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ON ABSOLUTE LIPSCHITZ NEIGHBOURHOOD RETRACTS, MIXERS, AND QUASICONVEXITY

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ABSTRACT. In this paper we study compact subsets of Euclidean spaces having Lipschitz continuous *local mixers*. This property, together with the condition of *local quasiconvexity*, characterizes those subspaces which are Lipschitz neighbourhood retracts. The spaces having Lipschitz continuous local mixers are Lipschitz LC^n spaces for $n > 0$, but in general not locally quasiconvex. By a suitable compatible re-metrization of a compact space, every (local) mixer of the space can be made Lipschitz continuous.

1. INTRODUCTION.

Since their first study by Borsuk [4] in 1931 absolute (neighbourhood) retracts have proved useful in several parts of topology, especially in approximation of maps, general position arguments and fixed point theory. Lipschitzian retracts have been considered in geometric measure theory (Federer–Fleming [6], Almgren [2]) and as such (Luukkainen [11]). Recently near relatives of these spaces (e. g., *linearly locally connected spaces*) have been studied in the context of quasiconformal mappings and uniform domains by Gehring [7] and others. It is not clear whether every ANR can be made a *Lipschitz* ANR by suitable metrization. There seem to be too many external relations which need attention. Thus, it is useful to obtain an efficient internal characterization of these spaces. There is a uniform and compact idea catching the essence of finite-dimensional retracts, called mixer — a sort of geometric trick first used in van Mill – van de Vel [13] — which we exploit here. The application of the methods from [*ibid*] for proving local Lipschitz connectedness in dimensions greater than zero is straightforward, but surprisingly the zero-dimensional case cannot be handled (see below). Furthermore, difficulties with canonical covers force us at this point to leave the general problem aside. We will prove a result for maps from subspaces (e.g. Lipschitz manifolds) of Euclidean spaces. The most general result one can state here concerns spaces with canonical covers whose nerves are sufficiently similar to Whitney triangulations of open subspaces of Euclidean spaces.

As was discovered in [13], the extension problem for maps from Euclidean spheres S^n can be reduced to considering three parameters. At some points only two of these are mutually desirable so that one needs a map $\mu : X^3 \to X$ onto the range X which at certain points forgets one of its variables. The map μ is called a *mixer* if $\mu(x,x,y) = \mu(x,y,x) = \mu(y,x,x) = x$ for all $x, y \in X$. In Euclidean spaces natural mixers are easy to come by; the middle point map yields a continuous elementary mixer on the reals and in the higher dimensional cases one can use products of elementary mixers. These mixers map Euclidean cubes onto themselves; hence any open subset of a Euclidean space has a mixer if local maps are allowed — more precisely, we have a local mixer $\mu : U \to X$ where $\mathrm{dom}(\mu) = U$ is a neighbourhood of the diagonal $\Delta(X^3)$ in X^3.

Since (by [13]) having a local mixer is equivalent to being an ANR for any finite-dimensional Peano continuum one might suppose that having a Lipschitz continuous mixer implies the property of being an "ALNE" for finite dimensional compacta. Indeed, this would suffice for extensions in all dimensions greater than 0 (here we mean extensions from spheres) but would not yield the essential property of (local) quasiconvexity considered in Luukkainen–Väisälä [12]. In fact, any compact ANR can be remetrized to have a Lipschitz continuous mixer without non-degenerate rectifiable paths! We shall discuss this matter at the end of our note. Even in Euclidean spaces, the mixer property does not imply quasiconvexity: the so-called Koch arcs (quasicirles [7]) are examples of spaces having Lipschitz continuous (local) mixers without having any non-degenerate rectifiable subarcs.

2. The Lipschitzian LC^n property.

In this section we consider the k-dimensional local connectivity properties of finite-dimensional absolute neighbourhood retracts in the setting of metric spaces and Lipschitz continuous mappings. For local connectivity and general properties of absolute neighbourhood retracts, the reader is referred to Hu [9]. We recall that a topological space X is a n-connected ($n \geq 0$) (in other words, X is a C^n space) if for every $0 \leq k \leq n$, and for every continuous map $f : S^k \to X$, there is a continuous extension $\bar{f} : E^{k+1} \to X$ (here E^{k+1} denotes the $(k+1)$-dimensional Euclidean ball with S^k as the boundary.) Similarly, X is called a locally n-connected (LC^n) space, if for every $0 \leq k \leq n$, for every $x \in X$ and for every neighbourhood U of x in X there is a neighbourhood $V \subset U$ of x such that every continuous map $f : S^k \to V$ has a continuous extension to a map $\bar{f} : E^{k+1} \to U$. A map $f : X \to Y$ is called a *Lipschitz* map if there is an L such that $d(f(x), f(y)) \leq L\varrho(x,y)$ for all $x, y \in X$, where d is a metric on X and ϱ is a metric on Y. The least such constant is called the Lipschitz constant of f. The Lipschitz constant of f is denoted by $\mathrm{Lip}(f)$. Let us now define a local connectivity property related to Lipschitz mappings. We say that (X, d) is an $LLC(n)$ space if there is a $K < \infty$ such that the following holds: given $p \in X, 0 < \epsilon \leq 1$ and a Lipschitz map $f : S^n \to B_d(p, \epsilon/K)$, we can find an extension $\bar{f} : E^{n+1} \to B_d(p, \epsilon)$ with $\mathrm{Lip}(\bar{f}) \leq K\mathrm{Lip}(f)$. (We denote by $B_d(p, \epsilon)$ the open ball of radius ϵ centered at p.) It is easily seen that every compact ALNE (i. e., an absolute Lipschitz neighbourhood extensor) is an $LLC(n)$ space for every n. We say that (X, d) is an LLC^n space if (X, d) is an $LLC(k)$

space for $0 \le k \le n$. For information on absolute Lipschitz (neighbourhood) extensors, we refer the reader to [11].

