Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Topology Proceedings
Vol 18, 1993

SPACES WITH A σ-HEREDITARILY CLOSURE-PRESERVING k-NETWORK*

CHUAN LIU

ABSTRACT. In this paper, we will survey some results of spaces with a σ-HCP k-network and pose some related problems.

σ-hereditarily closure-preserving (abbr. σ-HCP) k-network was first introduced by Guthrie for characterizing metrizable spaces. Afterwards, L. Foged Fo$_1$] characterized a Lašnev space as a Fréchet space with a σ-HCP k-network. In recent years, Junnila and Ziqiu Yun [JY] gave the relationship between $\aleph$-spaces and spaces with a σ-HCP k-network; Y. Tanaka characterized a g-metrizable spaces as a g-first countable space with a σ-HCP k-network, Hence, spaces with a σ-HCP k-network play an important role in studying generalized metric spaces.

Definitions: A collection $\mathcal{P}$ of subsets of topological space X is said to be a k-network for X, if, given any open set U and any compact set $K \subset U$, there is a finite subcollection $\mathcal{P}^*$ of $\mathcal{P}$such that $K \subset \cup\mathcal{P}^* \subset U$. A family $\{A_\alpha : \alpha \in I\}$ of subsets of a space X is said to be hereditarily closure-preserving (abbr. HCP) if $\bigcup_{\alpha\in J} \overline{B_\alpha} = \overline{\bigcup_{\alpha\in J} B_\alpha}$, whenever $J \subset I$ and $B_\alpha \subset A_\alpha$, for each $\alpha \in J$. If κ is a cardinal number, let S_κ be the space obtained from the disjoint union of κ many convergent sequences by identifying all the limit points to a single point x_0. S_2 denotes Aren's space.

* Partially supported by National Science Foundation of China.

All the spaces in this paper are regular and T_1. All mappings are continuous and surjective.

1. Lašnev spaces, $\aleph$-spaces, g-metrizable spaces and spaces with a σ-HCP k-network

First, Guthrie proved the following unpublished theorem

Theorem 1.1. *X is metrizable iff X is first countable and has a σ-HCP k-network.*

In 1985, L. Foged [Fo_2] gave a beautiful characterization of Lašnev space

Theorem 1.2. *X is a closed image of a metric space iff X is a Fréchet space and has a σ-HCP k-network .*

With this powerful result, H. Hung, Mizokami obtained some interesting characterizations of Lašnev space.

In 1988, S. Lin [L_1] showed that the following

Theorem 1.3. *Lašnev space X is not an $\aleph$-space iff X contains closed copy of S_{ω_1}.*

In 1992, he improved his result [L_2]

Theorem 1.4. *A k-space with a σ-HCP k-network is an $\aleph$-space iff it contains no closed copy of S_{ω_1}.*

But the most interesting result on σ-HCP k-network was obtained by Junnila and Ziqiu Yun, they pointed out the difference between $\aleph$-spaces and spaces with a σ-HCP k-network.

Theorem 1.5. [JY] *Let X have a σ-HCP k-network . Then X is an $\aleph$-space iff X contains no closed copy of S_{ω_1}.*

Theorem 1.5 has some corollaries

Corollary 1.1. *A space X is an $\aleph$-space iff X has a σ-HCP k-network and a point-countable closed k-network.*

Corollary 1.2. *If X is a space with a σ-HCP k-network and $\chi(X) \leq \omega_1$, then X is an $\aleph$-space.*

In 1991, Y. Tanaka [T1] and in 1992, S. Lin [L2] independently got the following result on g-metrizable space.

Theorem 1.6. *The following are equivalent for a space X*

(a) *X is g-metrizable.*
(b) *X is a k-space with a σ-HCP weak base.*
(c) *X is a weakly first countable space with a σ-HCP k-network.*

Recently, we improved Tanaka and S. Lin's Theorem.

Theorem 1.7. [LD] *A k-space with a σ-HCP k-network is g-metrizable iff it contains no closed copy of S_ω.*

Now we will discuss some special spaces with a σ-HCP k-network.

Recall that a space X is said to satisfy DCCC, if every discrete open collection in X is countable.

