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VERSIONS OF SHARKOVSKIĬ'S THEOREM ON TREES AND DENDRITES

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ABSTRACT. If T is a tree, and $f : T \to T$ is a continuous function having an orbit of size 3 contained in an arc, then a variation of Sharkovskiĭ's Theorem gives points of all periods for f. However, it is possible that no orbit of size 4 is contained in an arc. In this paper, we study what happens to Sharkovskiĭ's Theorem when periodic orbits of $[0, 1]$ or $\mathbb{R}$ are replaced by periodic orbits of a tree (or a dendrite) in which the orbit is contained in an arc.

1. INTRODUCTION

In the study of maps on the interval, one feature which has aroused great interest in recent years is the study of what has come to be known as the "forcing relation", in which certain patterns of orbits (or finite invariant sets) are seen to "force" other patterns (see [Be1], [Ba1], [Be2] and [MN], for example). If one is only interested in the size of the periodic orbit, then there are many similar results available on trees (see [ALM], [Ba2], [I], [IK], for example). However, if information about the pattern is desired as well, then it is often a more difficult problem, mainly because the most obvious definition of "pattern" in this setting does not always lead to the analogous "forcing" results which one would like to see. In this paper, we examine the problems which are apparent for the simplest type of pattern, namely those finite subsets of a tree which are contained in an arc. We shall see that we do not get the generalizations which might be expected, especially for periods which are a power of two. Although it is likely that most readers will be primarily interested in the results on trees, the generalizations to dendrites given here require little extra work.

A simple example can be given to illustrate the problem. Consider the interval $[1,7]$ on the x-axis, considered as a subset of the plane, and let $5'$ abbreviate the point $(5,1)$. We add to this set all points on the vertical line segment from 5 and $5'$ (i.e. $\{(5,y) : y \in [0,1]\}$), so that the given tree is homeomorphic to the simple triod (the tree with one node). The function f is defined by $f(1) = 3$, $f(2) = 5'$, $f(3) = 7$, $f(4) = 6$, $f(5) = f(5') = 4$, $f(6) = 2$, and $f(7) = 1$, with the obvious piecewise monotone extension. Using the usual Markov Graph arguments (see below), it is easy to see that f has points of all periods, but if we look at which orbits are contained in an arc, then f has no orbit of size four which is contained in an arc, although the x axis does contain orbits of all other periods in this example. Thus, the existence of an orbit of period three which is contained in an arc does not necessarily imply that there is an orbit of every period contained in an arc.

Definition. A *dendrite* is a locally connected, uniquely arcwise connected metric space. A subset X of a dendrite D will be called *lined* if X is contained in an arc. If D is a dendrite, we define $[x,y] = [y,x] \subseteq D$ to be the unique arc contained in D having x and y as endpoints. We let $(x,y] = [x,y] - \{x\}$ and $(x,y) = [x,y] - \{x,y\}$. Note that the "open" intervals (x,y) are not always open in the topology of D. If D and E are dendrites, with $D \subseteq E$, then there is a unique retraction $r : E \to D$ having the additional property that each point outside of D maps to the boundary of D. Such a retraction will be called the *natural retraction.* A metric d on D is called a *taxicab metric* if whenever $z \in [x,y]$, $d(x,y) = d(x,z) + d(z,y)$. If D and E are dendrites with taxicab metrics d and e, respectively, and $f : D \to E$, then f is called *linear* if there is a constant c so that for every $x, y \in D$, $e(f(x), f(y)) = cd(x,y)$. We say that f is *piecewise linear* if D is the union of finitely many dendrites $D_1, ..., D_n$ and $f \restriction D_i$ is linear for each i. If $X \subseteq D$, then we define $[X]$ to be the smallest subcontinuum of D which contains X. If X is finite, then $[X]$ is always a tree.