A metric space (X, d) is called *C-quasiconvex*, if for every pair (x, y) of points of X there is a path from x to y of length at most $C \cdot d(x, y)$. Local quasiconvexity is defined in a similar way. It is not difficult to prove that every compact, connected $LLC(0)$ space (in particular, every compact connected ALNE) is quasiconvex. It will be seen below that although the property of having a Lipschitz continuous local mixer implies the $LLC(n)$ property for every $n > 0$ (Theorem 2.1), it does not imply local quasiconvexity (Section 4). Anyhow, a metric space having a Lipschitz continuous local mixer is locally connected in a strong sense. Dropping an additional condition required in the definition given in [7], we say that a metric space (X, d) is *linearly locally connected* if there is a constant $L < \infty$ such that given $x \in X$ and $0 < \epsilon \le 1$, we can find a connected set C with $B_d(x, \epsilon/L) \subset C \subset B_d(x, \epsilon)$.

Suppose that (X, d) has a Lipschitz continuous local mixer $\mu : U \to X$, where U is a neighbourhood of $\Delta(X^3)$ in X^3. (In this paper, products of metric spaces are considered with the product metric for which the distance between two points is the maximum coordinate distance.) Let L be the Lipschitz constant of μ. We can assume that $L \ge 2$. As we are considering here only absolute neighbourhood retracts, we assume that X is locally connected. Given a point p of X, p has a connected neighbourhood V such that V^3 is contained in $\mathrm{dom}(\mu)$. Choose a $\delta > 0$ with $B_d(p, 2\delta) \subset V$; then for all $\epsilon > 0, q, r, s \in V$ such that $s \in B_d(p, \delta), d(q, s), d(r, s) < \epsilon/L$ and $\epsilon < \delta$, the inequality

$$d(\mu(q, r, x), \mu(s, s, x)) \le \max\{Ld(q, s), Ld(r, s)\} < \epsilon$$

holds for all $x \in V$, and as $\mu(s, s, x) = s$, therefore also does

$$B_d\Big(s, \frac{\epsilon}{L}\Big) \subset \mu\Big(B_d\Big(s, \frac{\epsilon}{L}\Big) \times B_d\Big(s, \frac{\epsilon}{L}\Big) \times V\Big) \subset B_d(s, \epsilon).$$

A similar relation holds for the sets

$$B_d(s, \epsilon/L) \times V \times B_d(s, \epsilon/L),$$

$$V \times B_d(s, \epsilon/L) \times B_d(s, \epsilon/L).$$

The union of these sets is connected and hence X is linearly locally connected at the points of $B_d(p, \delta)$. If X is compact, it follows that X is linearly locally connected. However, a linearly locally connected space need not be locally quasiconvex. For example, there are *quasiconformal* images of S^1 in R^2 (Koch arcs) which have no non-degenerate rectifiable subarcs. We will show later (Section 4) that the existence of a Lipschitz continuous mixer does not imply the existence of non-degenerate rectifiable arcs. Let us now consider the $LLC(n)$ property for $n > 0$.

Theorem 2.1. *Let (X, d) be a compact metric space with a Lipschitz continuous local mixer μ. Then (X, d) is an $LLC(n)$ space for all $n > 0$.*

Proof: Let $n > 0$ be fixed and write $A = \overline{E^{n+1} \setminus \frac{1}{2}E^{n+1}}$, where $\frac{1}{2}E^{n+1}$ denotes the subset $\{x \in E^{n+1} : \|x\| \leq 1/2\}$ of E^{n+1}. Then the radial projection $r : A \to S^n$ gives a Lipschitz retraction onto S^n. Let us define $S^n_+ = S^n \cap (\mathbb{R}^n \times [0, \infty))$ and $S^n_- = S^n \cap (\mathbb{R}^n \times (-\infty, 0])$ and let $p = (1, 0, \ldots, 0)$. Then $p \in S^n_+ \cap S^n_-$ and both S^n_+ and S^n_- are absolute Lipschitz extensors (being Lipschitz homeomorphic to E^n). Write $A_+ = S^n_+ \cup \frac{1}{2}E^{n+1}$ and $A_- = S^n_- \cup \frac{1}{2}E^{n+1}$. The map $A_+ \to S^n_+$ which maps $\frac{1}{2}E^{n+1}$ to p and is the identity in S^n_+ has an extension to a Lipschitz retraction $r_+ : E^{n+1} \to S^n_+$. Similarly, there is a Lipschitz retraction $r_- : E^{n+1} \to S^n_-$ such that $r_-(\frac{1}{2}E^{n+1}) = \{p\}$. Now define

$$L = \text{Lip}(\mu) \max\{\text{Lip}(r), \text{Lip}(r_+), \text{Lip}(r_-)\}.$$

As X is compact and the domain $\text{dom}(\mu)$ of μ is a neighbourhood of $\Delta(X^3)$ in X^3, there is $\delta > 0$ such that $B_d(x, \delta)^3 \subset$

dom(μ) for each $x \in X$. Suppose that $f : S^n \to B_d(x, \epsilon/L)$ is a Lipschitz map for some $0 < \epsilon < \delta$. Since

$$\mu\left(B_d(x,\epsilon/L) \times B_d(x,\epsilon/L) \times B_d(x,\epsilon/L)\right) \subset B_d(x,\epsilon),$$

we can define a map $\bar{f} : E^{n+1} \to B_d(x, \epsilon)$ by setting

$$\bar{f}(z) = \begin{cases} \mu(f(r(z)), f(r_+(z)), f(r_-(z))) & \text{for } z \in A, \\ f(p) & \text{for } z \in \frac{1}{2}E^{n+1}. \end{cases}$$