Theorem 1.8. *For a space X, the following are equivalent*

(1) $2^\omega \leq 2^{\omega_1}$
(2) *Every normal, character $\leq 2^\omega$, DCCC space with a σ-HCP k-network is an $\aleph_0$-space.*

Proof: (1) $\to$ (2) By Theorem 2.2 in [Ta], character $\leq 2^\omega$, normal space is ω_1-weakly collectionwise Hausdorff. It is easy to prove that a normal, ω_1-weakly collectionwise Hausdorff, DCCC space is $\aleph_1$-compact. Hence it is an $\aleph_0$-space [L3].

(2) $\to$ (1) If $2^\omega = 2^{\omega_1}$. We will modify Tall's example (Example D in [Ta]). Let $\{A_\alpha : \alpha \in 2^{\omega_1} = 2^\omega\}$ be a maximal family of independent subsets of ω. Let $\{H_\alpha : \alpha \in 2^{\omega_1} = 2^\omega\}$ enumerate $\mathcal{P}(\omega_1)$. Define $U'_\zeta \subset \mathcal{P}(\omega), \zeta \in \omega_1$, by $A_\alpha \in U'_\zeta$ if $\zeta \in H_\alpha; \omega - A_\alpha \in U'_\zeta$ if $\zeta \overline{\in} H_\alpha$. Then U'_ζ is a filter subbase. Let U'_ζ generate a filter on ω and extend it to an ultrafilter U_ζ. Consider the U_ζ's as points in $\beta\omega - \omega$. We endow $X = \omega \cup \{U_\zeta : \zeta < \omega_1\}$ with subspace topology of $\beta\omega$. Then X is normal, separable but it is not $\aleph_1$-compact. Since any convergent sequence in $\beta\omega$ is trivial sequence. Then so is X. Let $\mathcal{P}_0 = \{\{U_\zeta\} : \zeta < \omega_1\} \cup \{\{0\}\}$, $\mathcal{P}_n = \{\{n\}\}$ for each $n \in N$. Then $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$

is a σ-HCP network of X. By Proposition 1.2 in [T_6], $\mathcal{P}$ is a σ-HCP k-network. Thus X is a normal, character $\leq 2^\omega$, DCCC space with a σ-HCP k-network, but it is not an $\aleph_0$-space, a contradiction. Hence $2^\omega < 2^{\omega_1}$.

Corollary 1.3. *The following are equivalent*

(1) $2^\omega < 2^{\omega_1}$

(2) *Every normal, separable space with a σ-HCP k-network is an $\aleph_0$-space.*

From Corollary 1.2, we can get

Theorem 1.9. (CH) *Every separable space with a σ-HCP k-network is an $\aleph$-space.*

Remark Under $MA + \neg$CH, Z. Yun [Y_2] showed that there is a separable, normal space with a σ-HCP k-network which is not an $\aleph$-space, so whether every separable space with a σ-HCP k-network is an $\aleph$-space is independent of the axioms of set theory.

Theorem 1.10. [Lc_2] *Every normal, DCCC, k-space with a σ-HCP k-network is an $\aleph_0$-space.*

Also we showed that the space with a σ-HCP k-network has a σ-compact finite k-network and pointed out that space with a σ-compact finite k-network need not have a σ-HCP k-network [Lc_1].

2. The product of spaces with a σ-HCP k-network

It is well known that product of countable many $\aleph$-spaces is an $\aleph$-space. But the product of two spaces with a σ-HCP k-network need not have a σ-HCP k-network [L_4].

Junnila and Ziqiu Yun [JY] showed the following interesting theorem:

Theorem 2.1. *If X and Y have a σ-HCP k-network , then $X \times Y$ has a σ-HCP k-network iff either both X and Y are $\aleph$-space or in one of them every compact subset is finite.*

The following corollaries are from Theorem 2.1.

Corollary 2.1. *If $X \times Y$ has a σ-HCP k-network, then either X or Y is an $\aleph$-space.*

Corollary 2.2. *If X^2 has a σ-HCP k-network , then X is an $\aleph$-space.*

Corollary 2.3. *For each $n \in N$, let X_n have at least two points. If $\prod_{n \in N} X_n$ has a σ-HCP k-network , then it is an $\aleph$-space.*

Let S be the subspace $\{1/n : n \in N\} \cup \{0\}$ of R in usual topology.

