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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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TOWARDS A CLOSED STRING FIELD THEORY: TOPOLOGY AND CONVOLUTION ALGEBRA

JIM STASHEFF

In my Cortona talk [S:C], I developed an almost groupoid structure for open strings and thereon built a convolution operator. Since closed strings have no initial or terminal point, there are additional subtleties in making the space of closed strings $\mathcal{S}$ into a groupoid-like object. Subsequent work of several physicists, especially Saadi and Zwiebach and Kugo, Kunitomo and Suehiro, have lead me to the conviction that a groupoid-like object is NOT what the study of closed string configurations suggests. More importantly, the 'algebra' of string fields includes a binary operation but is NOT generated by that operation, but rather exhibits the structure of a strongly homotopy Lie algebra [**SS**].

String field theory is multi-layered, often presented as involving topology, geometry, algebra and analysis, especially analysis in the sense of Riemann surfaces. The bottom layer is the topology of string configurations and in the present paper I attempt to provide a topological interpretation of the algebra of string field theory directly in the space of free loops, that is, using as much as possible only the topology of string configurations. A background metric does make an appearance at one point and will be explicitly acknowledged. The Lagrangians of string field theory are built with this algebra, but may involve additional factors related to geometry (metric or conformal) as opposed to topology alone; these are not considered in the present paper. Nor is there any quantization involved nor any attempt to achieve reconciliation with (first quantized) string theory.

The obvious picture of a closed string is that of a closed curve in a (Riemannian) manifold M. The first subtlety one encounters with this picture is that the physics is often described in terms of a parameterization of such a curve, e.g. a map of the circle S^1 into M but the physics should not depend on the parameterization. Thus the space of closed strings $\mathcal{S}$ can be described as the space of equivalence classes (under reparameterization) of maps of the circle into M. Fields then refer to functions on $\mathcal{S}$ or sections of some bundle over $\mathcal{S}$. When particles are considered to be mathematical points, their interaction leads to consideration of the algebra of fields being given in terms of point-wise multiplication. For open strings, interactions are pictured in terms of contact between strings (Figure 1) and an algebra of fields which is given by convolution **[S:C]**.

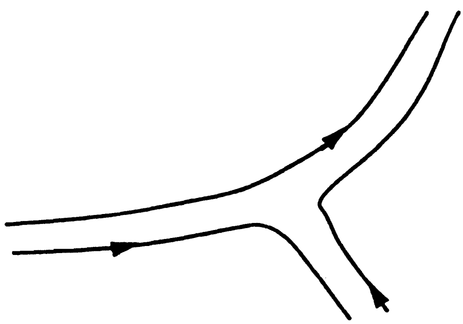

Figure 1.

For closed strings, there is in **[SZ]** **[KKS]** a similar convolution operation, but there is more to the algebra than that, namely N-ary operations corresponding to simultaneous interactions among (N + 1) strings which do NOT decompose in terms of the binary operation. Instead, the operations for various N are related via the so-called BRST operator in a relation which has precisely the form of (the Lie analog of) a strongly homotopy associative algebra (sha-algebra) [**S**:**I**], [**S**:**II**], [**SS**]. (This is also the dual of the structure that appears on the indecomposables of a (non-minimal) Sullivan model [**Su**].)

We begin by reviewing the joining of two strings to form a third (extending a method due to Lashof [**L**] in the case of based loops and to Witten [**W**] for strings). The idea is that two closed strings Y and Z join to form a third $Y * Z$ if a semi-circle of one agrees with a reverse oriented semi-circle of the other. (Notice this avoids Witten's marking of the circle.) The join $Y * Z$ is formed from the complementary semi-circles of each. To be more precise, consider the configuration of three arcs $A_i, i = 0, 1, 2$ with the three initial points identified and the three terminal points identified, as in a circle together with a diameter. (Figure 2) (One is tempted to call this a (theta) Θ curve, but string field theory is likely to involve theta functions in the sense of number theory; there's enough confusion of terminology already!)

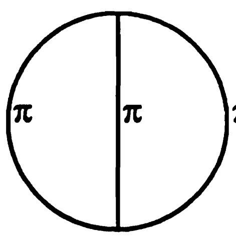

Figure 2.