Clearly $\bar{f}$ is Lipschitz in both A and $\frac{1}{2}E^{n+1}$, and since E^{n+1} is convex, Theorem 2.35 of [12] implies that $\bar{f}$ is Lipschitz (with $\mathrm{Lip}(\bar{f}) \leq L^2 \cdot \mathrm{Lip}(f)$). It follows that (X, d) is an $LLC(n)$ space. □

3. Extension of maps.

Granted that a space X with a local Lipschitz continuous mixer has the Lipschitzian LC^n properties, we shall next prove that such an X is a Euclidean Lipschitz neighbourhood extensor in the sense that every Lipschitz map $f : Y \to X$ from a subspace Y of a Euclidean space $\mathbb{R}^n$ can be extended to a Lipschitz map of a neighbourhood of Y in $\mathbb{R}^n$. In fact, we can prove more but we need a quantitative concept of dimension.

For convenience we say that Y is of *quantitative* dimension $\leq n$ if every cover of Y with Lebesgue number $l > 0$ has a refinement $\mathcal{U}$ satisfying the properties

(i) $\lambda(\mathcal{U})$ = the Lebesgue number of $\mathcal{U} \geq l/K$;
(ii) $\mathrm{ord}(\mathcal{U}) \leq n+1$

for some *fixed* $K \geq 1$. (Here $\mathrm{ord}(\mathcal{U})$ denotes the maximal number of elements of $\mathcal{U}$ containing any given point.) Thus, Y has a refining sequence $< \mathcal{U}_i >$ of open covers of order $\leq n+1$ with $\lambda(\mathcal{U}_i) \geq 1/K^{i+1}$ such that $\mathcal{U}_i$ refines the cover consisting of all balls of radius $1/K^i$. We can assume below – for purely technical reasons – that Y contains an isometric copy of the metric subspace $S = \{2^{-n} : n \in \mathbb{N}\}$ of the real line. (If necessary, we can replace Y by the cartesian product

$S \times Y$.) Under this assumption, Y has a constant K and a refining sequence $< \mathcal{U}_i >$ such that

$$\cdots \leq 1/K^{i+1} \leq \lambda(\mathcal{U}_i) \leq \text{mesh}(\mathcal{U}_i) \leq 1/K^i \leq \ldots.$$

For example, subspaces of $\mathbb{R}^n$ satisfy the above condition. (This claim can be proved as follows. It is obviously enough to establish it for the space $\mathbb{R}^n$ itself. It is also sufficient to show that $\mathbb{R}^n$ has a uniform open cover $\mathcal{U}$ of order $n+1$ with mesh($\mathcal{U}$) bounded, because then the *scalar multiples* $\{k^{-n}U : U \in \mathcal{U}\}$ of $\mathcal{U}$ (for a suitable positive integer k) provide a sequence satisfying the above definition. (We define $aU = \{ax : x \in U\}$ for $a \in \mathbb{R}$ and $U \subset \mathbb{R}^n$.) In order to show this, we can use, instead of a direct construction, the standard uniformly continuous covering map $f : \mathbb{R}^n \to T^n$, where T^n is the standard n-torus. As T^n is compact and n-dimensional, it has such a cover $\mathcal{U}$ with mesh($\mathcal{U}$) $< 1/2$. Then the pre-images $f^{-1}(U)$ of the elements of $\mathcal{U}$, divided into the separate leaves, define a cover of the desired form.)

Extensions from subspaces are frequently obtained via canonical covers but they do not suffice here. To obtain an adequate cover, we have to assume in an extension situation $f : A \to X$ (where A is a subspace of Y) that either A or $Y \setminus A$ is of finite quantitative dimension. Let us handle the first case. (The second case is similar but simpler.)

Theorem 3.1. *Suppose that (X, ρ) has the property LLC^n for every n. Let A be a closed subspace of a metric space (Y, d) of finite quantitative dimension. Then every Lipschitz map f : $A \to X$ has a Lipschitz extension $\bar{f} : W \to X$ to a (uniform) neighbourhood W of A in Y.*

Proof: Choose a sequence $< \mathcal{U}_i >$ with a constant K for A (as in the definition of finite quantitative dimension) such that $\text{ord}(\mathcal{U}_i) \leq n+1$ for all i. We will assume $K \geq 2$. Setting $\alpha_i = \lambda(\mathcal{U}_i)/2$, every $\mathcal{U}_i$ has an α_i-shrinking $\mathcal{U}_i'$ with $\lambda(\mathcal{U}_i') \geq \lambda(\mathcal{U}_i)/2$ and for which $B_d(y, \alpha_i)$ meets at most $n+1$ members of $\mathcal{U}_i'$ for all $y \in A$. We extend $\mathcal{U}_i'$ to a cover of a neighbourhood of A as follows. Enlarge each $U \in \mathcal{U}_i'$ by setting

$$\tilde{U} = U \cup \{y \in Y \setminus A : d(y, U) < \alpha_i/2\}.$$

Then $\tilde{\mathcal{U}}_i = \{\tilde{U} : U \in \mathcal{U}_i'\}$ covers $B_d(A, \alpha_i/2), \operatorname{ord}(\tilde{\mathcal{U}}_i) \leq n+1$ and $\lambda_y(\tilde{\mathcal{U}}_i) \geq \alpha_i/4$ for all $y \in B_d(A, \alpha_i/4)$. Here λ_y denotes the *local* Lebesgue number measured at y, i.e. $\lambda_y(\mathcal{V}) = \sup\{\delta > 0 : \text{there is a } V \in \mathcal{V} \text{ with } B_d(y,\delta) \subset V\}$. Thus, $\lambda_y(\tilde{\mathcal{U}}_i) \geq 1/(8K^{i+1})$ for all $y \in B_d(A, 1/(8K^{i+1}))$.