Corollary 2.4. *If $S \times X$ has a σ-HCP k-network , then X is an $\aleph$-space.*

Corollary 2.5. *If both X and Y are k-spaces and $X \times Y$ has a σ-HCP k-network , then both X and Y are $\aleph$-spaces or one of them is discrete.*

Now we consider when the product of spaces with a σ-HCP k-network is a k-space.

First, we need a Lemma. Combining with Gruenhage's Lemma [Gr] and the fact that S_ω is a perfect mapping image of Aren's space S_2. We have the following Lemma.

Lemma. *The following are equivalent*

(1) $\neg BF(\omega_2)$
(2) $S_\omega \times S_{\omega_1}$ *is not a k-space*
(3) $S_2 \times S_{\omega_1}$ *is not a k-space.*

A space X is said to belong to class $\mathcal{T}$ if it is the union of countably many closed and locally compact subset X_n such that $A \subset X$ is closed whenever $A \cap X_n$ is closed in X_n for all $n = 1, 2, 3, \ldots$.

Theorem 2.2. *Let $\mathcal{A}$ be the class which contains all k-spaces with a σ-HCP k-network. Then the following are equivalent*

(a) $\neg BF(\omega_2)$

(b) *For any* $X, Y \in \mathcal{A}$, $X \times Y$ *is a k-space iff one of the following holds*

(1) *Both* X *and* Y *are metrizable.*

(2) X *or* Y *is a locally compact, metrizable space.*

(3) X *and* Y *are* $\aleph$*-spaces and in the class* $\mathcal{T}$.

Proof: (b) → (a) $S_\omega, S_{\omega_1} \in \mathcal{A}$. But S_ω and S_{ω_1} don't satisfy (1), (2) and (3), thus $S_\omega \times S_{\omega_1}$ is not a k-space. Then (a) holds by Lemma.

(a) → (b) It is easy to prove that if X and Y satisfy one of (1), (2) and (3), then $X \times Y$ is a k-space.

Let $X, Y \in \mathcal{A}$ and $X \times Y$ be a k-space. If X and Y are $\aleph$-spaces, then one of (1), (2) and (3) holds [T_2]. If both X and Y are not $\aleph$-spaces, then X and Y contain a closed copy of S_{ω_1} by Theorem 1.5, thus $S_{\omega_1} \times S_{\omega_1}$ is a k-space. This contradicts with Lemma 5 in [Gr]. If one of them is an $\aleph$-space, without loss of generality, let Y be an $\aleph$-space. Then X contains a closed copy of S_{ω_1}. Y contains no clos ed copy of S_2 and no closed copy of S_ω. Otherwise $S_{\omega_1} \times S_2$ or $S_{\omega_1} \times S_\omega$ is a k-space, a contradiction. Since Y is a space in which every point is a G_δ. Then Y is a sequential space, hence Y is a strong Fréchet space [T_3]. Thus Y is metrizable [T_2]. Since $S_{\omega_1} \times Y$ is a k-space and S_{ω_1} is not a strong Fréchet. Then Y is locally compact by Theorem 4.5 [T_2]. □

Comparing with Corollary 2.1, we have the following

Theorem 2.3. *Let* X *and* Y *have a* σ*-HCP k-network. If* $X \times Y$ *is a k-space, then either* X *or* Y *is an* $\aleph$*-space.*

Proof: If neither X nor Y is an $\aleph$-space, then both X and Y contain a closed copy of S_{ω_1}, hence $S_{\omega_1} \times S_{\omega_1}$ is a k-space. This is a contradiction [Gr]. □

Remark: The referee informed the author that Theorem 2.3 also was shown by S. Lin in his 1988's preprint.

