpha_{r_p,p}\}$ is finite, or $\boldsymbol{Ps}(\beta_{k,p})$ holds.*

Theorems 3–5 say that, for a group G from the classes of free groups, free Abelian groups (or more generally, free groups in a precompact variety of groups) and torsion-free Abelian groups (or even the "wider" class of Abelian groups G with $|G| = r(G)$), the set-theoretic condition $\mathbf{Ps}(|G|)$ is not only a necessary condition for the existence of a pseudocompact group topology on G, but a sufficient condition as well. It follows from Theorem 6 that $\mathbf{Ps}(|G|)$ is no longer a sufficient condition for the existence of a pseudocompact group topology on a torsion Abelian group G. Now we consider other cases when this phenomenon occurs, thereby justifying the use of the more complicated condition in Theorems 7 and 8. (Recall that $\log \tau = \min\{\sigma \geq \omega : 2^\sigma \geq \tau\}$ for an infinite cardinal τ.)

7 Theorem. *A divisible Abelian group G admits a pseudocompact group topology if and only if $Ps(r(G), \log |G|)$ holds.*

Even though Problem 1 remains open in its full generality, we are able to solve it under the additional restriction that the pseudocompact topology is connected. Our next theorem characterizes pseudocompact connected Abelian groups.

8 Theorem. *An Abelian group G admits a pseudocompact connected group topology if and only if $Ps(r(G), \log |G|)$ holds.*

The reader definitely noticed an obvious similarity between Theorems 7 and 8. Surely this is not by accident. Indeed, Theorem 7 is a trivial corollary of Theorem 8 and a result of Wilcox [17] that pseudocompact divisible groups are connected. In view of Wilcox's theorem, the following is a generalization of the "in particular" part of Theorem 2:

9 Theorem. *The cardinality of a pseudocompact connected Abelian group G satisfies the inequality $|G| \leq 2^{2^{r(G)}}$.*

From the last two theorems it follows that both $\mathbf{Ps}(r(G))$ and $|G| \leq 2^{2^{r(G)}}$ are necessary conditions for the existence of a pseudocompact connected group topology on an Abelian group G. We show that the question of whether the conjunction of those two conditions is also sufficient for the existence of a pseudocompact group topology on an Abelian group G is independent of *ZFC*.

We provide a series of examples emphasizing the substantial difference between compact and pseudocompact topologizations of groups. For instance, a pseudocompact, torsion-free Abelian group admits a pseudocompact connected group topology (Theorem 5), while the group $\mathbb{Z}_p$ of p-adic integers is a compact torsion-free Abelian group which does not admit a compact connected group topology. A pseudocompact connected Abelian group admits a pseudocompact group topology which is both connected and locally connected, but the analogous statement with "pseudocompact" replaced by "compact" does not hold — the Pontryagin dual $\mathbb{Q}^*$ of the group of rationals $\mathbb{Q}$ serves as a counterexample. By contrast with the compact case, even if an Abelian group G admits a pseudocompact group topology, it may happen that either the divisible part $D(G)$ or the reduced part $G/D(G)$ of G does not admit such a topology. (We show however that if the direct sum $G \oplus H$ of Abelian groups G and H admits a pseudocompact connected group topology, then at least one of G and H admits a pseudocompact connected group topology.) Finally, according to Kaplansky and Los, an Abelian group is called algebraically compact if it is a direct summand of some compact group. Algebraic compactness is an extremely strong restriction on an Abelian group. For example, even the group $\mathbb{Z}$ of integer numbers is not algebraically compact. Quite the contrary, we establish that *every* Abelian group is a direct summand of some pseudocompact group, and so "algebraically pseudocompact".

It should be noted that some of our results were independently obtained by Comfort and Remus.

REFERENCES

1. F. S. Cater, P. Erdös and F. Galvin, *On the density of λ-box products*, Gen. Topol. Appl., **9** (1978), 307-312.
2. W. W. Comfort and J. van Mill, *Concerning connected, pseudocompact Abelian groups*, Topology Appl., **33** (1989), 21-45.
3. W. W. Comfort and L. C. Robertson, *Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups*, Pacific J. Math., **119** (1985), 265-285.
4. W. W. Comfort and L. C. Robertson, *Extremal phenomena in certain classes of totally bounded groups*, Dissertationes Mathematicae, **272**, 1988.
5. W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters*, Fund. Math., **55** (1964), 283-291.
6. W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math., **16** (1966), 483-496.
7. D. Dikranjan and D. Shakhmatov, *Pseudocompact topologizations of groups*, Zborinik radova Filozofskog fakulteta u Nišu, Ser. Mat., **4** (1990), 83-93.
8. D. Dikranjan and D. Shakhmatov, *Algebraic structure of pseudocompact groups*, preprint no. 91-19, Dept. of Math. and Stat. of York Univ., July 1991.
9. E. van Douwen, *The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality*, Proc. Amer. Math. Soc., **80** (1980), 678-682.
10. E. D. Gaughan, *Topological group structures on the infinite symmetric group*, Proc. Nat. Acad. Sci. U.S.A., **58** (1967), 907-910.
11. M. Hall, *A topology for the free group and related topics*, Annals of Math., **52** (1950), 127-139.
12. P. Halmos, *Comments on the real line*, Bull. Amer. Math. Soc., **50** (1944), 877-878.

13. D. Harrison, *Infinite abelian groups and homological methods*, Ann. Math., (2) **69** (1959), 366-391.
14. A. Hulanicki, *Algebraic structure of compact abelian groups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys., **6** (1958), 71-73.
15. I. Kaplansky, *Infinite abelian groups*, Ann Arbor: Univ. of Mich. Press, 1954.
16. J. von Neumann and E. Wigner, *Minimally almost periodic groups*, Annals of Math., (Ser. 2) **41** (1940), 746-750.
17. H. Wilcox, *Dense subgroups of compact groups*, Proc. Amer. Math. Soc., **28** (1971), 578-580.

University of Udine
via Zanon 6, Udine 33100, Italy
e-mail address: dikranja@udmi5400.cineca.it

Moscow State University
119 899 Moscow, Russia
Mailing address:
Department of Mathematics, Faculty of Science
Ehime University
Matsuyama 790, Japan
e-mail address: dmitri@dpcs4370.dpc.ehime-u.ac.jp