The main purpose of this paper is to investigate how Sharkovskiĭ's Theorem changes when an orbit in the unit interval is replaced by a lined orbit in a dendrite. In the remainder of this section, we give a few needed facts about dendrites, which are all obvious for trees. Section 2 gives a brief discussion the standard Markov Graph construction, which has only very minor changes in the dendrite setting. In Section 3, we discuss a method which will sometimes allow us to modify a lined orbit so that it is no longer lined, and Section 4 will have the additional results which are needed to lead up to the main theorem, which gives a complete characterization of all subsets of the positive integers of the form $\{n : f$ has a lined orbit of size $n\}$, where f ranges over all continuous functions on dendrites. It is assumed that the reader has some familiarity with previous results in the area, especially in the use of Markov Graphs to get periodic orbits, and the proofs of these routine results will sometimes be only outlined, or omitted altogether. Since our main interest is in trees, the proofs of some easy facts about dendrites will also be left to the reader.

Theorem 1.1. *If $D \subseteq E$ are dendrites, $r : E \to D$ is the natural retraction, and $f : E \to E$ is any continuous function on E, then $Per(r \circ f) \subseteq Per(f)$, where $Per(g)$ is defined to be $\{n : g$ has a point of period $n\}$, for any function g.*

Proof: A minor modification of Corollary 4.2 of [Ba2]. □.

From the above theorem, we get the following very weak version of Sharkovskiĭ's Theorem.

Corollary 1.2. *If f is a continuous function on a dendrite, and has a lined orbit of size n, then for all $m \lhd n$, f has an orbit (not necessarily lined) of size m, where $\lhd$ is the usual Sharkovskiĭ ordering* (see [Sa]).

Proof: Let r be the natural reatraction to an arc containing the orbit of size n. Apply Sharkovskiĭ's Theorem to this orbit, using the function $r \circ f$, and then apply Theorem 1.1. □

Of course, there are trivial maps on the triod having orbits of periods one and three, and no other periods, showing that the word "lined" is crucial in the above Corollary.

Proposition 1.3. *Every closed, connected subset of a dendrite is a dendrite.*

Proposition 1.4. *Every dendrite admits a taxicab metric.*

2. Markov Graphs for lined orbits on Dendrites

Definition. Let D be a dendrite. If $f : D \to D$ is continuous and X is a finite f-invariant lined subset of D, then one can form the *Markov Graph* of S in the usual way for intervals. Vertices of the Markov Graph are intervals $[x, y]$ (called *basic intervals*) such that $[x, y] \cap X = \{x, y\}$, and we say that $[x_1, y_1] \to [x_2, y_2]$ if and only if $[x_2, y_2] \subseteq [f(x_1), f(y_1)]$. We say that the relation $[x_1, y_1] \to [x_2, y_2]$ is *orientation preserving* if $x_2 \in [f(x_1), y_2]$, and *orientation reversing* otherwise. A *walk* is any sequence $[x_0, y_0] \to [x_1, y_1] \to \cdots \to [x_n, y_n]$ such that the indicated $\to$ relations hold, and a *loop* of length n is any walk as above having the additional property that $[x_n, y_n] = [x_0, y_0]$. For convenience, we generally let $[x_0, y_0] = [x_n, y_n]$ in this setting. A loop is said to be *orientation preserving* if an even number of the relations $[x_i, y_i] \to [x_{i+1}, y_{i+1}]$, $0 \leq i \leq n-1$, are orientation reversing, and the loop is called *orientation reversing* otherwise. Such a loop is called *repetitive* if there is a k dividing n, $1 < k < n$ such that $x_i = x_{i+k}$ and $y_i = y_{i+k}$ for all $i, 0 \leq i \leq n-k$, and is called *nonrepetitive* otherwise.

The main difference between arguments using Markov Graphs on the interval and Markov Graphs on trees (or dendrites) is that the orientation of the loops becomes a major factor. The following two lemmas are only minor variations of results which appeared in [Ba2], so only brief outlines of the proofs are given.

Lemma 2.1. *Suppose* $0 \leq a < b \leq 1$, $f : [a,b] \to [0,1]$ *is continuous, and* $f(a) = 0$, $f(b) = 1$. *Then* f *has a fixed point* x *such that* f *is not constant on any neighborhood of* x.