To emphasize the symmetry and to fix parameterizations, consider the arcs to be great semi-circles on the unit 2-sphere in R^3 parameterized by arc length from north pole to south pole. Denote the union of the three arcs by Θ. Denote by $\bar{A}_i$ the arc parameterized in the reverse direction. Let C_i denote any isometry $C_i : S^i \hookrightarrow \Theta$ which agrees with A_j on one semi-circle and with $\bar{A}_k$ on the other for a cyclic permutation (i, j, k) of $(0, 1, 2)$. (Up to rotation, C_i is A_j followed by $\bar{A}_k$.) Given any map $X : \Theta \to M$, let $X_i = X \circ C_i$.

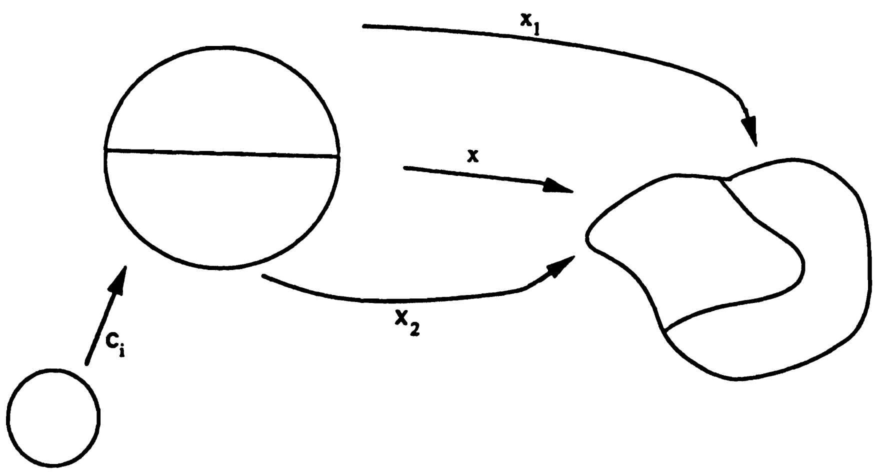

Figure 3

Now let ϕ, ψ be function(al)s on $\mathcal{S}$, the space of strings. Define the **convolution product** $\phi * \psi$ as follows:

$$(\phi * \psi)(X_0) = \int \phi(X_1)\psi(X_2)$$

where the integral is over all isometries C_i and all maps $X : \Theta \to M$ such that $X_0 = X \circ C_0$. (Thus $\phi * \psi$ depends on all ways of **decomposing** X_0 into two loops X_1 and X_2.)

RESTRICTED POLYHEDRA

There are lots of ways to generalize this convolution product to more than three loops. Motivated by the physical interpretation which sees the theta curve as imbedded in a world sheet, several physicists, starting with Kaku [**K**], have considered a tetrahedral configuration in which the perimeter of each face is regarded as a circle to be mapped via a closed string. Extension to polyhedra with 5 faces was worked out by Saadi and

Zwiebach [**SZ**] and their lead was carried through to general polyhedra by Kugo, Kunitomo and Suehiro [**KKS**]. Here polyhedra refer to cell decompositions of the oriented 2-sphere in which each face (=2-cell) has boundary (perimeter) consisting of a finite number of edges (1-cells). Each face and hence its perimeter carries the orientation induced from S^2. The polyhedra are restricted geometrically in that each edge is assigned a length such that:

(1) Saadi and Zwiebach: the perimeter of each face has length 2π (This implies each edge has length $\leq \pi$), and
(2) Kugo, Kunitomo and Suehiro: any simple closed edge path has length $\geq 2\pi$.

There is only one restricted trihedron, Θ.

For tetrahedra, the restrictions are precisely that opposite edges have equal lengths, say (a_1, a_2, a_3), with $0 \leq a_i \leq \pi$ and $\Sigma a_i = 2\pi$

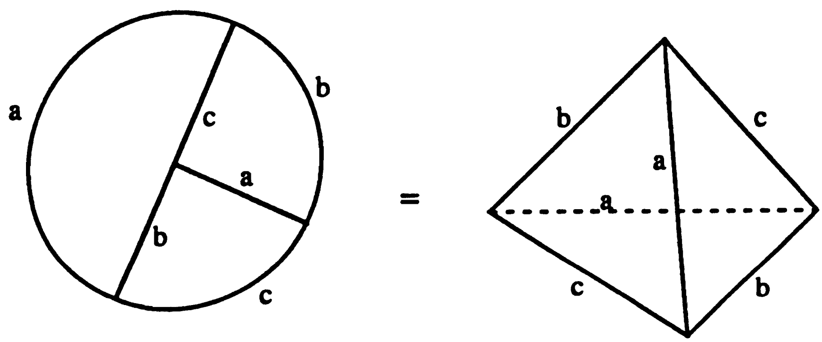

Figure 4.