For $i = 1, 2, \ldots$ put

$$G_i = \{y \in Y \setminus A : 1/(8K^{i+4}) < d(y, A) < 1/(8K^{i+2})\}$$

and let $G_0 = \{y \in Y \setminus A : 1/(8K^4) < d(y, A)\}$. We obtain a cover $\mathcal{G} = \{G_i\}$ of $Y \setminus A$ with $\operatorname{ord}(\mathcal{G}) \leq 2$ and $\lambda_y(\mathcal{G}) \geq 1/(8K^{i+5})$ for all $y \in G_i$, $i > 0$. Therefore,

$$\lambda_y(\mathcal{G}) \geq d(y, A)/K^3$$

whenever $y \in Y \setminus A$. Define a family $\mathcal{V}$ covering $U = B_d(A, 1/4K)$ by letting $\mathcal{V}$ consist of all nonempty traces $\tilde{\mathcal{U}}_i \cap G_i, i \in \mathbb{N}$. Then $\operatorname{ord}(\mathcal{V}) \leq 2n + 2$ and $y \in G_i$ implies

$$1/(8K^{i+5}) \leq \lambda_y(\mathcal{V}) \leq \operatorname{mesh}_y(\mathcal{V}) \leq 3/(2K^{i-1})$$

and thus

$$d(y, A)/K^3 \leq \lambda_y(\mathcal{V}) \leq \operatorname{mesh}_y(\mathcal{V}) \leq 12K^5 d(y, A).$$

(The symbol $\operatorname{mesh}_y(\mathcal{V})$ denotes the number $\sup\{\operatorname{diam}(V) : y \in V\}$.) By the last inequality we can assume that $\mathcal{V}$ does not contain any element V for which there is no $y \in U$ such that $B_d(y, d(y, A)/K^3) \subset V$. Thus, $\operatorname{diam}(V) \geq d(V, A)/K^3$ for all $V \in \mathcal{V}$. Write $\mathcal{V} = \{V_i : i \in I\}$. For each $y \in Y$, let $I_y = \{\, i \in I : y \in V_i \,\}$.

Next we note that we can assume $Y \setminus A$ is quasiconvex. Indeed, we can even assume that each set of the form $B_d(y, \alpha) \setminus A$, where $y \in Y$ and $\alpha > 0$, is quasiconvex. (In fact, embed Y isometrically into a normed space E and let $Y' = E \times \mathbb{R}, A' = A \times \{0\}$ and denote the resulting metric of Y' likewise by d.

Then $B_d(y,\alpha)\setminus A'$ is $(1+\epsilon)$–quasiconvex for all $\epsilon > 0$ and for all $\alpha > 0$.)

As in the classical use of canonical covers in Dugundji [5], choose for each V_i an $a_i \in A$ with $d(V_i, a_i) \leq 2d(V_i, A)$. Let N denote the nerve of $\mathcal{V}$ and metrize N so that it becomes a uniform complex in the sense of Isbell [10]: the distance between two points is defined as the maximum difference between the corresponding barycentric coordinates. However, the nerve has to be modified slightly. The vertices must be assigned "weights" roughly equal to the diameters of their counterparts in $\mathcal{V}$. More precisely, we assume that N lies in the space c_0 over the set $\mathcal{V} = \{V_i\}$. If e_i is the base element of $c_0(\mathcal{V})$ assigned to V_i (i. e., $e_i(V_i) = 1$ and $e_i(V_j) = 0$ for $i \neq j$), then the corresponding vertex will be identified with $\operatorname{diam}(V_i)\cdot e_i$.

The simplices of N are not anymore standard; anyhow, we rely on the fact that for any simplex Δ of N, the diameters of the elements of $\mathcal{V}$ in the carrier of Δ deviate from each other at most by the multiplicative factor $12K^9$. For each $i \in I$, let $\tilde{e}_i = \operatorname{diam}(V_i)\cdot e_i$. Let Δ_x denote the simplex generated (in $c_0(\mathcal{V})$) by $\{\, \tilde{e}_i : i \in I_x \,\}$. Thus, $\operatorname{diam}_{c_0(\mathcal{V})}(\Delta_x) \leq \operatorname{mesh}_x(\mathcal{V})$.

We note that for all $V_i, V_j \in \mathcal{V}$, $i \neq j$, with $V_i \cap V_j \neq \emptyset$, we have $d(a_i,a_j) \leq d(a_i,V_i) + d(a_j,V_j) + \operatorname{diam}(V_i) + \operatorname{diam}(V_j) \leq 2\,(d(V_i,A) + d(V_j,A)) + \operatorname{diam}(V_i) + \operatorname{diam}(V_j)$. On the other hand

$$\|\tilde{e}_i - \tilde{e}_j\|_{c_0} \geq (1/2)(\operatorname{diam}(V_i) + \operatorname{diam}(V_j)) \geq$$
$$(d(V_i,A) + d(V_j,A))\,/(4K^3) + \operatorname{diam}(V_i)/4 + \operatorname{diam}(V_j)/4 \geq$$
$$d(a_i,a_j)/(8K^3).$$