Proof: Let a' be largest so that $a \leq a' < b$ and $f(a') = 0$. Let x be the smallest fixed point greater than or equal to a'. □

Lemma 2.2. *Suppose* $P \subseteq D$ *is lined,* $f : D \to D$ *continuous, with* $f(P) \subseteq P$, *and suppose* $I_1 \to I_2 \to \cdots \to I_n \to I_1$ *is an orientation preserving closed loop in the Markov Graph of* P. *Then there is a point* $x \in I_n$ *such that* $f^n(x) = x$ *and* $f^j(x) \in I_j$ *for all* j, $1 \leq j \leq n$.

Proof: Identify I_n with the unit interval $[0,1]$, and for each j, let $\pi_j : D \to I_j$ be the natural retraction. Let $g_j = \pi_{j+1} \circ f$, and let $g = g_{n-1} \circ g_{n-2} \circ \ldots \circ g_2 \circ g_1 \circ g_0$, and note that $g : [0,1] \to [0,1]$ satisfies the hypothesis of Lemma 5. Let x be the resulting fixed point of g having the property that g is not constant on any neighborhood of x. Then x (and its images) were not moved by π_j, and it is easy to show that x is as desired. □

Definition. Let $p : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a pattern. We say that p is a *doubling* if $n = 2k$ for some k and $|p(2i) - p(2i-1)| = 1$ for all $i, 1 \leq i \leq k$. In this case, identifying $2i-1$ and $2i$ gives a new pattern q at size k (which can be defined using the formula $g(i) = \frac{1}{4}(p(2i-1) + p(2i) + 1))$. In this case, p is called a doubling of q. For any p, the resulting q is unique, but a fixed q may have many different doublings.

One well-known characterization of a doubling is the existence of a certain type of loop in the Markov Graph $G(p)$. Thus, p is a doubling iff there is a closed loop $I_1 \to I_2 \to \cdots \to I_n \to I_1$ such that once this loop is entered it can never be exited, so that the loop could be thought of as a "sink loop". It is also well-known that if this "sink loop" is removed from $G(p)$, then what remains is isomorphic (including orientation) to $G(q)$, where p is a doubling of q. (See [Ba2], for example.)

Definition. Given $p : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ a permutation cycle, the *induced loop* of p can be defined as follows. Let $f_p; [1, n] \to [1, n]$ be the piecewise linear extension of p, pick $x \in [1, 2]$ such that for all $j, 1 \leq j \leq n$, $f^j(x)$ is contained in a basic interval, and $f^n[1, x] = [1, 2]$. Such x is uniquely defined, and the loop $[1, 2] = I_0 \to I_1 \to I_2 \to \cdots \to I_n = I_0$ such that $f^j[1, x] \subseteq I_j$ is called the induced loop.

Proposition 2.3. *p is a doubling iff the induced loop of p is repetitive.*

Proposition 2.4. *The induced loop of p is always orientation preserving.*

The following theorem is one of the basic tools for generating periodic orbits of a given period, and the corresponding version on the unit interval is well known (see [Str], [BGMY], for example) The generalization to dendrites requires a slightly modified proof, and a weaker conclusion, than the case on the interval.

Theorem 2.5. *Let D be a dendrite. Suppose $P \subseteq D$ is a lined orbit, $f : D \to D$ continuous, with $f(P) \subseteq P$, and suppose $I_0 \to I_2 \to \cdots \to I_n = I_0$ is a nonrepetitive orientation reversing closed loop in the Markov Graph of P. Then either*

(i) *f has an orbit of size n contained in $[P]$, or*
(ii) *f has an orbit of size $2n$ contained in $[P]$ which is a doubling of a doubling.*