In other words, the "moduli" space of restricted tetrahedra is given by the union of two 2-simplices (one for each orientation of the tetrahedron) with vertices in common

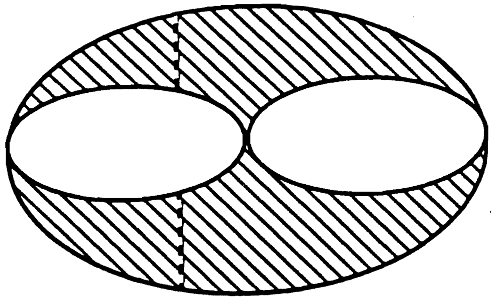

Figure 5.

More generally, let $\mathcal{M}_N$ be the "moduli" space of all restricted $(N+1)$-hedra P with an arbitrary ordering of the faces from 0 to N. As a space, $\mathcal{M}_N$ is given the topology of the local coordinates which are the edge lengths - described in more detail below.

We are going to analyze the KKS-operations $\lceil \phi_1 \ldots \phi_N \rceil$ and $(\phi_0 \ldots \phi_N)$ in terms of integration over $\mathcal{M}_N$ with respect to specified parameterizations $C_i : S^1 \hookrightarrow P$ which are isometries with the perimeter of the i-th face - analogous to a local coordinate at one of $(N+1)$ punctures on a Riemann surface.

Let $\mathcal{P}_N$ (homeomorphic to $\mathcal{M}_N \times (S^1)^{N+1}$) denote the space of ordered restricted $(N+1)$-hedra with such specified isometries $C_i : S^1 \hookrightarrow P$. We will use the notation $\bar{P} = (P, C_0, \ldots, C_N) \in \mathcal{P}_N$ and $|P|$ will denote the underlying topological space of P, ignoring the "metric" given by the lengths of the edges. The moduli space $\mathcal{M}_N$ of all restricted $(N+1)$-hedra as defined is manifestly a union of pieces indexed by the topological type of the polyhedron. In fact, for a given topological type, the restrictions 1) and 2) with edge lengths > 0 describe a convex subspace (polytope) $\mathcal{M}_N$ of R^E where E is the number of edges of the $(N+1)$-hedron. Strictly speaking, when an edge length goes to zero, the topological type of the

$(N+1)$-hedron changes and thus two cells are glued together along a cell of one lower dimension.

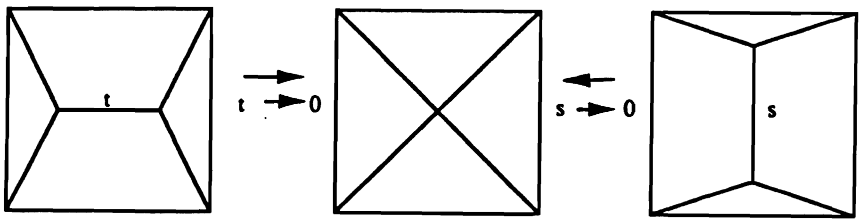

Figure 6.

Thus $\mathcal{M}_N$ is described as a finite cell complex in which each cell has boundary composed of a finite number of cells of lower dimension. The cells of maximal dimension correspond to 3-valent polyhedra and the dimension of these cells is $2N-4$, with faces corresponding to $(N+1)$-hedra with one 4-valent vertex, etc. when an edge length goes to zero.

The cell complex $\mathcal{M}_N$ does have a "boundary" corresponding to saturation of the inequalities 2). KKS refer to such polyhedra as **critical**, i.e. if there is an edge path enclosing at least two faces and of length precisely 2π. Separating the polyhedron P along this edge path produces two restricted polyhedra Q and R (Figure 7) The separation can be regarded as giving a partition of the set $\{0, \ldots, N\}$ or, if the faces of P are ordered, as an **unshuffle** giving induced orderings on Q and R.