Corollary 2.6. *Let* X *have a* σ*-HCP k-network. If* X^2 *is a k-space, then* X *is an* $\aleph$*-space.*

Also, Tanaka [T_4] showed the following

Theorem 2.4. *Let X and Y have a σ-HCP. Then $X \times Y$ is weakly first countable iff one of the following holds*

(a) *X and Y are metrizable*
(b) *X or Y is a locally compact, metrizable space and both X and Y are weakly first countable*
(c) *X and Y have a σ-locally finite weak base consisting of compact subsets*

3. Some mapping theorems

First, Tanaka [T_1] showed the following

Theorem 3.1. *Let $f : X \to Y$ be a closed mapping. If X has a σ-HCP k-network. Then so does Y.*

Comparing with Theorem 3.4 in [L_5], we have the following

Theorem 3.2. *There is a space X with no σ-HCP k-network which has a G_δ-diagonal, a space Y which has a σ-HCP k-network, and a perfect mapping $f : X \to Y$.*

Proof: Let I denote the unit closed interval. By Theorem 2.1, $S_{\omega_1} \times I$ has not σ-HCP k-network. But it is a σ-space. Let $f : S_{\omega_1} \times I \to S_{\omega_1}$ be a project mapping. Then f is a perfect mapping. □

Theorem 3.3 *Let $f : X \to Y$ be a closed mapping of a paracompact $\aleph$-space X onto a space Y. If Y^2 is a k-space, then $\partial f^{-1}(y)$ is Lindelöf for each $y \in Y$.*

Proof: By Theorem 3.1 and Corollary 2.6, Y is an $\aleph$-space, hence Y has a point-countable closed k-network. Thus $\partial f^{-1}(y)$ is Lindelöf for each $y \in Y$ by Proposition 6.4 in [GMT]. □

Remark: Theorem 3.3 gives an affirmative answer to Tanaka's question [T_6]: "Let Y be a closed image, under a map f, of a metric space, then, is each $\partial f^{-1}(y)$ a Lindelöf space when

Y^2 is a k-space" G. Gruenhage [Gr] answers this question in another way.

Zimin Gao [Ga] and S. Lin [L_5] proved independently the following:

Theorem 3.4. *Let $f : X \to Y$ be a closed Lindelöf mapping. If X is an $\aleph$-space; then so is Y.*

We could prove this theorem in another way.

Proof: By Theorem 3.1, Y has a σ-HCP k-network. If Y has a closed copy of S_{ω_1}, then $f|_{f^{-1}(S_{\omega_1})} : f^{-1}(S_{\omega_1}) \to S_{\omega_1}$ also is a closed Lindelöf mapping. So $f^{-1}(S_{\omega_1})$ is a paracompact $\aleph$-space (Note: S_{ω_1} is a paracompact space). By Proposition 6.4 in [GMT], S_{ω_1} contains a point-countable closed k-network. This contradicts with Example 9.2 in [GMT]. Hence Y is an $\aleph$-space by Theorem 1.5. □

Theorem 3.5. *$\aleph$-spaces are preserved by open and closed mappings.*

Proof: Let $f : X \to Y$ be an open and closed mapping, X be an $\aleph$-space. By Theorem 3.1, Y has a σ-HCP k-network. If Y contains a closed copy of S_{ω_1}, then for each $\alpha < \omega_1$, let $S^\alpha = \{x_n^\alpha : n \in N\}$ be one of the spines of $S_{\omega_1} \cdot x_n^\alpha \to x_0$. Pick $y \in f^{-1}(x_0)$. Since X is perfect. There exists $\{G_k : k \in N\}$ with G_k open such that $\{y\} = \bigcap_{k\in N} \overline{G}_k = \bigcap_{k\in N} G_k$ and $\overline{G}_{k+1} \subset G_k$. then $f(G_k)$ is open in Y. For each $\alpha < \omega_1$, we choose a subsequence $\{x_{n,k}^\alpha : k \in N\}$ of $\{x_n^\alpha : n \in N\}$ such that $x_{n,k}^\alpha \in f(G_k)$. Pick $y_k^\alpha \in G_k \cap f^{-1}(x_{n,k}^\alpha)$. It is easy to show that $y_k^\alpha \to y$. Put $X_1 = \{y\} \cup \{y_k^\alpha : \alpha < \omega_1, k \in N\}$. Obviously, X_1 is homemorphic to S_{ω_1}, a contradiction. Hence Y is an $\aleph$-space by Theorem 1.5. □

Remark: Theorem 3.5 also was proved by Z. Yun in another way [Y_1]

4. Some Problems

Problem 4.1. Is a separable k-space with a σ-HCP k-network an $\aleph_0$-space? (Yes, if (CH))

Problem 4.2. Are k-spaces with a σ-HCP k-network hereditarily meta-Lindelöf?