Proof: Repeating the loop twice gives a loop which is orientation preserving, but repetitive. For convenience, call these intervals $I_0, \cdots, I_{2n}$. By backwards induction, define minimal intervals $J_k \subseteq I_k$ such that $J_{k+1} \subseteq f(J_k)$. With care (see Prop. 3.7 of [Bal]), this can be done so that $J_k \subseteq J_{k+n}$, $J_k, k = 0, 1, 2, ..., n-1$ are pairwise disjoint, and $f^{2n} : J_0 \to J_{2n}$ satisfies the hypothesis of Lemma 2.1. The resulting periodic point x must then be either of period n, or a doubling of period $2n$. If neither (i) nor (ii) holds for x, then x has pattern p, a doubling of some pattern q, which is not a doubling. We now let $Q = \{f^j(x) : 1 \leq j \leq 2n\}$ and look at the Markov Graph for Q, which has a nonrepetitive orientation preserving closed loop of size n, since the Markov Graph of q does. Therefore, by Lemma 2.2, there is a lined orbit of size n. □

3. Moving Periodic Points off of a Line

Now that we have seen that lined periodic orbits arising from nonrepetitive orientation preserving closed loops must exist, we examine the orientation reversing case in more detail, showing that the conclusion of Theorem 2.5 cannot be improved. While lined orbits may exist in this case, and cannot actually be "removed" as periodic orbits, we shall show that they can often be moved in such a way that they are no linger lined. There are two steps to the procedure. In the first step, a well-known trick is used to expand the desired orbit to a set of closed intervals of orbits. In the next step, modification on these intervals produces the desired result. The orientation reversing assumption will be easily seen to be crucial.

Lemma 3.1. *Let $f : D \to D$ be continuous and piecewise monotone, let $I \subseteq D$ be an interval containing an orbit $\{x_1, x_2, \cdots, x_n\}$ of size $n \geq 3$, and assume that every element of I has only finitely many f-preimages in I. Suppose, in addition, that f is monotone on some neighborhood of each x_j (i.e. the x_j's are not on the boundary of the maximal monotone pieces). Then there is a dendrite E, a continuous function $g : E \to E$, and a continuous surjection $h : E \to D$ such that $h \circ g = f \circ h$ and*

- (i) *Each element of $h^{-1}(I)$ has only finitely many g-preimages in $h^{-1}(I)$.*
- (ii) *If $x \in D$ and f is monotone on some neighborhood of x, then g is monotone on some neighborhood of $h^{-1}(x)$.*
- (iii) *For each $j, 1 \leq j \leq n, h^{-1}(x_j)$ is homeomorphic to $[0, 1]$.*
- (iv) *If $y \in E$ and $h(y) \notin \{x_1, \cdots, x_n\}$, then y is a point of period k iff $h(y)$ is a point of period k.*

Proof: Let Y be the smallest subset of I which contains $\{x_1, \cdots, x_n\}$ and is closed under both f-images and f preimages in I. Clearly, Y is countable, since each element of I has only finitely many relevant preimages. Replace each point of Y by a closed interval, making sure that sum of the lengths of these intervals is finite, and call the new dendrite E. Define the continuous function $g : E \to E$ by using f on the 'old' points, and letting g be one-to-one on the new intervals. We then let h be the function which swithches the new intervals back to the old points which they came from. □

Lemma 3.2. *Assume all of the hypotheses of Lemma 3.1, and let E, g and h be as in the conclusion. In addition, suppose that f^n is orientation reversing on some neighborhood of x_1. Then there is a dendrite E' containing E (where E' is E with one copy of $[0,1]$ attached) and $g' : E' \to E'$ such that*

(i) $g' \restriction E - \cup_{j=1}^{n} h^{-1}(x_j) = g \restriction E - \cup_{j=1}^{n} h^{-1}(x_j)$

(ii) *The only periodic points in $(E' - E) \cup \bigcup_{j=1}^{n} h^{-1}(x_j)$ belong to the union of an unlined orbit of period of period n and a lined orbit of period $2n$.*

Proof: Pick i, j, k such that x_j is between x_i and x_k. By the orientation reversing hypothesis, there are points $y_1, y_2 \cdots y_n \in E$ such that $h(y_s) = x_s, 1 \le s \le n, y_s$ is in the interior of $h^{-1}(x_s)$, and $\{y_1, \cdots, y_n\}$ is a g-orbit. Attach a copy of $[0,1]$ to y_j at an endpoint of $[0,1]$ and define $g'(y) = g(y_j)$ for all new points. Let m be such that $g(y_m) = y_j$, and let z be the unattached endpoint of the new copy of $[0,1]$. Define $g'(y_m) = z$. Define g' on $\bigcup_{s=1}^{n} h^{-1}(x_s)$ as in the above triod example so that (i) holds. □

Note that if one tries to do this starting with orientation preserving orbits, then the new function produces two new orbits of size n rather than one new orbit of size $2n$.