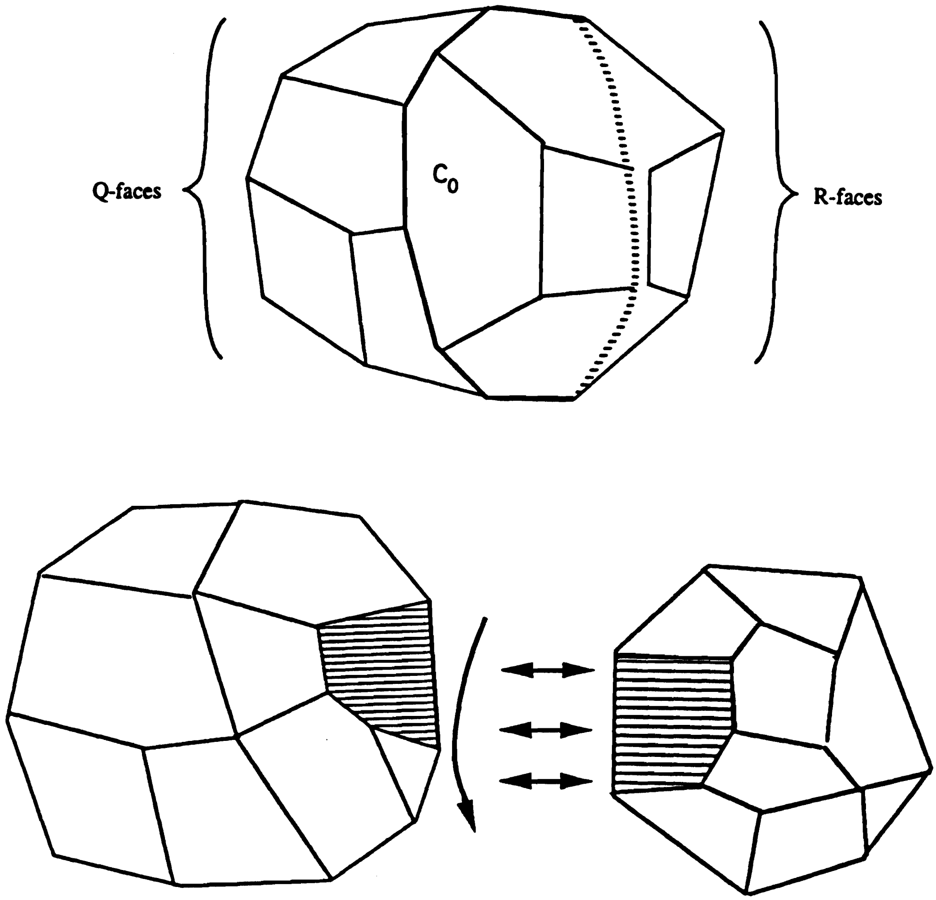

Figure 7.

Conversely, if we have two restricted polyhedra Q and R, we can form a twisted connected sum $Q\#R$ by deleting two faces $F_Q \in Q$ and $F_R \in R$ and identifying their perimeters by an arbitrary orientation reversing isometry. Generically, the result will be a restricted polyhedron P (although occasionally the identification may produce vertices of valence greater than 3). If the faces of Q and R are ordered, we can define such a twisted connected sum for each shuffle.

Thus we can describe boundary "facets" of $\mathcal{M}_N$ by inclusions

$$(*) \qquad S^1 \times \mathcal{M}_Q \times \mathcal{M}_R \hookrightarrow \mathcal{M}_N$$

where $Q + R = N + 1$, the inclusions being indexed by (Q, R)-unshuffles and the S^1 coordinate giving the twist.

THE N-ARY OPERATIONS

Now we are ready to suggest a primarily topological interpretation of the N-ary operations of [**SZ**] and [**KKS**] which will include a modified form of the $*$-product of Kaku and of Witten.