Remark: An affirmative answer to problem 4.2 implies an affirmative answer to the problem 4.1.

Problem 4.3. [L_6] Is every space with a σ-HCP k-network a closed image of an $\aleph$-space?

Problem 4.4. Is a k-space with a σ-HCP k-network a closed image of a g-metrizable space?

Remark: A negative answer to problem 4.2 implies a negative answer to problem 4.4.

Problem 4.5. Is a stratifiable Fréchet space a Lašnev space if the space has a point-countable k-network?

References

[Fo_1] L. Foged, *On g-metrizability*, Pacific J. Math., **98** (1982), 327-332.

[Fo_2] ——, *A characterization of closed images of metric spaces*, Proc. Amer. Math. Soc., **95** (1985), 487-490.

[Ga] Z. min Gao, *ℵ-spaces is invariant under perfect mapping*, Q. & A. in General Topology, **5** (1987), 271-279.

[Gr] G. Gruenhage, *k-spaces and products of closed images of metric space*, Proc. Amer. Math. Soc., **80** (1980), 478-482

[GMT] ——, E. Michael and Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math., **113** (1984), 303-332.

[JY] H. Junnila and Yun Ziqiu, *ℵ-spaces and spaces with a σ-hereditarily closure preserving k-network*, Top. Appl., **44** (1992), 209-215.

[Lc_1] C. Liu, *Spaces with a σ-compact finite k-network*, Q & A. in General Topology, **10** (1992), 81-87.

[Lc_2] ——, *Notes on spaces with a σ-HCP k-network*, to appear.

[LD] C. Liu and M. Dai, *G-metrizability and* S_ω, to appear.

[L_1] S. Lin, *On Lašnev spaces*, Acta. Math. Sinica, (PRC) **34** (1991), 222-225.

[L_2] ——, *On g-metrizable spaces*, Chinese Ann. of Math., 13A: **3** (1992), 403-409.

[L3] ——, *A study of pseudobases*, Q. & A. in General Topology, **6** (1988), 81-97.

[L4] ——, *On a problem of K. Tamano*, Q. & A. in General Topology, **6** (1988), 99-102.

[L5] ——, *Mapping theorems on $\aleph$-spaces*, Top. Appl., **30** (1988), 159-164.

[L6] ——, *A survey of the theory of $\aleph$-spaces*, Q. & A. in General Topology, **8** (1990), 405-419.

[T1] Y. Tanaka, *σ-hereditarily closure-preserving k-networks and g-metrizability*, Proc. Amer. Math. Soc., **112** (1991), 283-290.

[T2] ——, *A characterization for the products of k- and $\aleph_0$-spaces and related results*, Proc. Amer. Math. Soc., **59** (1976), 149-155.

[T3] ——, *Metrization II*, Topics in General Topology, Eds. M. Morita and J. Nagata, Amsterdam, 1989, 275-314.

[T4] ——, *Symmetric spaces, g-developable spaces and g-metrizable spaces*, Math. Japonica, **36** (1991), 71-84.

[T5] ——, *On the k-ness for the products and closed images of metric spaces*, General Top. Appl., **9** (1978), 175-183.

[T6] ——, *Point-countable covers and k-networks*, Top. Proc., **12** (1987), 327-349.

[Ta] F. Tall, *Normality versus collectionwise normality*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 685-732.

[Y1] Z. Yun, *A new characterization of $\aleph$-spaces*, Top. Proc., **16** (1991), 253-256.

[Y2] ——, *Separability and $\aleph$-spaces*, Q. & A. in General Topology, **11** (1993), 109-111.

Guangxi University
Nanning, Guangxi 530004
P. R. China