By the LLC^0 condition, we can find $\epsilon > 0$ and a $C < \infty$ such that for every ball $B_\rho(x,\epsilon/C)$ in X, every pair a, b of points of $B_\rho(x,\epsilon/C)$ can be joined by a C-Lipschitz path $g : [0,1] \to B_\rho(x,\epsilon)$. (Notice that this condition is also implied by quasiconvexity; any rectifiable path $g : [0,1] \to X$ determines a Lipschitz path $h : [0,1] \to g([0,1])$ with $\operatorname{Lip}(h) = \operatorname{length}(g)$.) Also there is $\delta > 0$ such that $V_i, V_j \in \mathcal{V}, V_i, V_j \subset B_d(A,\delta), V_i \cap V_j \neq \emptyset$ imply $\rho(f(a_i), f(a_j)) < \epsilon/C$. Let $N_0 = \{\Delta \in N^{(0)} :$ the carrier of Δ is contained in $B_d(A,\delta)\}$. By this condition the map $h_0 : (N_0)^{(0)} \to X$ given by $h_0(\tilde{e}_i) = f(a_i)$ can be extended to a map $\tilde{h}_0 : (N_0)^{(1)} \to X$ with

$$\mathrm{Lip}(\tilde{h}_0|\Delta) \leq C\mathrm{Lip}(\tilde{h}_0|\partial\Delta) \leq 8CK^3\mathrm{Lip}(f)$$

on each 1–simplex Δ of N_0. Exactly as in the classical case, because $\dim(N) \leq 2n+1$, we inductively obtain a neighbourhood B of the neutral element in $c_0(\mathcal{V})$ and a map $\varphi : B \cap N \to X$ which is *simplexwise* uniformly (but perhaps not globally) Lipschitz and for which $\rho(\varphi(\tilde{e}_i), f(a_i)) \to 0$ whenever $d(V_i, A) \to 0$. Moreover, $\mathrm{Lip}(\varphi) \leq 8C'K^3\mathrm{Lip}(f)$ (where C' does not depend on f) on each simplex $\Delta \subseteq B \cap N$.

Let $\alpha > 0$ be such that φ is defined in Δ_x for all $x \in B_d(A, 2\alpha)$ and for which $B_d(A, 2\alpha) \subset U$. We have extended f as soon as we have defined a suitable map $\phi : B_d(A,\alpha)\backslash A \to N$. For each $V_i \in \mathcal{V}$, let $\xi_i(y) = d(y, B_d(A,\alpha) \setminus V_i)$ and put

$$\phi(y) = \sum_i \left(\frac{\xi_i(y)}{\sum_j \xi_j(y)}\right)\tilde{e}_i.$$

Notice that for all $x \in B_d(A,\alpha) \setminus A$, we have $\phi(x) \in \Delta_x$. By our assumption, $B_d(x,\alpha) \setminus A$ is $(1+\epsilon)$–quasiconvex for all $\epsilon > 0$ whenever $x \in B_d(A,\alpha) \setminus A$. We will prove that ϕ is locally L-Lipschitz in $B_d(A,\alpha) \setminus A$ for some L. Then $\varphi \circ \phi$ is locally uniformly Lipschitz in $B_d(A,\alpha) \setminus A$ and hence by [12] (Theorem 2.35) it is L'-Lipschitz in each ball $B_d(x,\alpha) \setminus A$ for some L', where $x \in B_d(A,\alpha)$. (Here L' can be taken to be the local Lipschitz constant of $\varphi \circ \phi$.) Thus, let $x, y \in Y$ belong to a carrier of a simplex of N. Since $\mathrm{mesh}_x\mathcal{V} \leq 12K^9\lambda_y(\mathcal{V})$, we have

$$\max_{x\in V_i}\left|\frac{\xi_i(x)\mathrm{diam}(V_i)}{\sum_j \xi_j(x)} - \frac{\xi_i(y)\mathrm{diam}(V_i)}{\sum_j \xi_j(y)}\right| =$$

$$\max_{x\in V_i} \mathrm{diam}(V_i)\left|\frac{\xi_i(x)\sum_j \xi_j(y) - \xi_i(y)\sum_j \xi_j(x)}{\left(\sum_j \xi_j(x)\right)\left(\sum_j \xi_j(y)\right)}\right| \leq$$

$$\max_{x\in V_i} \operatorname{diam}(V_i)\frac{\sum_j |\xi_i(x)\xi_j(y)-\xi_i(y)\xi_j(x)|}{\left(\sum_j \xi_j(x)\right)\left(\sum_j \xi_j(y)\right)} \leq$$

$$\max_{x\in V_i} \operatorname{diam}(V_i)\frac{\sum_j (\xi_i(x)|\xi_j(y)-\xi_j(x)|+\xi_j(x)|\xi_i(x)-\xi_i(y)|)}{\left(\sum_j \xi_j(x)\right)\left(\sum_j \xi_j(y)\right)} \leq$$

$$\max_{x\in V_i} \operatorname{diam}(V_i)\frac{\left((2(2n+2))\xi_i(x)+\sum_j \xi_j(x)\right)d(x,y)}{\left(\sum_j \xi_j(x)\right)\left(\sum_j \xi_j(y)\right)} \leq$$

$$\max_{x\in V_i} \frac{\operatorname{diam}(V_i)}{\sum_j \xi_j(y)}\cdot(4n+5)d(x,y) \leq 60K^9(n+1)d(x,y)$$