For $\bar{P} \in \mathcal{P}_N$, i.e. for any restricted $(N+1)$-hedron P with isometries $C_i : S^1 \hookrightarrow P$ with image the perimeter of the i-th face , consider

$$Map(P, M) \to (LM)^{N+1}$$

by

$$(X : P \to M) \to (X \circ C_i, i = 0, \ldots, N).$$

For fixed $P \in \mathcal{M}_N$, as the C_i vary, we have

$$Map(P, M) \times (S^1)^{N+1} \to (LM)^{N+1}$$

by

$$(X : P \to M, \theta_0, \ldots, \theta_N) \to (X \circ C_i \circ \theta_i),$$

interpreting $\theta \in S^1$ as a rotation $\theta \in Aut(S^1)$. As P varies by varying the metric on the underlying ordered $(N+1)$-hedron $|P|$, we have a $(3N-3)$-dimensional family

$$Map(P, M) \times (S^1)^{N+1} \times \mathcal{M}_{|P|} \to (LM)^{N+1}.$$

As $|P|$ varies over all $(N+1)$-hedra, the $\mathcal{M}_{|P|}$ fit together to form $\mathcal{M}_N$ and the above families fit together to form a "bundle"

$$\mathcal{Q}_N = \cup_{|P|} Map(P, M) \times (S^1)^{N+1} \times \mathcal{M}_{|P|}$$

with a map into $(LM)^{N+1}$. (The boundary of each piece $\mathcal{M}_{|P|}$ is approached via isometries so the spaces of maps pass nicely to the limit.)

Now consider "fields" ϕ_i on LM. What this means is usually understood by physicists and deduced from context by mathematicians. I will specify ϕ_i as a q_i-form on LM. (For $q_i = 0$, this is a function(al) on LM.)

In accordance with the usual paradigm, KKS desire an "action" of the form

$$S = \phi \cdot Q_B\phi + \Sigma_{N=3}^{\infty}(\phi \dots \phi) \qquad (\phi \textit{ repeated } N \textit{ times})$$

invariant with respect to a variation

$$\delta\phi = Q_B\Lambda + \Sigma_{N=1}^{\infty}\lceil \phi \dots \phi\Lambda \rceil \qquad (\phi \textit{ repeated } N \textit{ times}).$$

We can simplify the formulas by setting $\lceil \Lambda \rceil := Q_B\Lambda$ and $(\phi_0\phi_1) := \phi_0 \cdot Q_B\phi_1$. The formulas then become

$$S = \Sigma_{N=2}^{\infty}(\phi \dots \phi) \qquad (\phi \textit{ repeated } N \textit{ times})$$

and

$$\delta\phi = \Sigma_{N=0}^{\infty}\lceil \phi \dots \phi\Lambda \rceil \qquad (\phi \textit{ repeated } N \textit{ times}).$$

The operator Q_B is referred to in the physics literature as a BRST-operator. We interpret this to mean that ϕ_i is a longitudinal differential form along the reparameterization orbits, and Q_B is the exterior derivative d along the orbits. Henceforth we will write d instead of Q_B. If the formula for $\delta\phi$ is to be homogeneous, ϕ must be of degree 3 and Λ of degree 2.

Here is a topolgical construction of an operation $\lceil \phi_1 \dots \phi_N \rceil$ which produces a form on LM of dimension $\Sigma_1^N q_i - 3(N-1)$. First define $\lfloor \phi_1 \dots \phi_N \rfloor$ on $Map(|P|, M)$ by pulling back $\phi_1 \times \dots \times \phi_N$ from $(LM)^N$ to $(LM)^{N+1}$ (via the projection off the 0-th factor) and thence to $Map(|P|, M) \times (S^1)^{N+1} \times \mathcal{M}_{|P|}$, then fibre integrating over $(S^1)^{N+1} \times \mathcal{M}_{|P|}$. Now let $Map_{X_0}(|P|, M)$ denote the space of maps $X \to M$ with specified X_0. This subspace can be considered as the "fibre" of the projection $Map(P, M) \to LM$ given by composition with C_0. (Suitably defined) generalized path integration over this fibre and summation over ordered topological types $|P|$ will produce the desired form $\lceil\phi_1\dots\phi_N\rceil_{X_0}$ evaluated at the loop X_0. (To be somewhat more in keeping with the physics literature, we can denote this fibre integration as

$$\int_{X_0}\lfloor\phi_1\dots\phi_N\rfloor DX$$

to mean an integral over all $X : P \to M$ such that $X \circ C_0 = X_0, P \in \mathcal{M}_{N-1}$.)