It is easy to see that successive applications of Lemmas 3.1 and 3.2 can be used to change finitely many lined orbits to unlined orbits, provided they satisfy the orientation reversing hypothesis, and that if one starts with a tree, then one ends with a tree, the number of arcs added being the same as the number of orbits changed. It is also clear that by making the added arcs arbitrarily small, one can produce a dendrite which alters infinitely many such orbits.

4. Possible Sets of Lined Orbits

We now want to see exactly what sets of lined orbits are possible, and to do this, we need to look more closely at the structure of the patterns.

Definition. Let $n = 2k+1$ be odd. Then an orbit having pattern $1 \to k+1 \to k+2 \to k \to k+3 \to k-1 \to k+4 \to k-2 \cdots \to 2k \to 2 \to 2k+1 \to 1$ is called a *Štefan orbit* (see [Ste]). If we let p be the permutation described in the previous sentence, then any orbit having pattern $p'(i) = n+1-p(n+1-i)$ is also called a *Štefan orbit.* The corresponding cycles p and p' will be called *Štefan cycles.*

Definition. We define the *primary* permutation cycles on $\{1, 2, \ldots, n\}$ by induction on n. To start the induction, the cycle of length one is primary. Let $n = (2k+1)2^j > 1$, and suppose that the term *primary* has been defined for all cycles having length less than n. Let p be a cycle of length n. If $j = 0$, then p is *primary* iff p is Štefan. If $j > 0$, then let $N(s) = \{s(2^j) + t : t = 1, 2, \ldots, 2k+1\}$, for each s, $0 \le s \le 2^j - 1$. The p is *primary* iff

(1) $p\{1, 2, \ldots, n/2\} = \{(n/2)+1, \ldots, n\}$
(2) $p^2 \restriction \{1, 2, \ldots, n/2\}$ is primary
(3) For each s, $0 \le s \le 2k+1$, $p(N(s)) = N(s')$, for some s'
(4) $p \restriction N(s)$ is either strictly increasing or strictly decreasing, for all but one value of s.

Theorem 4.1. (Block-Coppel)*: Each permutation cycle forces a primary permutation cyle of the same size.* (See [BC])

An equivalent definition, but less useful in this setting, would be to define a cycle p to be primitive iff it forces no other cycles of the same length. Several different terms have been used for this property, and the above definition follows [ALM], which used the term *primary*. Other terms which have been used for the same property (or eqivalent properties) include *strongly simple* [BC] and *minimal* [Bal].

If n is not a power of two, then a primary pattern of size n is clearly not a doubling. Thus, we get

Theorem 4.2. *Let $f : D \to D$ be continuous, and have a lined orbit of size n, and suppose $n \rhd k$, where k is not a power of two. Then f has a lined orbit of size k.*

Proof: The pattern of size n must force a primary pattern of size k, which is not a doubling, and this pattern must correspond to a loop in the Markov Graph. Therefore, by either Lemma 2.2 or Theorem 2.5 (depending on the orientation of the loop), f has a lined orbit of period k. □

Note that this theorem does not say there are no orbits of size k which can be altered. In general, there will be many such alterable orbits, but at least one lined orbit of size k will exist which cannot be charged to unlined (without introducing more lined orbits of size k). This tells us that we have a Sharkovskiĭ type theorem for lined orbits on dendrites, except for the powers of two.