By symmetry we obtain that ϕ is locally $60K^9(n+1)$-Lipschitz. (Notice how the quantitative dimension of A is involved.) We shall prove that $\varphi\circ\phi$ is Lipschitz in a neighbourhood of A. Write $C''=8C'K^3\operatorname{Lip}(f)$. For each $x\in B_d(A,\alpha)\setminus A$ we have $\varphi(\phi(x))\in\varphi(\Delta_x)$, and hence $\rho\big(\varphi(\phi(x)),f(a_i)\big) = \rho\big(\varphi(\phi(x)), \varphi(\tilde{e}_i)\big) \leq C''\cdot\operatorname{mesh}_x(\mathcal{V})$ for all $i\in I_x$. (Recall that $\operatorname{diam}_{c_0(\mathcal{V})}(\Delta_x) \leq \operatorname{mesh}_x(\mathcal{V})$.) On the other hand, $\operatorname{mesh}_x(\mathcal{V})\leq 12K^5 d(x,A)$, and for each $i\in I_x$ $d(x,a_i)\leq d(V_i,a_i)+\operatorname{diam}(V_i)\leq 2d(x,A)+\operatorname{diam}(V_i)\leq(2+12K^5)d(x,A)$. Put $\beta=\alpha/(64C''K^5)$, and let x, $y\in B_d(A,\beta)\setminus A$. If $d(x,y)<\alpha/2$, then there is $z\in A$ such that x, $y\in B_d(z,\alpha)$ and so (by $(1+\epsilon)$-quasiconvexity for all $\epsilon>0$) $\rho\big(\varphi(\phi(x)),\varphi(\phi(y))\big)\leq C''Ld(x,y)$, where by the above $L=60K^9(n+1)$. On the other hand, suppose that $d(x,y)\geq\alpha/2$. We have $d(x,a_i)$, $d(y,a_j)\leq\alpha/16$ for all $i\in I_x$, $j\in I_y$. Similarly $\rho\big(\varphi(\phi(x)),f(a_i)\big)$, $\rho\big(\varphi(\phi(y)),f(a_j)\big)\leq\alpha/4$. Then

$$\frac{\rho\big(\varphi(\phi(x)),\varphi(\phi(y))\big)}{d(x,y)}\leq\frac{\rho(f(a_i),f(a_j))+\dfrac{\alpha}{2}}{d(a_i,a_j)-\dfrac{\alpha}{8}}\leq 2(\operatorname{Lip}(f)+2)$$

because $d(a_i,a_j)\geq\alpha/4$.

Finally, let $y\in A$ and $x\in B_d(A,\beta)\setminus A$. Let $i\in I_x$; then by the preceding paragraph $\rho(\varphi(\phi(x)),f(a_i))\leq L''d(x,A)\leq$

$L''d(x,y)$, where $L'' = 12K^5C''$. As $d(a_i,x) \leq 2d(x,A) \leq 2d(x,y)$, we have $\rho(f(a_i),f(y)) \leq \mathrm{Lip}(f)d(a_i,y) \leq 3\mathrm{Lip}(f)d(x,y)$, and hence $\rho(\varphi(\phi(x)),f(y)) \leq L'''d(x,y)$ for $L''' = 12K^5C'' + 3\mathrm{Lip}(f)$.

It follows that the extension of f by $\varphi \circ \phi$ to $W = B_d(A,\beta)$ is a Lipschitz map. □

Corollary 3.2. *Suppose that X is an LLC^n space for every n. Let A be a closed subspace of a metric space Y, where Y has finite quantitative dimension. Then every Lipschitz map $f : A \to X$ has a Lipschitz extension $\bar{f} : W \to X$ to a neighbourhood W of A in Y.*

The property of being of finite quantitative dimension does not imply Lipschitz embeddability into some $\mathbb{R}^n$. Consider the disjoint union of 1-skeleta of cubical subdivisions of Euclidean cubes converging to some point in an infinite-dimensional Banach space where the dimensions of the cubes (and the orders of the subdivisions) tend to infinity. If properly defined, such a space is of quantitative dimension 1 but has infinite metric dimension. However, we have not been able to show that the dimension condition in Theorem 3 is necessary. Anyhow, we get the following result.

Corollary 3.3. *Let X be a compact metric space with a Lipschitz continuous local mixer. If X is locally C-quasiconvex for some $C < \infty$, then X is an ALNE for all metric spaces of finite quantitative dimension.*

Proof: The claim follows from Theorems 2.1 and 3.1. □

Theorem 3.4. *A quasiconvex compact subset X of a Euclidean space $\mathbb{R}^n$ is an ALNE if and only if X has a Lipschitz continuous local mixer (with respect to the standard Euclidean metric).*

Proof: By Theorem 2.1, quasiconvex metric subspaces of Euclidean spaces with Lipschitz continuous local mixers are LLC^k spaces for all $k \in \mathbb{N}$. Thus, by 3.1 the identity mapping of such a subspace A onto itself extends to a Lipschitz mapping of a neighbourhood U onto A, whence A is a Lipschitz retract of an open subset of a Euclidean space (with respect to the standard metric) and therefore A is an ALNE. On the other hand, it is easy to see that an ALNE in a Euclidean space has a Lipschitz continuous local mixer. □

4. Metrization.

By a theorem of Aharoni [1], and Assouad [3], and Pelant [15], every compact metric space can be Lipschitz embedded into a Lipschitz cube (a Lipschitz cube is the Hilbert cube Q with a metric $d(x,y) = \max\{a_n|x_n - y_n| : n \in \mathbb{N}\}$, where $a_n \geq a_{n+1} \to 0$ and $\sup\{a_n/a_{n+1} : n \in \mathbb{N}\} < \infty$) which obviously has a Lipschitz continuous mixer, and it follows that every compact Lipschitz extensor has a Lipschitz continuous mixer. Here we show that every compact absolute retract X has a Lipschitz continuous mixer with respect to a specific compatible metric of X; an analogous statement holds for local mixers.

Given a compact metrizable space X and a continuous map $f : X^n \to Y$ into a metric space Y, it is not difficult to find a compatible metric of X such that f becomes Lipschitz continuous (with respect to the product metric). However, if $X = Y$, then the problem is more difficult.