It is this generalized path integration which apparently involves a metric on M. In the case of the standard metric on $M = R^d$ and $N = 2$, notice that this integral is over paths from $X_0(\theta)$ to $X_0(\theta+\pi)$ and hence is well defined as a Brownian bridge.

For $N = 2$, we have $\lceil\phi_1\phi_2\rceil = 6\ \phi_1 * \phi_2$ since there are 6 orderings of the faces of Θ and as we integrate over all $X : \Theta \to M$, we repeat a given value 6 times.

Now we wish to define the operation $(\phi_0\dots\phi_N)$ to be a generalized scalar product. The case $(\phi_0\phi_1)$ is related to a more ordinary scalar product and sometimes written $\phi_0 \cdot d\phi_1$. Assume that forms on LM admit the analog of a Hodge star operator, but with 0 as 'middle' dimension. As $*$ has several other meanings in string field theory, we will denote by $\check{\phi}$ the Hodge dual of ϕ. [One example might be the semi-infinite forms of Feigin [F] with basic forms e_i for all NON-ZERO integers. For the Feigin semi-infinite forms, the duality will replace e_i by e_{-i}.] Now define $(\phi_0\phi_1)$ as

$$\int_{S^1}\int \check{\phi}_0(X)d\phi_1(\bar{X}_\theta)DX$$

where the inside integral is over all $X : S^1 \to M$ and $\bar{X}_\theta$ has the opposite orientation given specifically by composing X with the reflection fixing θ and $\theta + \pi$. Thus if ϕ_0, ϕ_1 have the same degree as forms, the result before integration is a 1-form, yielding a number after integration.

To define $(\phi_0 \ldots \phi_N)$ in general, first define $< \phi_0 \ldots \phi_N >$ (X) on $Map(|P|, M)$ by pulling back $\check{\phi}_0 \times \phi_1 \times \cdots \times \phi_N$ and integrating over $(S^1)^{N+1} \times \mathcal{M}_{|P|}$. Since $\mathcal{M}_N$ has dimension $2N-4$, the result is a function(al) on $Map(|P|, M)$ if all the $q_i = 3$. Now define $(\phi_0 \ldots \phi_N)$ by generalized path integration over $Map(|P|, M)$and add the results for the finite number of ordered topological types $|P|$:

$$(\phi_0 \ldots \phi_N) = \Sigma_{|P|} \int (\int_{(S^1)^{N+1} \times \mathcal{M}_{|P|}} \check{\phi}_0 \times \ldots \phi_N) DX.$$

Since $\mathcal{M}_N$ has dimension $2N-4$, the result is a number if all the $q_i = 3$.

THE STRONGLY HOMOTOPY LIE STRUCTURE

As KKS assert, the invariance of S is a consequence of

$$(16.N) \quad d\lceil \phi_1 \ldots \phi_N \rceil + \textstyle\sum_1^N \lceil \phi_1 \ldots d\phi_i \ldots \phi_N \rceil =$$

$$\textstyle\sum_{Q=2}^{N-1} \lceil \lceil \phi_{i_1} \cdots \phi_{i_Q} \rceil \phi_{i_{Q+1}} ots \phi_{i_N} \rceil$$

where the sum is over all unshuffles of $1, \ldots, N$ (signs and factorial fudge factors being ignored for now). In particular, the binary operation $\lceil \phi_1 \phi_2 \rceil$ has the graded symmetry

$$\lceil \phi_1 \phi_2 \rceil = (-1)^{(q_1-1)(q_2-1)} \lceil \phi_2 \phi_1 \rceil,$$

but need not satisfy the Jacobi identity. Instead (16.3) asserts (writing d for Q_B and ignoring signs):

$$d\lceil \phi_1 \phi_2 \phi_3 \rceil + \lceil d\phi_1 \phi_2 \phi_3 \rceil + \lceil \phi_1 d\phi_2 \phi_3 \rceil + \lceil \phi_1 \phi_2 d\phi_3 \rceil = \Sigma \lceil \lceil \phi_{i_1} \phi_{i_2} \rceil \phi_{i_3} \rceil,$$

the sum again being over unshuffles of $1, 2, 3$. If the LHS were zero, we would have a form of the Jacobi identity. In the language of homotopy theory, $\lceil \phi_1 \phi_2 \rceil$ satisfies the Jacobi identity **up to homotopy** . The conditions (16.N) specify that the

homotopy $\lceil\phi_1\phi_2\phi_3\rceil$ satisfies higher order homotopy conditions - to all orders. These are precisely the conditions which in mathematics are summarized in the name "strongly homotopy Lie algebra", abbreviated "sh Lie algebra" [**SS**]. This is essentially related to the fact that $\delta^2 = 0$.