Theorem 4.3. *Let $j \geq 1$, and let p be a primary cycle of length 2^j. Then the Markov Graph of p has exactly one nonrepetitive orientation reversing closed loop for each smaller power of 2 (i.e. $1, 2, 4, ..., 2^{j-1}$), and no other nonrepetitive closed loops.*

Proof: By induction on j. It is clear for $j = 1$, so suppose $j \geq 1$ and the theorem is true for j. Let $k = j + 1$, and number the intervals of the Markov Graph from left to right in the usual manner as $1, 2, \cdots, 2^k - 1$. Then the odd numbers form a nonrepetitive orientation reversing closed loop of size 2^j. Any walk through the Markov Graph either becomes trapped in this loop, or stays in the even-numbered intervals. But the even numbers form a subgraph isomorphic (including orientation) to the Markov Graph of the corresponding primary cycle of length 2^j, and the rest follows immediately from the induction hypothesis. □

Theorem 4.4. *Let $n \geq 3$ be odd, and let p be a Štefan cycle of size n. Then the Markov Graph of p has one nonrepetitive closed loop each of length* $1, 2,$ *and* 4*, which are all orientation reversing, and for any positive integer $k \lhd n$ other than* $1, 2,$ *or* 4*, the Markov Graph has a nonrepetitive orientation preserving closed loop of length k.*

Proof: Since the Markov Graphs of Štefan cycles are well known, this is an easy exercise. □

Corollary 4.5. *If $n \geq 3$ is odd, and $f : D \to D$ has a lined orbit of size n, then for all $k \lhd n$, with the possible exception of $k = 4$, f has a lined orbit of size k. In addition, examples exist for all such n to show that f need not have a lined orbit of period 4.*

Proof: Let p be the pattern (in the sense of interval maps) of the lined orbit of size n given in the hypothesis. By Theorem 4.1, p forces a Štefan pattern q, which must correspond to a loop in the Markov graph. Since a Štefan pattern is not a doubing, the same argument as Theorem 4.2 now gives a Štefan orbit of size n, so the rest follows from Theorem 4.4. To get the examples, start with the usual piecewise linear Štefan orbit on an arc, add an arc to form a simple triod, and use Lemmas 3.1 and 3.2 to change the orbit of period 4 from lined to unlined. □

Theorem 4.6. *Let $n \geq 3$ be odd, $j \geq 1$, and let p be a primary orbit of size $2^j \cdot n$. Then the Markov Graph of p has one nonrepetitive closed loop each of length* $1, 2, 2^2, \cdots, 2^{j+2}$*, all of which are orientation reversing, and for any positive integer*

$k \lhd 2^j \cdot n$ other than $1, 2, 2^2, \cdots, 2^{j+2}$, the Markov Graph has a nonrepetitive orientation preserving closed loop of length k.

Proof: An easy proof by induction, using Theorem 4.4 and the definition of primary. □

Combining these theorems gives us our main result.

Main Theorem. (1) *Let D be a dendrite, and let $f : D \to D$ be continuous and have a lined orbit of size $n = m \cdot 2^j$, where m is odd. Then for some $S \subseteq \{4, 8, \cdots, 2^j, 2^{j+1}, 2^{j+2}\}$, where S does not contain two consecutive powers of 2, f has a lined orbit of size k for all k such that $k \lhd n$ and $k \notin S$.*

(2) *Conversely, for any set X of positive integers not ruled out by (1), there is a dendrite and a continuous function $f : D \to D$ such that $\{k : f$ has a lined orbit of size $k\} = X$. If S is not an infinite subset of $\{2^j : j = 1, 2, 3 \cdots\}$, then D may be chosen to be a tree.*

Proof: Most cases have already been proven in the previous results. Using Theorems 4.3 and 4.6, the remaining cases require only trivial modifications of the argument in Corollary 4.5. □

It is easy to adjust the main theorem to trees in general (rather than dendrites), or to a specific tree. If T is a tree, and m is the largest number of nodes contained in some lined subset of T, the the statement of the theorem is essentially the same, subject to the additional restriction that the number of 'missing' powers of two can be no more than m.

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