We consider a more general situation where $\phi : X^p \to X$ is a continuous map, $p > 1$ and where X is a compact metric space with a metric d. We denote the group of all permutations of the set $\{1, \dots, p\}$ by S_p. As ϕ is not assumed to be symmetric, it is necessary to consider permuted maps ϕ_π defined by

$$\phi_\pi(u_1, \dots, u_p) = \phi(u_{\pi(1)}, \dots, u_{\pi(p)})$$

for all $\pi \in S_p$. For each n-tuple $\bar{\pi} = (\pi_1, \dots, \pi_n) \in (S_p)^n$ (let $(S_p)^0 = \{*\}$, where $*$ is the 0-tuple) we define a sequence of maps $f_{n,\bar{\pi}} : X^{n(p-1)+1} \to X$ by recursion as follows. Let $f_{0,*}$ be the identity map of X, and if $f_{n,\bar{\tau}}$ is already defined for all $\bar{\tau} \in (S_p)^n$, let

$$f_{n+1,\bar{\pi}}(u_1, \dots, u_{n(p-1)+1}, \dots, u_{(n+1)(p-1)+1}) =$$

$$f_{n,\bar{\pi}'}(u_1, \ldots, u_{n(p-1)}, \phi_{\pi_{n+1}}(u_{n(p-1)+1}, \ldots, u_{n(p-1)+p})),$$

where $\bar{\pi} = (\pi_1, \ldots, \pi_n, \pi_{n+1}) \in (S_p)^{n+1}$ and where $\bar{\pi}' = (\pi_1, \ldots, \pi_n)$.

The maps $f_{n,\bar{\pi}}$ witness the way ϕ is "entangled" with itself — we kill this effect by introducing a decreasing sequence (a_n) of positive real numbers such that $a_n \to 0$ damping down the maps $f_{n,\bar{\pi}}$; to do everything smoothly enough we require that $\sup\{a_n/a_{n+1} : n \in \mathbb{N}\} < 1+\epsilon$, where ϵ is a preassigned positive real number.

It is easy to prove that the family of all maps $g^{(n)}_{\bar{u},\bar{\pi}} : X \to X, \bar{u} \in X^{n(p-1)}$, where

$$g^{(n)}_{\bar{u},\bar{\pi}}(x) = f_{n,\bar{\pi}}(u_1, \ldots, u_{n(p-1)}, x),$$

and where $\bar{\pi} \in (S_p)^n$, is an equicontinuous subset of $\mathcal{C}(X, X)$ (with the compact-open topology). (Use the facts that X is compact and the set of all the $\bar{\pi}$ is finite.) Therefore, the formula

$$\sigma_n(x, y) = \sup\{a_n d(g^{(n)}_{\bar{u},\bar{\pi}}(x), g^{(n)}_{\bar{u},\bar{\pi}}(y)) : \bar{u} \in X^{n(p-1)}, \bar{\pi} \in (S_p)^n\}$$

defines a continuous pseudometric on X. Since $g^{(0)}_*$ is the identity map, σ_0 is even topologically stronger than d and hence a compatible metric on X. (A pseudometric ρ is called here *topologically stronger* than a pseudometric τ in case τ is continuous relative to ρ, in other words, if the topology generated by τ is contained in that generated by ρ.) Now let

$$\sigma(x, y) = \max\{\sigma_n(x, y) : n \in \mathbb{N}\}.$$

Clearly σ is a compatible metric on X. Recall that we use on X^p the product metric $\tilde{\sigma}$ given by $\tilde{\sigma}((x_1, \ldots, x_p), (y_1, \ldots, y_p)) = \max\{\sigma(x_1, y_1), \ldots, \sigma(x_p, y_p)\}$. Now

$$\sigma(\phi(x_1, \ldots, x_p), \phi(y_1, \ldots, y_p)) \leq$$
$$\sigma(\phi(x_1, \ldots, x_{p-1}, x_p), \phi(x_1, \ldots, x_{p-1}, y_p)) +$$
$$\sigma(\phi(x_1, \ldots, x_{p-2}, x_{p-1}, y_p), \phi(x_1, \ldots, x_{p-2}, y_{p-1}, y_p)) + \cdots +$$
$$\sigma(\phi(x_1, y_2, \ldots, y_p), \phi(y_1, y_2, \ldots, y_p)).$$

To estimate the above summands, note for example that there is an n such that

$$\sigma(\phi(x_1,\dots,x_{p-2},x_{p-1},y_p),\phi(x_1,\dots,x_{p-2},y_{p-1},y_p)) =$$

$$\sigma_n(\phi(x_1,,\dots,x_{p-2},x_{p-1},y_p),\phi(x_1,\dots,x_{p-2},y_{p-1},y_p)) =$$

$$a_n \sup\Big\{ d(f_{n,\bar\pi}(u_1,\dots,u_{n(p-1)},\phi(x_1,\dots,x_{p-2},x_{p-1},y_p)), f_{n,\bar\pi}\left(u_1,\dots,u_{n(p-1)},\phi(x_1,\dots,x_{p-2},y_{p-1},y_p))\right) : \left(u_1,\dots,u_{n(p-1)}\right) \in X^{n(p-1)}, \bar\pi \in (S_p)^n \Big\} \leq$$

$$a_n \sup\Big\{ d(f_{n,\bar\pi}(u_1,\dots,u_{n(p-1)},\phi_\tau(x_1,\dots,x_{p-2},y_p,x_{p-1})), f_{n,\bar\pi}\left(u_1,\dots,u_{n(p-1)},\phi_\tau(x_1,\dots,x_{p-2},y_p,y_{p-1}))\right) : \left(u_1,\dots,u_{n(p-1)}\right) \in X^{n(p-1)}, \bar\pi \in (S_p)^n, \tau \in S_p \Big\} =$$