After introducing (16), KKS remark "the LHS is expected to lead to a total derivative form" with respect to the modular parameters of restricted $(N+1)$-hedra "as was the case in the 4-string vertex in the Kyoto group's open string field theory" [**HIKKO**]. For closed string field theory, a related argument is sketched speculatively for N = 3 by Zwiebach [**Z**] . We now validate the value of such expectations.

THE JACOBI HOMOTOPY CONDITION

Considering the six orderings of the faces of a tetrahedron, we see the moduli space $\mathcal{M}_3$ consists of 3 copies of two 2-simplices

$$\Delta^2 = \{(a,b,c)|0 \leq a,b,c \leq \pi, a+b+c = 2\pi\}$$

with vertices identified in pairs (Figure 5). The definition of $\lceil\phi_1\phi_2\phi_3\rceil$ is

$$\lceil\phi_1\phi_2\phi_3\rceil(X) = \Sigma_{|P|}\int_{\mathcal{M}_3\times(S^1)^3}\phi_1(X_1)\phi_2(X_2)\phi_3(X_3)$$

for $X : P \to M$ where P is an ordered tetrahedron with edge lengths (a,b,c). Applying the exterior derivative gives the appropriate terms $\lceil\ldots d\phi_i\ldots\rceil$ plus boundary terms

$$(***)\ \int_{\partial\mathcal{M}_3\times(S^1)^3}\phi_1(X_1)\phi_2(X_2)\phi_3(X_3)$$

and $\partial\mathcal{M}_3$ consists of the six edges in Figure 5 corresponding to setting a, b or c equal to π.

Consider $c = \pi$. This tetrahedron can be assembled from two Θ's as follows:

From $(\Theta_1, C_0', C_1', C_2')$ and $(\Theta_2, C_0'', C_1'', C_2'')$, form a tetrahedron P by identifying the image of C_0'' with that of C_1' with an arbitrary rotation by $0 \leq a \leq \pi$ (Figure 8). We order and parameterize the face perimeters of P by $(C_0', C_1'', C_2'', C_2')$. Thus Θ_i can be regarded as a subspace of P. Now given $X : P \to M$, we have induced maps

$$Y, Z : \Theta \to M$$

defined by $Y = X|\Theta_1$ and $Z = X|\Theta_2$.

Thus for $c = \pi$, we have a piece of the boundary term (***) which can be written

$$\int_{(S^1)^2} \lceil \phi_1 \phi_2 \rceil (Y_1) \phi_3(Y_2),$$

noting that $Y_2 = X_3$.

Together with the corresponding piece

$$\int_{(S^1)^2} \phi_3 \lceil \phi_1 \phi_2 \rceil,$$

we have $\int \lceil \lceil \phi_1 \phi_2 \rceil \phi_3 \rceil$, being careful about signs. Adding together the contributions for a or $b = \pi$, we have (16.3).

THE N-ARY CONDITION

The proof of (16.N) for $N > 3$ follows the same pattern, although the bookkeeping / combinatorics is more complicated. Again there are boundary terms in $d\lceil \phi_1 \ldots \phi_2 \rceil$ which are given by integration over the facets of $\mathcal{M}_N \times (S^1)^N$,which are of the form $S^1 \times \mathcal{M}_Q \times \mathcal{M}_R \times (S^1)^N$ and can be reordered, according to the twisted connected sum indexed by a shuffle, as

$$\mathcal{M}_Q \times (S^1)^Q \times \mathcal{M}_R \times (S^1)^R$$

(indeed, $Q + R = N + 1$). Thus this boundary term can be written as

$$\int_{\mathcal{M}_R \times (S^1)^R} \int_{\mathcal{M}_Q \times (S^1)^Q}$$

giving rise to the terms

$$[[\phi_{i_1} \cdots \phi_{i_Q}]\phi_{i_{Q+1}} \cdots \phi_{i_N}].$$

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University of North Carolina
Chapel Hill, NC 27599-3250
e-mail address jds@math.unc.edu