$$a_n \sup\Big\{ d(f_{n+1,\bar\pi}(\bar u,x_1,\dots,x_{p-2},y_p,x_{p-1}), f_{n+1,\bar\pi}(\bar u,x_1,\dots,x_{p-2},y_p,y_{p-1})) : \bar u \in X^{n(p-1)}, \bar\pi \in (S_p)^{n+1} \Big\} \leq$$

$$a_n \sup\Big\{ d(f_{n+1,\bar\pi}(\bar u,x_{p-1}), f_{n+1,\bar\pi}(\bar u,y_{p-1})) : \bar u \in X^{(n+1)(p-1)}, \bar\pi \in (S_p)^{n+1} \Big\} \leq$$

$$= (a_n/a_{n+1})\,\sigma_{n+1}(x_{p-1},y_{p-1}) \leq (1+\epsilon)\sigma(x_{p-1},y_{p-1}).$$

The $p-1$ other summands are treated in an analogous way. We get

$$\sigma(\phi(x_1,\dots,x_{p-1},x_p),\phi(y_1,\dots,y_{p-1},y_p)) \leq$$

$$p(1+\epsilon)\max\{\sigma(x_1,y_1),\dots,\sigma(x_p,y_p)\} = \qquad ,$$

$$p(1+\epsilon)\tilde{\sigma}((x_1,\ldots,x_p),(y_1,\ldots,y_p))$$

proving that σ is a compatible metric on X such that $\phi : (X^p,\tilde{\sigma}) \to (X,\sigma)$ is a Lipschitz mapping, where $\tilde{\sigma}$ is the product metric. □

Thus, we have proved the following result:

Theorem 4.1. *Let X be a compact metrizable space and let $\phi : X^p \to X$ be a continuous mapping. Then X has a compatible metric σ such that $\phi : (X,\sigma)^p \to (X,\sigma)$ is Lipschitz continuous.*

Corollary 4.2. *Let X be a compact metrizable space and let $\mu : X^3 \to X$ be a continuous mixer. Then X has a compatible metric σ such that $\mu : (X,\sigma)^3 \to (X,\sigma)$ is Lipschitz continuous.*

We can also state a similar result for local mixers.

Corollary 4.3. *Let X be a compact metrizable space and let μ be a continuous local mixer defined in a neighbourhood V of $\Delta(X^3)$ in X^3. Then X has a compatible metric σ and $\Delta(X^3)$ has a neighbourhood $W \subseteq V$ such that μ is Lipschitz continuous in W with respect to the product metric σ^3.*

Proof: We can assume that X is a closed subspace of the Hilbert cube Q. Let W be a closed neighbourhood of $\Delta(X^3)$ in X^3 with $W \subseteq V$. As Q is an absolute extensor, the restriction $\mu|W : W \to Q$ has a continuous extension to a map $\phi : Q^3 \to Q$. By Corollary 4.2 Q has a compatible metric σ such that $\phi : (Q,\sigma)^3 \to (Q,\sigma)$ is Lipschitz continuous. Then $\sigma|(X \times X)$ is a compatible metric of X making μ Lipschitz continuous in W. □

Indeed, we can prove more. Let ρ be a compatible metric for the unit interval [0,1] such that $([0,1],\rho)$ contains no non-degenerate rectifiable arc and therefore there is no non-constant Lipschitz mapping $f : ([0,1],d) \to ([0,1],\rho)$ where d is the standard metric on $[0,1]$. (In order to obtain such a metric for $[0,1]$, first find an embedding $h : [0,1] \to \mathbb{R}^2$ such that for all $0 < a < b < 1$, the arc $h([a,b])$ has infinite length. Then define $\rho(x,y) = ||h(x) - h(y)||_2$, where $||\cdot||_2$ is any of the standard norms for $\mathbb{R}^2$.) Equip the Hilbert cube Q with the metric ρ_∞ given by $\rho_\infty(x,y) = \max\{2^{-n}\rho(x_n,y_n) : n \in \mathbb{N}\}$. Then (Q,ρ_∞) does not contain any non-degenerate rectifiable path. Indeed, suppose that $\gamma : ([0,1],d) \to (Q,\rho_\infty)$ is a non-constant Lipschitz map. Then there exist $n \in \mathbb{N}$ and $x, y \in [0,1]$ such that $\pi_n(\gamma(x)) \neq \pi_n(\gamma(y))$. The projection map $\pi_n : (Q,\rho_\infty) \to ([0,1],\rho)$ is clearly Lipschitz (with $\mathrm{Lip}(\pi_n) = 2^n$) and hence the composed map $\pi_n \circ \gamma : ([0,1],d) \to ([0,1],\rho)$ would be Lipschitz, which is impossible.

Now let X be a compact AR. We can assume that X is embedded in Q as a closed subspace; let d denote the metric also for the subspace X. Let $\mu : X^3 \to X$ be a continuous mixer. It follows from the proof of Theorem 4.1 that X has a compatible metric σ, stronger than d, such that $\mu : (X,\sigma)^3 \to (X,\sigma)$ is Lipschitz continuous. The identity $(X,\sigma) \to (X,d)$ is a Lipschitz map, too, from which it follows that (X,σ) cannot have any non-degenerate rectifiable paths and therefore, (X,σ) is not (locally) quasiconvex. As a consequence, the existence of a Lipschitz continuous mixer does not in general imply that the space is an ALE. A similar statement holds for local mixers and ALNEs.

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